QUADRATIC EQUATION & EXPRESSION

1. Quadratic expression :

A polynomial of degree two of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression in x.

2. Quadratic equation :

An equation $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in R$ has two and only two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

3. Nature of roots :

Nature of the roots of the given equation depends upon the nature of its discriminant D i.e. $b^2 - 4ac$.

Suppose $a, b, c \in R$, $a \neq 0$ then

(i) If $D > 0$ $\Rightarrow$ roots are real and distinct (unequal)

(ii) If $D = 0$ $\Rightarrow$ roots are real and equal (Coincident)

(iii) If $D < 0$ $\Rightarrow$ roots are imaginary and unequal i.e. non real complex numbers.

Suppose $a, b, c \in Q$ $a \neq 0$ then

(i) If $D > 0$ and D is a perfect square $\Rightarrow$ roots are rational & unequal

(ii) If $D > 0$ and D is not a perfect square $\Rightarrow$ roots are irrational and unequal.

For a quadratic equation their will exist exactly 2 roots real or imaginary. If the equation $ax^2 + bx + c = 0$ is satisfied for more than 2 distinct values of x, then it will be an identity & will be satisfied by all x. Also in this case $a = b = c = 0$.

4. Conjugate roots :

Irrational roots and complex roots occur in conjugate pairs i.e.

if one root $\alpha + i\beta$, then other root $\alpha - i\beta$

if one root $\alpha + \sqrt{\beta}$, then other root $\alpha - \sqrt{\beta}$

5. Sum of roots :

$$S = \alpha + \beta = \frac{-b}{a} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

Product of roots :

$$P = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{Coefficient of } x^2}$$

6. Formation of an equation with given roots :

$x^2 - Sx + P = 0$

$\Rightarrow$ x^2 – (Sum of roots) x + Product of roots = 0

7. Roots under particular cases :

For the equation $ax^2 + bx + c = 0$, $a \neq 0$

(i) If $b = 0 \Rightarrow$ roots are of equal magnitude but of opposite sign.

(ii) If $c = 0 \Rightarrow$ one root is zero and other is $-b/a$

(iii) If $b = c = 0 \Rightarrow$ both roots are zero

(iv) If $a = c \Rightarrow$ roots are reciprocal to each other.

(v) If $a > 0, c < 0$ or $a < 0, c > 0 \Rightarrow$ roots are of opposite signs

(vi) If $a > 0, b > 0, c > 0$ or $a < 0, b < 0, c < 0 \Rightarrow$ both roots are –ve

(vii) If $a > 0, b < 0, c > 0$ or $a < 0, b > 0, c < 0 \Rightarrow$ both roots are +ve.

8. Symmetric function of the roots :

If roots of quadratic equation $ax^2 + bx + c, a \neq 0$ are α and β, then

(i) $(\alpha - \beta) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \pm \dfrac{\sqrt{b^2 - 4ac}}{a}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \dfrac{b^2 - 2ac}{a^2}$

(iii) $\alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \dfrac{-b\sqrt{b^2 - 4ac}}{a^2}$

(iv) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3(\alpha + \beta)\alpha\beta = \dfrac{-b(b^2 - 3ac)}{a^3}$

(v) $\alpha^3 - \beta^3 = (\alpha - \beta)[\alpha^2 + \beta^2 - \alpha\beta]$

$= \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\,[\alpha^2 + \beta^2 - \alpha\beta]$

$= \dfrac{(b^2 - ac)\sqrt{b^2 - 4ac}}{a^3}$

(vi) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$= \{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 2\alpha^2\beta^2 = \left(\dfrac{b^2 - 2ac}{a^2}\right)^2 - \dfrac{2c^2}{a^2}$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha^2 - \beta^2) = \dfrac{-b(b^2 - 2ac)\sqrt{b^2 - 4ac}}{a^4}$

(viii) $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta = \dfrac{b^2 + ac}{a^2}$

(ix) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

(x) $\left(\dfrac{\alpha}{\beta}\right)^2 + \left(\dfrac{\beta}{\alpha}\right)^2 = \dfrac{\alpha^4 + \beta^4}{\alpha^2\beta^2} = \dfrac{[(b^2 - 2ac)^2 - 2a^2c^2]}{a^2c^2}$

9. Condition for common roots :

The equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have

(i) One common root if $\dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

(ii) Both roots common if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

10. Maximum and Minimum value of quadratic expression :

In a quadratic expression $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$,

Where $D = b^2 - 4ac$

(i) If $a > 0$, quadratic expression has minimum value $\frac{4ac - b^2}{4a}$ at $x = \frac{-b}{2a}$ and there is no maximum value.

(ii) If $a < 0$, quadratic expression has maximum value $\frac{4ac - b^2}{4a}$ at $x = \frac{-b}{2a}$ and there is no minimum value.

11. Location of roots :

Let $f(x) = ax^2 + bx + c$, $a \neq 0$ then w.r.to $f(x) = 0$

(i) If k lies between the roots then $a.f(k) < 0$ (necessary & sufficient)

(ii) If between k_1 & k_2 their is exactly one root of k_1, k_2 themselves are not roots

$f(k_1) . f(k_2) < 0$ (necessary & sufficient)

(iii) If both the roots are less than a number k

$D \geq 0$, $a.f(k) > 0$, $\frac{-b}{2a} < k$ (necessary & sufficient)

(iv) If both the roots are greater than k

$D \geq 0$, $a.f(k) > 0$, $\frac{-b}{2a} > k$ (necessary & sufficient)

(v) If both the roots lies in the interval (k_1, k_2)

$D \geq 0$, $a.f(k_1) > 0$, $a.f(k_2) > 0$, $k_1 < \frac{-b}{2a} < k_2$

(vi) If k_1, k_2 lies between the roots

$a.f(k_1) < 0$, $a.f(k_2) < 0$

(vii) λ will be the repeated root of $f(x) = 0$ if

$f(\lambda) = 0$ and $f'(\lambda) = 0$

12. For cubic equation $ax^3 + bx^2 + cx + d = 0$:

We have $\alpha + \beta + \gamma = \frac{-b}{a}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and $\alpha\beta\gamma = \frac{-d}{a}$

where α, β, γ are its roots.

13. For biquadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$:

We have $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$, $\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \gamma\delta\beta = \frac{-d}{a}$

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$ and $\alpha\beta\gamma\delta = \frac{e}{a}$

COMPLEX NUMBER

1. Complex Number :

A number of the form z = x + iy (x, y $\in$ R, i = $\sqrt{-1}$) is called a complex number, where x is called a real part i.e. x = Re(z) and y is called an imaginary part i.e. y = Im(z).

Modulus $|z| = \sqrt{x^2 + y^2}$,

amplitude or amp(z) = arg(z) = $\theta = \tan^{-1}\frac{y}{x}$.

(i) Polar representation :

$x = r\cos\theta$, $y = r\sin\theta$, $r = |z| = \sqrt{x^2 + y^2}$

(ii) Exponential form :

$z = re^{i\theta}$, where r = |z|, θ = amp.(z)

(iii) Vector representation :

P(x, y) then its vector representation is $z = \overrightarrow{OP}$

2. Integral Power of Iota :

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Hence $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, i^{4n} or $i^{4(n+1)} = 1$

3. Complex conjugate of z :

If z = x + iy, then $\bar{z}$ = x – iy is called complex conjugate of z

* $\bar{z}$ is the mirror image of z in the real axis.
* $|z| = |\bar{z}|$
* $z + \bar{z} = 2Re(z)$ = purely real

* $z - \bar{z} = 2i\,I_m(z)$ = purely imaginary
* $z\bar{z} = |z|^2$
* $\overline{z_1 + z_2 + \dots + z_n} = \bar{z}_1 + \bar{z}_2 + \dots\dots\dots + \bar{z}_n$
* $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
* $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
* $\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\bar{z}_1}{\bar{z}_2}\right)$ (provided $z_2 \neq 0$)
* $\overline{(z^n)} = (\bar{z})^n$
* $\overline{(\bar{z})} = z$
* If $\alpha = f(z)$, then $\bar{\alpha} = f(\bar{z})$

  Where α = f(z) is a function in a complex variable with real coefficients.
* $z + \bar{z} = 0$ or $z = -\bar{z}$ $\Rightarrow$ z = 0 or z is purely imaginary
* $z = \bar{z}$ $\Rightarrow$ z is purely real

4. Modulus of a complex number :

Magnitude of a complex number z is denoted as |z| and is defined as

$$|z| = \sqrt{(Re(z))^2 + (Im(z))^2}\ ,\ |z| \geq 0$$

(i) $z\bar{z} = |z|^2 = |\bar{z}|^2$

(ii) $z^{-1} = \frac{\bar{z}}{|z|^2}$

(iii) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\,Re\,(z_1\bar{z}_2)$

(iv) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$

(v) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(vi) $|z_1 \pm z_2| \ge |z_1| - |z_2|$

5. Argument of a complex number :

Argument of a complex number z is the $\angle$ made by its radius vector with +ve direction of real axis.

$$\arg z = \theta, \quad z \in 1^{st} \text{ quad.}$$
$$= \pi - \theta, \quad z \in 2^{nd} \text{ quad.}$$
$$= -\theta, \quad z \in 3^{rd} \text{ quad.}$$
$$= \theta - \pi, \quad z \in 4^{th} \text{ quad.}$$

(i) arg (any real + ve no.) = 0

(ii) arg (any real – ve no.) = π

(iii) $\arg(z - \bar{z}) = \pm \pi/2$

(iv) $\arg(z_1.z_2) = \arg z_1 + \arg z_2 + 2k\pi$

(v) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$

(vi) $\arg(\bar{z}) = -\arg z = \arg\left(\frac{1}{z}\right)$, if z is non real

$= \arg z$, if z is real

(vii) $\arg(-z) = \arg z + \pi, \arg z \in (-\pi, 0]$

$= \arg z - \pi, \arg z \in (0, \pi]$

(viii) $\arg(z^n) = n \arg z + 2k\pi$

(ix) $\arg z + \arg \bar{z} = 0$

argument function behaves like log function.

6. Square root of a complex no.

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b > 0$$

$$= \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b < 0$$

7. De-Moiver's Theorem :

It states that if n is rational number, then

$(\cos\theta + i\sin\theta)^n = \cos\theta + i\sin n\theta$

and $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i \sin n\theta$

8. Euler's formulae as $z = re^{i\theta}$, where

$e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore \quad e^{i\theta} + e^{-i\theta} = 2\cos\theta$ and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

9. n^{th} roots of complex number $z^{1/n}$

$$= r^{1/n}\left[\cos\left(\frac{2m\pi+\theta}{n}\right) + i\sin\left(\frac{2m\pi+\theta}{n}\right)\right],$$

where m = 0, 1, 2, ...(n – 1)

(i) Sum of all roots of $z^{1/n}$ is always equal to zero

(ii) Product of all roots of $z^{1/n} = (-1)^{n-1} z$

10. Cube root of unity :

cube roots of unity are 1, ω, ω^2 where

$\omega = \frac{-1+i\sqrt{3}}{2}$ and $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$

11. Some important result :

If $z = \cos\theta + i\sin\theta$

(i) $z + \frac{1}{z} = 2\cos\theta$

(ii) $z - \frac{1}{z} = 2\,i\sin\theta$

(iii) $z^{n} + \frac{1}{z^{n}} = 2\cos n\theta$

(iv) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ & $z = \cos\gamma + i\sin\gamma$ and given $x + y + z = 0$, then

(a) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ (b) $yz + zx + xy = 0$

(c) $x^2 + y^2 + z^2 = 0$ (d) $x^3 + y^3 + z^3 = 3xyz$

12. Equation of Circle :

* $|z - z_1| = r$ represents a circle with centre z_1 and radius r.
* $|z| = r$ represents circle with centre at origin.
* $|z - z_1| < r$ and $|z - z_1| > r$ represents interior and exterior of circle $|z - z_1| = r$.
* $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents a general circle where $a \in c$ and $b \in R$.
* Let $|z| = r$ be the given circle, then equation of tangent at the point z_1 is $z\bar{z}_1 + \bar{z}z_1 = 2r^2$
* diametric form of circle :

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm\frac{\pi}{2},$$

or $$\frac{z - z_1}{z - z_2} + \frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2} = 0$$

or $$\left|z - \frac{z_1 + z_2}{2}\right| = \frac{|z_1 - z_2|}{2}$$

or $$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

Where z_1, z_2 are end points of diameter and z is any point on circle.

13. Some important points :

(i) Distance formula $PQ = |z_2 - z_1|$

(ii) Section formula

For internal division $= \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$

For external division $= \frac{m_1z_2 - m_2z_1}{m_1 - m_2}$

(iii) Equation of straight line.

* Parametric form $z = tz_1 + (1 - t)z_2$ where $t \in R$

* Non parametric form $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$

* Three points z_1, z_2, z_3 are collinear if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

or slope of AB = slope of BC = slope of AC.

(iv) The complex equation $\left|\frac{z - z_1}{z - z_2}\right| = k$ represents a circle if $k \neq 1$ and a straight line if $k = 1$.

(v) The triangle whose vertices are the points represented by complex numbers z_1, z_2, z_3 is equilateral if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

i.e. if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_1z_3$.

(vi) $|z - z_1| = |z - z_2| = \lambda$, represents an ellipse if $|z_1 - z_2| < \lambda$, having the points z_1 and z_2 as its foci and if $|z_1 - z_2| = \lambda$, then z lies on a line segment connecting z_1 & z_2

(vii) $|z - z_1| \sim |z - z_2| = \lambda$ represents a hyperbola if $|z_1 - z_2| > \lambda$, having the points z_1 and z_2 as its foci, and if $|z_1 - z_2| = \lambda$, then z lies on the line passing through z_1 and z_2 excluding the points between z_1 & z_2.

(viii) If four points z_1, z_2, z_3, z_4 are concyclic,

then $\left(\frac{z_1 - z_2}{z_1 - z_4}\right)\left(\frac{z_3 - z_4}{z_3 - z_2}\right)$ is purely real.

(ix) If three complex numbers are in A.P., then they lie on a straight line in the complex plane.

(x) If z_1, z_2, z_3 be the vertices of an equilateral triangle and z_0 be the circumcentre,

then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

(xi) If z_1, z_3, z_3 .... z_n be the vertices of a regular polygon of n sides & z_0 be its centroid, then

$z_1^2 + z_2^2 + ... + z_n^2 = nz_0^2$.

(xii) If z_1, z_2, z_3 be the vertices of a triangle, then the triangle is equilateral

iff $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0$.

(xiii) If z_1, z_2, z_3 are the vertices of an isosceles triangle, right angled at z_2,

then $z_1^2 + z_2^2 + z_3^2 = 2z_2 (z_1 + z_3)$.

(xiv) $z_1, z_2, z_3. z_4$ are vertices of a parallelogram then

$z_1 + z_3 = z_2 + z_4$

PERMUTATION & COMBINATION

1. Factorial notation -

The continuous product of first n natural numbers is called factorial

i.e. $\lfloor n$ or $n! = 1.\ 2.\ 3........(n-1).n$

$n! = n(n-1)! = n(n-1)(n-2)!$ & so on

or $n(n-1).........(n-r+1) = \dfrac{n!}{(n-r)!}$

Here $0! = 1$ and $(-n)!$ = meaningless.

2. Fundamental principle of counting -

(i) **Addition rule** : If there are two operations such that they can be done independently in m and n ways respectively, then either (any one) of these two operations can be done by $(m + n)$ ways.

Addition $\Rightarrow$ OR (or) Option

(ii) **Multiplication rule :** Let there are two tasks of an operation and if these two tasks can be performed in m and n different number of ways respectively, then the two tasks together can be done in $m \times n$ ways.

Multiplication $\Rightarrow$ And (or) Condition

(iii) **Bijection Rule :** Number of favourable cases

= Total number of cases

− Unfavourable number of cases.

3. Permutations (Arrangement of objects) -

(i) The number of permutations of n different things taken r at a time is $^np_r = \dfrac{n!}{(n-r)!}$

(ii) The number of permutations of n dissimilar things taken all at a time is $^np_n = n!$

(iii) The number of permutations of n distinct objects taken r at a time, when repetition of objects is allowed is n^r.

(iv) If out of n objects, 'a' are alike of one kind, 'b' are alike of second kind and 'c' are alike of third kind and the rest distinct, then the number of ways of permuting the n objects is $\dfrac{n!}{a!\,b!\,c!}$

4. Restricted Permutations -

(i) The number of permutations of n dissimilar things taken r at a time, when m particular things always occupy definite places $= {}^{n-m}p_{r-m}$

(ii) The number of permutations of n different things taken r at a time, when m particular things are always to be excluded (included)

$= {}^{n-m}P_r \; ({}^{n-m}C_{r-m} \times r!)$

5. Circular Permutations -

When clockwise & anticlockwise orders are treated as different.

(i) The number of circular permutations of n different things taken r at a time $\dfrac{^nP_r}{r}$

(ii) The number of circular permutations of n different things taken altogether $\dfrac{^nP_n}{n} = (n-1)!$

When clockwise & anticlockwise orders are treated as same.

(i) The number of circular permutations of n different things taken r at a time $\dfrac{^nP_r}{2r}$

(ii) The number of circular permutations of n different things taken all together $\dfrac{^nP_n}{2n} = \dfrac{1}{2}(n-1)!$

6. Combination (selection of objects) -

The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ or C (n, r)

$$ {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{{}^nP_r}{r!} $$

(i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) ${}^nC_r = {}^nC_s \Rightarrow r = s$ or $r + s = n$

(iv) ${}^nC_0 = {}^nC_n = 1$

(v) ${}^nC_1 = {}^nC_{n-1} = n$

(vi) ${}^nC_r = \frac{n}{r}\ {}^{n-1}C_{r-1}$

(vii) ${}^nC_r = \frac{1}{r}(n - r + 1)\ {}^nC_{r-1}$

7. Restricted combinations -

The number of combinations of n distinct objects taken r at a time, when k particular objects are always to be

(i) included is ${}^{n-k}C_{r-k}$

(ii) excluded is ${}^{n-k}C_r$

(iii) included and s particular things are to be excluded is ${}^{n-k-s}C_{r-k}$

8. Total number of combinations in different cases -

(i) The number of selections of n identical objects, taken at least one = n

(ii) The number of selections from n different objects, taken at least one

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$

(iii) The number of selections of r objects out of n identical objects is 1.

(iv) Total number of selections of zero or more objects from n identical objects is n + 1.

(v) Total number of selections of zero or more objects out of n different objects

$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n$

(vi) The total number of selections of at least one out of $a_1 + a_2 + ... + a_n$ objects where a_1 are alike (of one kind), a_2 are alike (of second kind), ... a_n are alike (of n^{th} kind) is

$[(a_1 + 1)(a_2 + 1)(a_3 + 1) + ... + (a_n + 1)] - 1$

(vii) The number of selections taking atleast one out of $a_1 + a_2 + a_3 + ... + a_n + k$ objects when a_1 are alike (of one kind), a_2 are alike (of second kind), ... a_n are alike (of k^{th} kind) and k are distinct is

$[(a_1 + 1)(a_2 + 1)(a_3 + 1) .... (a_n + 1)]\ 2^k - 1$

9. Division and distribution -

(i) The number of ways in which (m + n + p) different objects can be divided into there groups containing m, n, & p different objects respectively is $\frac{(m+n+p)!}{m!\,n!\,p!}$

(ii) The total number of ways in which n different objects are to be divided into r groups of group sizes n_1, n_2, n_3, ... n_r respectively such that size of no two groups is same is $\frac{n!}{n_1!\ n_2!............n_r!}$.

(iii) The total number of ways in which n different objects are to be divided into groups such that k_1 groups have group size n_1, k_2 groups have group size n_2 and so on, k_r groups have group size n_r, is given as

$$ \frac{n!}{(n_1!)^{k_1}(n_2!)^{k_2}............(n_r!)^{k_r}\ k_1!\ k_2!............\ k_r!}. $$

(iv) The total number of ways in which n different objects are divided into k groups of fixed group size and are distributed among k persons (one group to each) is given as
(number of ways of group formation) × k!

10. Selection of light objects and multinomial theorem -

(i) The coefficient of x^n in the expansion of $(1 - x^{-r})$ is equal to ${}^{n+r-1}C_{r-1}$

(ii) The number of solution of the equation $x_1 + x_2 + \ldots\ldots\ldots + x_r = n$, $n \in N$ under the condition $n_1 \le x_1 \le n'_1$, $n_2 \le x_2 \le n'_2$, ... $n_r \le x_r \le n'_r$
where all x'_is are integers is given as
Coefficient of x^n is

$$\left[\left(x^{n_1} + x^{n_1+1} + \ldots + x^{n'_1}\right)\left(x^{n_2} + x^{n_2+1} + \ldots + x^{n'_2}\right)\ldots\left(x^{n_r} + x^{n_r+1} + \ldots + x^{n'_r}\right)\right]$$

11. Derangement Theorem -

(i) If n things are arranged in a row, then the number of ways in which they can be rearranged so that no one of them occupies the place assigned to it is

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}\right]$$

(ii) If n things are arranged at n places then the number of ways to rearrange exactly r things at right places is

$$= \frac{n!}{r}\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$$

12. Some Important results -

(a) Number of total different straight lines formed by joining the n points on a plane of which m(<n) are collinear is ${}^nC_2 - {}^mC_2 + 1$.

(b) Number of total triangles formed by joining the n points on a plane of which m(< n) are collinear is ${}^nC_3 - {}^mC_3$.

(c) Number of diagonals in a polygon of n sides is ${}^nC_2 - n$.

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelogram so formed is ${}^mC_2 \times {}^nC_2$.

(e) Given n points on the circumference of a circle, then
number of straight lines ${}^nC_2$
number of triangles ${}^nC_3$
number of quadrilaterals ${}^nC_4$

(f) If n straight lines are drawn in the plane such that no two lines are parallel and no three lines are concurrent. Then the number of part into which these lines divide the plane is $= 1 + \Sigma n$

(g) Number of rectangles of any size in a square of n × n is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(h) Number of rectangles of any size in a rectangle of n × p is $\frac{np}{4}(n + 1)(p + 1)$ and number of squares of any size is $\sum_{r=1}^{n}(n + 1 - r)(p + 1 - r)$.

PROBABILITY

1. Mathematical definition of probability :

Probability of an event

$$= \frac{\text{No. of favourable cases to event A}}{\text{Total no. of cases}}$$

Note : (i) $0 \le P(A) \le 1$

(ii) Probability of an impossible event is zero

(iii) Probability of a sure event is one.

(iv) P(A) + P(Not A) = 1 i.e. $P(A) + P(\bar{A}) = 1$

2. Odds for an event :

If $P(A) = \frac{m}{n}$ and $P(\bar{A}) = \frac{n-m}{n}$

Then odds in favour of A = $\frac{P(A)}{P(\bar{A})} = \frac{m}{n-m}$

and odds in against of A = $\frac{P(\bar{A})}{P(A)} = \frac{n-m}{m}$

3. Set theoretical notation of probability and some important results :

(i) $P(A + B) = 1 - P(\bar{A}\,\bar{B})$

(ii) $P(A/B) = \frac{P(AB)}{P(B)}$

(iii) $P(A + B) = P(AB) + P(\bar{A}B) + P(A\bar{B})$

(iv) $A \subset B \Rightarrow P(A) \le P(B)$

(v) $P(\bar{A}B) = P(B) - P(AB)$

(vi) $P(AB) \le P(A)\,P(B) \le P(A + B) \le P(A) + P(B)$

(vii) P(Exactly one event) = $P(A\bar{B}) + P(\bar{A}B)$

(viii) $P(\bar{A} + \bar{B}) = 1 - P(AB) = P(A) + P(B) - 2P(AB)$

$= P(A + B) - P(AB)$

(ix) P(neither A nor B) = $P(\bar{A}\,\bar{B}) = 1 - P(A + B)$

(x) When a coin is tossed n times or n coins are tossed once, the probability of each simple event is $\frac{1}{2^n}$.

(xi) When a dice is rolled n times or n dice are rolled once, the probability of each simple event is $\frac{1}{6^n}$.

(xii) When n cards are drawn ($1 \le n \le 52$) from well shuffled deck of 52 cards, the probability of each simple event is $\frac{1}{{}^{52}C_n}$.

(xiii) If n cards are drawn one after the other with replacement, the probability of each simple event is $\frac{1}{(52)^n}$.

(xiv) P(none) = 1 – P (atleast one)

(xv) Playing cards :

(a) Total cards : 52 (26 red, 26 black)

(b) Four suits : Heart, diamond, spade, club (13 cards each)

(c) Court (face) cards : 12 (4 kings, 4 queens, 4 jacks)

(d) Honour cards : 16 (4 Aces, 4 kings, 4 queens, 4 Jacks)

(xvi) Probability regarding n letters and their envelopes :

If n letters corresponding to n envelopes are placed in the envelopes at random, then

(a) Probability that all the letters are in right envelopes = $\frac{1}{n!}$

(b) Probability that all letters are not in right envelopes = $1 - \frac{1}{n!}$

(c) Probability that no letters are in right envelope

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \ldots\ldots + (-1)^n \frac{1}{n!}$$

(d) Probability that exactly r letters are in right envelopes = $\frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots + (-1)^{n-r}\frac{1}{(n-r)!}\right]$

4. Addition Theorem of Probability :

(i) When events are mutually exclusive

i.e. $n(A \cap B) = 0 \quad \Rightarrow \quad P(A \cap B) = 0$

$\therefore\ P(A \cup B) = P(A) + P(B)$

(ii) When events are not mutually exclusive i.e.

$P(A \cap B) \neq 0$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or $\quad P(A + B) = P(A) + P(B) - P(AB)$

(iii) When events are independent i.e. $P(A \cap B) = P(A)\,P(B)$

$\therefore \quad P(A + B) = P(A) + P(B) - P(A)\,P(B)$

5. Conditional probability :

P(A/B) = Probability of occurrence of A, given that B has already happened = $\frac{P(A \cap B)}{P(B)}$

P(B/A) = Probability of occurrence of B, given that A has already happened = $\frac{P(A \cap B)}{P(A)}$

Note : If the outcomes of the experiment are equally likely, then P(A/B) = $\frac{\text{No. of sample pts. in } A \cap B}{\text{No. of pts. in } B}$.

(i) If A and B are independent event, then P(A/B) = P(A) and P(B/A) = P(B)

(ii) Multiplication Theorem :

$P(A \cap B) = P(A/B).\ P(B),\ P(B) \neq 0$

or $\quad P(A \cap B) = P(B/A)\ P(A),\ P(A) \neq 0$

Generalized :

$P(E_1 \cap E_2 \cap E_3 \cap \ldots\ldots\ldots\ldots\ldots \cap E_n)$

$= P(E_1)\ P(E_2/E_1)\ P(E_3/E_1 \cap E_2)\ P(E_4/E_1 \cap E_2 \cap E_3) \ldots\ldots\ldots$

If events are independent, then

$P(E_1 \cap E_2 \cap E_3 \cap \ldots\ldots\ldots \cap E_n) = P(E_1)\ P(E_2) \ldots\ldots\ldots P(E_n)$

6. Probability of at least one of the n Independent events :

If $P_1, P_2, \ldots\ldots\ P_n$ are the probabilities of n independent events $A, A_2, \ldots.\ A_n$ then the probability of happening of at least one of these event is.

$$1 - [(1 - P_1)(1 - P_2) \ldots\ldots (1 - P_n)]$$

or $\quad P(A_1 + A_2 + \ldots + A_n) = 1 - P(\overline{A}_1)\ P(\overline{A}_2) \ldots.\ P(\overline{A}_n)$

7. Total Probability :

Let $A_1, A_2,$ .... A_n are n mutually exclusive & set of exhaustive events and event A can occur through any one of these events, then probability of occurence of A

$$P(A) = P(A \cap A_1) + P(A \cap A_2) + \ldots\ldots + P(A \cap A_n)$$

$$= \sum_{r=1}^{n} P(A_r)\, P(A/A_r)$$

8. Baye's Rule :

Let A_1, A_2, A_3 be any three mutually exclusive & exhaustive events (i.e. $A_1 \cup A_2 \cup A_3$ = sample space & $A_1 \cap A_2 \cap A_3 = \phi$) an sample space S and B is any other event on sample space then,

$$P(A_i/B) = \frac{P(B/A_i)P(A_i)}{P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + P(B/A_3)P(A_3)},$$

i = 1, 2, 3

9. Probability distribution :

(i) If a random variable x assumes values $x_1, x_2, \ldots\ldots x_n$ with probabilities $P_1, P_2, \ldots\ldots P_n$ respectively then

(a) $P_1 + P_2 + P_3 + \ldots\ldots + P_n = 1$

(b) mean $E(x) = \Sigma P_i x_i$

(c) Variance $= \Sigma x^2 P_i - (\text{mean})^2 = \Sigma(x^2) - (E(x))^2$

(ii) **Binomial distribution :** If an experiment is repeated n times, the successive trials being independent of one another, then the probability of -

r success is ${}^nC_r\, P^r\, q^{n-r}$

atleast r success is $\sum_{k=r}^{n} {}^nC_k\, P^k\, q^{n-k}$

where p is probability of success in a single trial, q = 1 – p

(a) mean E(x) = np

(b) $E(x^2) = npq + n^2 p^2$

(c) Variance $E(x^2) - (E(x))^2 = npq$

(d) Standard deviation $= \sqrt{npq}$

10. Truth of the statement :

(i) If two persons A and B speaks truth with the probability p_1 & p_2 respectively and if they agree on a statement, then the probability that they are speaking truth will be given by

$$\frac{p_1p_2}{p_1p_2 + (1-p_1)(1-p_2)}.$$

(ii) If A and B both assert that an event has occurred, probability of occurrence of which is α then the probability that event has occurred.

Given that the probability of A & B speaking truth is p_1, p_2.

$$\frac{\alpha p_1p_2}{\alpha p_1p_2 + (1-\alpha)(1-p_1)(1-p_2)}$$

(iii) If in the second part the probability that their lies (jhuth) coincides is β then from above case required probability will be

$$\frac{\alpha p_1p_2}{\alpha p_1p_2 + (1-\alpha)(1-p_1)(1-p_2)\beta}.$$

PROGRESSION AND SERIES

1. Arithmetic Progression (A.P.) :

(a) General A.P. — a, a + d, a + 2d, ... , a + (n − 1) d
where a is the first term and d is the common difference

(b) General (n^{th}) term of an A.P. —
$T_n = a + (n - 1)d$ [n^{th} term from the beginning]
If an A.P. having m terms, then n^{th} term from end
$= a + (m - n)d$

(c) Sum of n terms of an A.P. —

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + T_n]$$

Note : If sum of n terms i.e. S_n is given then $T_n = S_n - S_{n-1}$ where S_{n-1} is sum of (n − 1) terms.

(d) Supposition of terms in A.P. —

(i) Three terms as a - d, a, a + d

(ii) Four terms as a − 3d, a − d, a + d, a + 3d

(iii) Five terms as a − 2d, a − d, a, a + d, a + 2d

(e) **Arithmetic mean (A.M.) :**

(i) A.M. of n numbers $A_1, A_2, \ldots\ldots A_n$ is defined as

$$\text{A.M.} = \frac{A_1 + A_2 + \ldots\ldots + A_n}{n} = \frac{\Sigma A_i}{n} = \frac{\text{Sum of numbers}}{n}$$

(ii) For an A.P., A.M. of the terms taken symmetrically from the beginning and from the end will always be constant and will be equal to middle term or A.M. of middle term.

(iii) If A is the A.M. between two given nos. a and b, then

$$A = \frac{a+b}{2} \quad \text{i.e.} \quad 2A = a + b$$

(iv) If $A_1, A_2, \ldots\ldots A_n$ are n A.M's between a and b, then $A_1 = a + d, A_2 = a + 2d, \ldots\ldots A_n = a + nd$,

where $d = \frac{b-a}{n+1}$

(v) Sum of n A.M's inserted between a and b is $\frac{n}{2}(a + b)$

(vi) Any term of an A.P. (except first term) is equal to the half of the sum of term equidistant from the term i.e. $a_n = \frac{1}{2}(a_{n-r} + a_{n+r}), r < n$

2. Geometric Progression (G.P.)

(a) General G.P. — a, ar, ar^2 , ...
where a is the first term and r is the common ratio

(b) General (n^{th}) term of a G.P. — $T_n = ar^{n-1}$
If a G.P. having m terms then n^{th} term from end $= ar^{m-n}$

(c) Sum of n terms of a G.P. —

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a - T_n r}{1-r}, \quad r < 1$$

$$= \frac{a(r^n - 1)}{r-1} = \frac{T_n r - a}{r-1}, \quad r > 1$$

(d) Sum of an infinite G.P. — $S_\infty = \frac{a}{1-r}, \quad |r|<1$

(e) Supposition of terms in G.P. —

(i) Three terms as $\frac{a}{r}$, a, ar

(ii) Four terms as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3

(iii) Five terms as $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2

(f) Geometric Mean (G.M.) —

(i) Geometrical mean of n numbers $x_1, x_2, \ldots\ldots\ldots\ldots x_n$ is defined as

G.M. = $(x_1 x_2 \ldots\ldots\ldots\ldots x_n)^{1/n}$.

(i) If G is the G.M. between two given numbers a and b, then

$G^2 = ab \Rightarrow G = \sqrt{ab}$

(ii) If $G_1, G_2, \ldots\ldots\ldots\ldots G_n$ are n G.M's between a and b, then

$G_1 = ar, G_2 = ar^2, \ldots\ldots G_n = ar^n$, where $r = \left(\frac{b}{a}\right)^{1/n+1}$

(iii) Product of the n G.M.'s inserted between a & b is $(ab)^{n/2}$

3. Arithmetico - Geometric Progression (A.G.P.) :

(a) General form — a, (a + d)r, (a + 2d) r^2, ...

(b) General (n^{th}) term — $T_n = [a + (n - 1) d] r^{n-1}$

(c) Sum of n terms of an A.G.P — $S_n = \frac{a}{1-r} + r.\frac{d(1-r^{n-1})}{(1-r)^2}$

(d) Sum of infinite terms of an A.G.P.

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

4. Sum standard results :

(a) $\Sigma n = 1 + 2 + 3 + \ldots\ldots + n = \frac{n(n+1)}{2}$

(b) $\Sigma n^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\Sigma n^3 = 1^3 + 2^3 + 3^3 + \ldots. + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

(d) $\Sigma a = a + a + \ldots. + (n \text{ times}) = na$

(e) $\Sigma(2n - 1) = 1 + 3 + 5 + \ldots. (2n - 1) = n^2$

(f) $\Sigma 2n = 2 + 4 + 6 + \ldots. + 2n = n(n + 1)$

5. Harmonic Progression (H.P)

(a) General H.P. — $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d} + \ldots\ldots$

(b) General (n^{th} term) of a H.P. — T_n

$= \frac{1}{a+(n-1)d} = \frac{1}{n^{th}\text{term coresponding to A.P.}}$

(c) Harmonic Mean (H.M.)

(i) If H is the H.M. between a and b, then $H = \frac{2ab}{a+b}$

(ii) If $H_1, H_2, \ldots\ldots, H_n$ are n H.M's between a and b,

then $H_1 = \frac{ab(n+1)}{bn+a}, \ldots\ldots, H_n = \frac{ab(n+1)}{na+b}$

or first find n A.M.'s between $\frac{1}{a}$ & $\frac{1}{b}$, then their reciprocal will be required H.M's.

6. Relation Between A.M., G.M. and H.M.

(i) $AH = G^2$

(ii) $A \geq G \geq H$

(iii) If A and G are A.M. and G.M. respectively between two +ve numbers, then these numbers are

$A \pm \sqrt{A^2 - G^2}$

BINOMIAL THEOREM

1. Binomial Theorem for any +ve integral index :

$(x + a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \ldots\ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_n\, a^n$

$$= \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

(i) General term - $T_{r+1} = {}^nC_r\, x^{n-r} a^r$ is the $(r + 1)^{th}$ term from beginning.

(ii) $(m + 1)^{th}$ term from the end = $(n - m + 1)^{th}$ from beginning = T_{n-m+1}

(iii) middle term

(a) If n is even then middle term = $\left(\frac{n}{2}+1\right)^{th}$ term

(b) If n is odd then middle term = $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term

Binomial coefficient of middle term is the greatest binomial coefficient.

2. To determine a particular term in the given expasion :

Let the given expansion be $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$, if x^n occurs in T_{r+1} $(r + 1)^{th}$ term then r is given by $n\alpha - r(\alpha + \beta) = m$ and for x^0, $n\alpha - r(\alpha + \beta) = 0$

3. Properties of Binomial coefficients :

For the sake of convenience the coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$ ..... ${}^nC_r$ ..... ${}^nC_n$ are usually denoted by $C_0, C_1, \ldots, C_r \ldots\ldots C_n$ respectively.

* $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$
* $C_0 - C_1 + C_2 - C_3 + \ldots + C_n = 0$
* $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$
* ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\frac{n-1}{r-1}\,{}^{n-2}C_{r-2}$ and so on ...
* ${}^{2n}C_{n+r} = \frac{2n!}{(n-r)!(n+r)!}$
* ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
* $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n.2^{n-1}$
* $C_1 - 2C_2 + 3C_3 \ldots\ldots = 0$
* $C_0 + 2C_1 + 3C_2 + \ldots + (n + 1)C_n = (n + 2)2^{n-1}$
* $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{(2n)!}{(n!)^2} = {}^{2n}C_n$
* $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots$

$$= \begin{cases} 0, & \text{if n is odd} \\ (-1)^{n/2}\,{}^nC_{n/2}, & \text{if n is even} \end{cases}$$

Note : ${}^{2n+1}C_0 + {}^{2n+1}C_1 + \dots + {}^{2n+1}C_n = {}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \dots\dots {}^{2n+1}C_{2n+1} = 2^{2n}$

* $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots\dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

* $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \dots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

4. Greatest term :

(i) If $\frac{(n+1)a}{x+a} \in Z$ (integer) then the expansion has two greatest terms. These are k^{th} and $(k+1)^{th}$ where x & a are +ve real nos.

(ii) If $\frac{(n+1)a}{x+a} \notin Z$ then the expansion has only one greatest term. This is $(k+1)^{th}$ term $k = \left[\frac{(n+1)a}{x+a}\right]$,

{[.] denotes greatest integer less than or equal to x}

5. Multinomial Theorem :

(i) $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r,\ n \in N$

$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\,r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{s!\,r!} x^s a^r$,

where s = n – r

(ii) $(x+y+z)^n = \sum_{r+s+t=n} \frac{n!}{s!\,r!\,t!} x^r y^s z^t$

Generalized $(x_1 + x_2 + \dots\dots x_k)^n$

$$= \sum_{r_1+r_2+\dots r_k=n} \frac{n!}{r_1!\,r_2!\dots\dots r_k!} x_1^{r_1} x_2^{r_2} \dots\dots x_k^{r_k}$$

6. Total no. of terms in the expansion $(x_1 + x_2 + \dots x_n)^m$ is ${}^{m+n-1}C_{n-1}$

TRIGONOMETRIC RATIO AND IDENTITIES

1. Some important results :

(i) Arc length AB = $r\theta$

Area of circular sector = $\frac{1}{2}r^2\theta$

(ii) For a regular polygon of side a and number of sides n

(a) Internal angle of polygon = $(n-2)\frac{\pi}{n}$

(b) Sum of all internal angles = $(n-2)\pi$

(c) Radius of incircle of this polygon $r = \frac{a}{2}\cot\frac{\pi}{n}$

(d) Radius of circumcircle of this polygon R

$= \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

(e) Area of the polygon = $\frac{1}{4}na^2\cot\left(\frac{\pi}{n}\right)$

(f) Area of triangle = $\frac{1}{4}a^2\cos\frac{\pi}{n}$

(g) Area of incircle = $\pi\left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2$

(h) Area of circumcircle = $\pi\left(\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\right)^2$

2. Relation between system of measurement of angles :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi} \qquad \& \ \pi \text{ radian} = 180^0$$

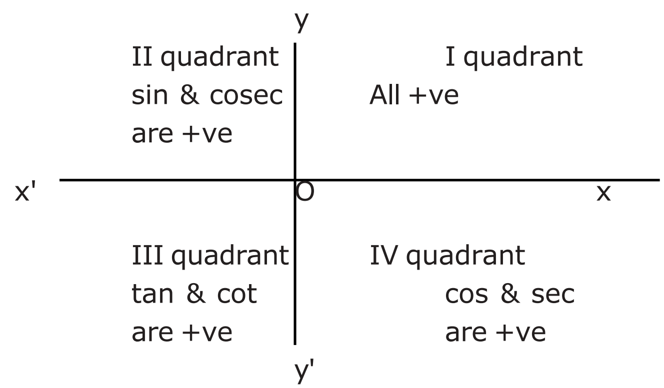
3. Trigonometric identities :

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

(iii) $\sec^2\theta - \tan^2\theta = 1$

4. Sign convention :



5. T-ratios of allied angles : The signs of trigonometrical ratio in different quadrant.

Allied∠ of T-ratios	$(-\theta)$	$90^0 \pm \theta$	$180^0 \pm \theta$	$270^0 \pm \theta$	$360^0 \pm \theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
$\cos\theta$	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$
$\cot\theta$	$-\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$
$\sec\theta$	$\sec\theta$	$\mp\operatorname{cosec}\theta$	$-\sec\theta$	$\pm\operatorname{cosec}\theta$	$\sec\theta$
$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta$	$\sec\theta$	$\mp\operatorname{cosec}\theta$	$-\sec\theta$	$\pm\operatorname{cosec}\theta$

6. Sum & differences of angles of t-ratios :

(i) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

(ii) $\cos(A \pm B) = \cos A\cos B \pm \sin A\sin B$

(iii) $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

(iv) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(v) $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(vi) $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(vii) $\tan(A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$$

Generalized tan (A + B + C + ...)

$$= \frac{S_1 - S_3 + S_5 - S_7 + ...}{1 - S_2 + S_4 - S_6 + S_8 - ......}$$

Where $S_1 = \Sigma \tan A$

$S_2 = \Sigma \tan A \tan B$,

$S_3 = \Sigma \tan A \tan B \tan C$ & so on

(viii) $\sin(A + B + C) = \Sigma \sin A \cos B \cos C - \Pi \sin A$

$= \Pi \cos A$ (Numerator of tan (A + B + C))

(ix) $\cos(A + B + C) = \Pi \cos A - \Sigma \sin A \sin B \cos C$

$= \Pi \cos A$ (Denominator of tan (A + B + C))

for a triangle $A + B + C = \pi$

$\Sigma \tan A = \Pi \tan A$

$\Sigma \sin A = \Sigma \sin A \cos B \cos C$

$1 + \Pi \cos A = \Sigma \sin A \sin B \cos C$

(viii) $\sin 75^0 = \dfrac{\sqrt{3} + 1}{2\sqrt{2}} = \cos 15^0$

(ix) $\cos 75^0 = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \sin 15^0$

(x) $\tan 75^0 = 2 + \sqrt{3} = \cot 15^0$

(xi) $\cot 75^0 = 2 - \sqrt{3} = \tan 15^0$

7. Formulaes for product into sum or difference and vice-versa :

(i) $2\sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $2\cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $2\cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

(v) $\sin C + \sin D = 2\sin\left(\dfrac{C + D}{2}\right)\cos\left(\dfrac{C - D}{2}\right)$

(vi) $\sin C - \sin D = 2\cos\left(\dfrac{C + D}{2}\right)\sin\left(\dfrac{C - D}{2}\right)$

(vii) $\cos C + \cos D = 2\cos\left(\dfrac{C + D}{2}\right)\cos\left(\dfrac{C - D}{2}\right)$

(viii) $\cos C - \cos D = 2\sin\left(\dfrac{C + D}{2}\right)\sin\left(\dfrac{D - C}{2}\right)$

(ix) $\tan A + \tan B = \dfrac{\sin(A + B)}{\cos A \cos B}$

8. T-ratios of multiple and submultiple angles :

(i) $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1 + \tan^2 A}$

$= (\sin A + \cos A)^2 - 1 = 1 - (\sin A - \cos A)^2$

$\Rightarrow \sin A = 2\sin A/2 \cos A/2 = \dfrac{2\tan A/2}{1 + \tan^2 A/2}$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$ $\Rightarrow$ $\tan A = \dfrac{2\tan A/2}{1-\tan^2 A/2}$

(iv) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = 4\sin(60^0 - \theta)\sin(60^0 + \theta)\sin\theta$

$= \sin\theta\ (2\cos\theta - 1)(2\cos\theta + 1)$

(v) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$= 4\cos(60^0 - \theta)\cos(60^0 + \theta)\cos\theta$

$= \cos\theta\ (1 - 2\sin\theta)(1 + 2\sin\theta)$

(vi) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

$= \tan(60^0 - A)\tan(60^0 + A)\tan A$

(vii) $\sin A/2 = \sqrt{\dfrac{1-\cos A}{2}}$

(viii) $\cos A/2 = \sqrt{\dfrac{1+\cos A}{2}}$

(ix) $\tan A/2 = \sqrt{\dfrac{1-\cos A}{1+\cos A}} = \dfrac{1-\cos A}{\sin A}$, $A \neq (2n+1)\pi$

9. Maximum and minimum value of the expression :

$a\cos\theta + b\sin\theta$

Maximum (greatest) Value = $\sqrt{a^2+b^2}$

Minimum (Least) value = $-\sqrt{a^2+b^2}$

10. Conditional trigonometric identities :

If A, B, C are angles of triangle i.e. $A + B + C = \pi$, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

i.e. $\sum \sin 2A = 4 \prod (\sin A)$

(ii) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

(iii) $\sin A + \sin B + \sin C = 4\cos A/2 \cos B/2 \cos C/2$

(iv) $\cos A + \cos B + \cos C = 1 + 4\sin A/2 \sin B/2 \sin C/2$

(v) $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \cos C$

(vi) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

(vii) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(viii) $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$

(ix) $\sum \tan A/2 \tan B/2 = 1$

(x) $\sum \cot A \cot B = 1$

(xi) $\sum \cot A/2 = \prod \cot A/2$

11. Some useful series :

(i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \text{to } n\text{terms}$

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left[\frac{n\beta}{2}\right]}{\sin\frac{\beta}{2}},\ \beta \neq 2n\pi$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \text{to } n\text{terms}$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\ \beta \neq 2n\pi$$

(iii) $\cos\alpha . \cos 2\alpha . \cos 2^2\alpha \ldots \cos(2^{n-1}\alpha) = \dfrac{\sin 2^n\alpha}{2^n \sin\alpha}$, $\alpha \neq n\pi$

$= 1$, $\alpha = 2k\pi$

$= -1$, $\alpha = (2k+1)\pi$

TRIGONOMETRIC EQUATIONS

1. General solution of the equations of the form

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I$

(ii) $\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2}, \quad n \in I$

(iii) $\tan\theta = 0 \Rightarrow \theta = n\pi, \quad n \in I$

(iv) $\sin\theta = 1 \Rightarrow \theta = 2n\pi + \frac{\pi}{2}$

(v) $\cos\theta = 1 \Rightarrow \theta = 2\pi n$

(vi) $\sin\theta = -1 \Rightarrow \theta = 2n\pi - \frac{\pi}{2}$ or $2n\pi + \frac{3\pi}{2}$

(vii) $\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$

(viii) $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$

(ix) $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$

(x) $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$

(xi) $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

(xii) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

(xiii) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$

2. For general solution of the equation of the form

$a\cos\theta + b\sin\theta = c$, where $c \le \sqrt{a^2+b^2}$, divide both side by $\sqrt{a^2+b^2}$

and put $\frac{a}{\sqrt{a^2+b^2}} = \cos\alpha$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\alpha$.

Thus the equation reduces to form

$$\cos(\theta - \alpha) = \frac{c}{\sqrt{a^2+b^2}} = \cos\beta(\text{say})$$

now solve using above formula

3. Some important points :

(i) If while solving an equation, we have to square it, then the roots found after squaring must be checked wheather they satisfy the original equation or not.

(ii) If two equations are given then find the common values of θ between 0 & 2π and then add $2n\pi$ to this common solution (value).

INVERSE TRIGONOMETRIC FUNCTIONS

1. If y = sin x, then x = $\sin^{-1}$ y, similarly for other inverse T-functions.

2. Domain and Range of Inverse T-functions :

Function	Domain (D)	Range (R)
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\cot^{-1} x$	$-\infty < x < \infty$	$0 < \theta < \pi$
$\sec^{-1} x$	$x \le -1, x \ge 1$	$0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$
$\text{cosec}^{-1} x$	$x \le -1, x \ge 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$

3. Properties of Inverse T-functions :

(i) $\sin^{-1}(\sin\theta) = \theta$ provided $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\cos^{-1}(\cos\theta) = \theta$ provided $\theta \le \theta \le \pi$

$\tan^{-1}(\tan\theta) = \theta$ provided $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\cot^{-1}(\cot\theta) = \theta$ provided $0 < \theta < \pi$

$\sec^{-1}(\sec\theta) = \theta$ provided $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

$\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$ provided $-\frac{\pi}{2} \le \theta < 0$

or $0 < \theta \le \frac{\pi}{2}$

(ii) $\sin(\sin^{-1} x) = x$ provided $-1 \le x \le 1$

$\cos(\cos^{-1} x) = x$ provided $-1 \le x \le 1$

$\tan(\tan^{-1} x) = x$ provided $-\infty < x < \infty$

$\cot(\cot^{-1} x) = x$ provided $-\infty < x < \infty$

$\sec(\sec^{-1} x) = x$ provided $-\infty < x \le -1$ or $1 \le x < \infty$

$\text{cosec}(\text{cosec}^{-1} x) = x$ provided $-\infty < x \le -1$

or $1 \le x < \infty$

(iii) $\sin^{-1}(-x) = -\sin^{-1} x$,

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

$\tan^{-1}(-x) = -\tan^{-1} x$

$\cot^{-1}(-x) = \pi - \cot^{-1} x$

$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$

$\sec^{-1}(-x) = \pi - \sec^{-1} x$

(iv) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \quad \forall x \in [-1, 1]$

$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \quad \forall x \in R$

$\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$

4. Value of one inverse function in terms of another inverse function :

(i) $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}} = \cot^{-1} \dfrac{\sqrt{1-x^2}}{x}$

$= \sec^{-1} \dfrac{1}{\sqrt{1-x^2}} = \text{cosec}^{-1} \dfrac{1}{x},\ 0 \le x \le 1$

(ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1} \dfrac{\sqrt{1-x^2}}{x} = \cot^{-1} \dfrac{x}{\sqrt{1-x^2}}$

$= \sec^{-1} \dfrac{1}{x} = \text{cosec}^{-1} \dfrac{1}{\sqrt{1-x^2}},\ 0 \le x \le 1$

(iii) $\tan^{-1} x = \sin^{-1} \dfrac{x}{\sqrt{1+x^2}} = \cos^{-1} \dfrac{1}{\sqrt{1+x^2}} = \cot^{-1} \dfrac{1}{x}$

$= \sec^{-1} \sqrt{1+x^2} = \text{cosec}^{-1} \dfrac{\sqrt{1+x^2}}{x},\ x \ge 0$

(iv) $\sin^{-1} \left(\dfrac{1}{x}\right) = \text{cosec}^{-1} x,\ \forall\ x \in (-\infty, 1] \cup [1, \infty)$

(v) $\cos^{-1} \left(\dfrac{1}{x}\right) = \sec^{-1} x,\ \forall\ x \in (-\infty, 1] \cup [1, \infty)$

(vi) $\tan^{-1} \left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1} x & \text{for } x > 0 \\ -\pi + \cot^{-1} x & \text{for } x < 0 \end{cases}$

5. Formulae for sum and difference of inverse trigonometric function :

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$; if $x > 0, y > 0, xy < 1$

(ii) $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$; if $x > 0, y > 0, xy > 1$

(iii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$; if $xy > -1$

(iv) $\tan^{-1}x - \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$; if $x > 0, y < 0, xy < -1$

(v) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(vi) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$;

if $x,y \ge 0$ & $x^2 + y^2 \le 1$

(vii) $\sin^{-1}x \pm \sin^{-1}y = \pi - \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$;

if $x,y \ge 0$ & $x^2 + y^2 > 1$

(viii) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$;

if $x,y > 0$ & $x^2 + y^2 \le 1$

(ix) $\cos^{-1}x \pm \cos^{-1}y = \pi - \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$;

if $x,y > 0$ & $x^2 + y^2 > 1$

6. Inverse trigonometric ratios of multiple angles

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$, if $-1 \le x \le 1$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$, if $-1 \le x \le 1$

(iii) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(iv) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

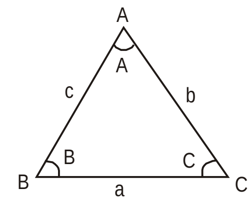
(v) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

(vi) $3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

PROPERTIES & SOLUTION OF TRIANGLE

Properties of triangle :

1. A triangle has three sides and three angles.
In any ΔABC, we write $BC = a$, $AB = c$, $AC = b$



and $\angle BAC = \angle A$, $\angle ABC = \angle B$, $\angle ACB = \angle C$

2. In ΔABC :

(i) $A + B + C = \pi$

(ii) $a + b > c$, $b + c > a$, $c + a > b$

(iii) $a > 0$, $b > 0$, $c > 0$

3. Sine formula :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k(\text{say})$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k\ (\text{say})$$

4. Cosine formula :

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

5. Projection formula :

$a = b \cos C + c \cos B$

$b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

6. Napier's Analogies :

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$$

7. Half angled formula - In any ΔABC :

(a) $$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

$$\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$ where $2s = a + b + c$

(b) $$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$$

$$\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

(c) $$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan \frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}$$

8. Δ, Area of triangle :

(i) $$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$

(ii) $$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

9. $$\tan \frac{A}{2} \tan \frac{B}{2} = \frac{s-c}{s}$$

$$\tan \frac{B}{2} \tan \frac{C}{2} = \frac{s-a}{s}$$

$$\tan \frac{C}{2} \tan \frac{A}{2} = \frac{s-b}{s}$$

10. Circumcircle of triangle and its radius :

(i) $$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

(ii) $$R = \frac{abc}{4\Delta}$$ Where R is circumradius

11. Incircle of a triangle and its radius :

(iii) $r = \frac{\Delta}{s}$

(iv) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(v) $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

(vi) $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

(vii) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

12. The radii of the escribed circles are given by :

(i) $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}$, $r_2 = s\tan\frac{B}{2}$, $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$,

$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$,

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

(iv) $r_1 + r_2 + r_3 - r = 4R$

(v) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$

(vi) $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

(vii) $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}$

(viii) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

(ix) $\Delta = 2R^2\sin A\sin B\sin C = 4Rr\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

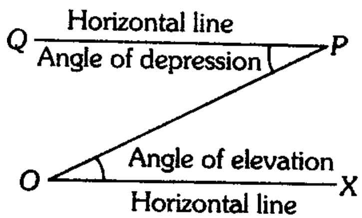
(x) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$, $r_2 = \frac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$,

$r_3 = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

HEIGHT AND DISTANCE

1. Angle of elevation and depression :

If an observer is at O and object is at P then $\angle XOP$ is called angle of elevation of P as seen from O.



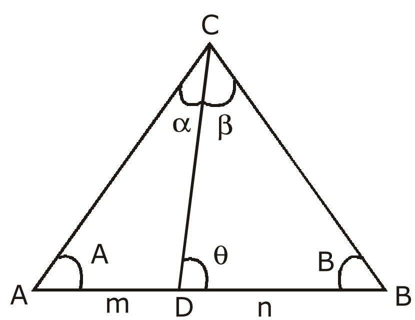
If an observer is at P and object is at O, then $\angle QPO$ is called angle of depression of O as seen from P.

2. Some useful result :

(i) In any triangle ABC if AD : DB = m : n

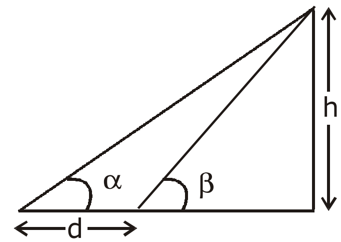
$\angle ACD = \alpha$, $\angle BCD = \beta$ & $\angle BDC = \theta$

then $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$



$= n\cot A - m\cot B$ [m – n Theorem]

(ii) $d = h(\cot\alpha - \cot\beta)$



POINT

1. Distance formula :

Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by d(P, Q) = PQ

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\text{Difference of } x \text{ coordinate})^2 + (\text{Difference of } y \text{ coordinate})^2}$$

Note : (i) $d(P, Q) \geq 0$

(ii) $d(P, Q) = 0 \Leftrightarrow P = Q$

(iii) d(P, Q) = d(Q, P)

(iv) Distance of a point (x, y) from origin

$$(0, 0) = \sqrt{x^2 + y^2}$$

2. Use of Distance Formula :

(a) **In Triangle :**

Calculate AB, BC, CA

(i) If AB = BC = CA, then Δ is equilateral.

(ii) If any two sides are equal then Δ is isosceles.

(iii) If sum of square of any two sides is equal to the third, then Δ is right triangle.

(iv) Sum of any two equal to left third they do not form a triangle

i.e. AB = BC + CA or BC = AC + AB

or AC = AB + BC. Here points are collinear.

(b) **In Parallelogram :**

Calculate AB, BC, CD and AD.

(i) If AB = CD, AD = BC, then ABCD is a parallelogram.

(ii) If AB = CD, AD = BC and AC = BD, then ABCD is a rectangle.

(iii) If AB = BC = CD = AD, then ABCD is a rhombus.

(iv) If AB = BC = CD = AD and AC = BD, then ABCD is a square.

(C) **For circumcentre of a triangle :**

Circumcentre of a triangle is equidistant from vertices i.e. PA = PB = PC.

Here P is circumcentre and PA is radius.

(i) Circumcentre of an acute angled triangle is inside the triangle.

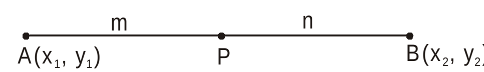
(ii) Circumcentre of a right triangle is mid point of the hypotenuse.

(iii) Circumcentre of an obtuse angled triangle is outside the triangle.

3. Section formula :

(i) **Internally :**

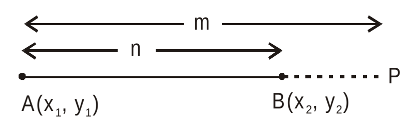
$$\frac{AP}{BP} = \frac{m}{n} = \lambda, \text{ Here } \lambda > 0$$



$$P\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

(ii) **Externally :**

$$\frac{AP}{BP} = \frac{m}{n} = \lambda$$



$$P\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

(iii) Coordinates of mid point of PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(iv) The line ax + by + c = 0 divides the line joining the points

(x_1, y_1) & (x_2, y_2) in the ratio $= -\frac{(ax_1 + by + c)}{(ax_2 + by_2 + c)}$

(v) For parallelogram – midpoint of diagonal AC = mid point of diagonal BD

(vi) Coordinates of centroid G $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

(vii) Coordinates of incentre I

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

(viii) Coordinates of orthocentre are obtained by solving the equation of any two altitudes.

4. Area of Triangle :

The area of triangle ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ (Determinant method)}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2}[x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3]$$

[Stair method]

Note :

(i) Three points A, B, C are collinear if area of triangle is zero.

(ii) If in a triangle point arrange in anticlockwise then value of Δ be +ve and if in clockwise then Δ will be –ve.

5. Area of Polygon :

Area of polygon having vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) .... (x_n, y_n) is given by area

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}.$$ Points must be taken in order.

6. Rotational Transformation :

If coordinates of any point P(x, y) with reference to new axis will be (x', y') then

	$x\downarrow$	$y\downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

7. Some important points :

(i) Three pts. A, B, C are collinear, if area of triangle is zero

(ii) Centroid G of ΔABC divides the median AD or BE or CF in the ratio 2 : 1

(iii) In an equilateral triangle, orthocentre, centroid, circumcentre, incentre coincide.

(iv) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(v) Area of triangle formed by coordinate axes & the line $ax + by + c = 0$ is $\frac{c^2}{2ab}$.

STRAIGHT LINE

1. Slope of a Line : The tangent of the angle that a line makes with +ve direction of the x-axis in the anticlockwise sense is called slope or gradient of the line and is generally denoted by m. Thus $m = \tan\theta$.

(i) Slope of line || to x-axis is $m = 0$

(ii) Slope of line || to y-axis is $m = \infty$ (not defined)

(iii) Slope of the line equally inclined with the axes is 1 or − 1

(iv) Slope of the line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{y_2 - y_1}{x_2 - x_1}$.

(v) Slope of the line $ax + by + c = 0$, $b \neq 0$ is $-\frac{a}{b}$

(vi) Slope of two parallel lines are equal.

(vii) If m_1 & m_2 are slopes of two $\perp$ lines then $m_1m_2 = -1$.

2. Standard form of the equation of a line :

(i) Equation of x-axis is $y = 0$

(ii) Equation of y-axis is $x = 0$

(iii) Equation of a straight line || to x-axis at a distance b from it is $y = b$

(iv) Equation of a straight line || to y-axis at a distance a from it is $x = a$

(v) **Slope form :** Equation of a line through the origin and having slope m is $y = mx$.

(vi) **Slope Intercept form :** Equation of a line with slope m and making an intercept c on the y-axis is $y = mx + c$.

(vii) **Point slope form :** Equation of a line with slope m and passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

(viii) **Two point form :** Equation of a line passing through the points (x_1, y_1) & (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

(ix) **Intercept form :** Equation of a line making intercepts a and b respectively on x-axis and y-axis is $\frac{x}{a} + \frac{y}{b} = 1$.

(x) **Parametric or distance or symmetrical form of the line :** Equation of a line passing through (x_1, y_1) and making an angle θ, $0 \le \theta \le \pi$, $\theta \neq \frac{\pi}{2}$ with the +ve direction of x-axis is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$\Rightarrow \quad x = x_1 + r\cos\theta,\ y = y_1 + r\sin\theta$

Where r is the distance of any point P(x, y) on the line from the point (x_1, y_1)

(xi) **Normal or perpendicular form :** Equation of a line such that the length of the perpendicular from the origin on it is p and the angle which the perpendicular makes with the +ve direction of x-axis is α, is

$$x\cos\alpha + y\sin\alpha = p.$$

3. Angle between two lines :

(i) Two lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are

(a) Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(b) Perpendicular if $a_1a_2 + b_1b_2 = 0$

(c) Identical or coincident if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(d) If not above three, then $\theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 - b_1b_2}\right|$

(ii) Two lines $y = m_1x + c$ and $y = m_2x + c$ are

(a) Parallel if $m_1 = m_2$

(b) Perpendicular if $m_1m_2 = -1$

(c) If not above two, then $\theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

4. Position of a point with respect to a straight line :
The line $L(x_i, y_i)$ i = 1, 2 will be of same sign or of opposite sign according to the point $A(x_1, y_1)$ & B (x_2, y_2) lie on same side or on opposite side of L (x, y) respectively.

5. Equation of a line parallel (or perpendicular) to the line
$ax + by + c = 0$ is $ax + by + c' = 0$ (or $bx - ay + \lambda = 0$)

6. Equation of st. lines through (x_1, y_1) making an angle α with y = mx + c is

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

7. length of perpendicular from (x_1, y_1) on ax + by + c = 0

is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

8. Distance between two parallel lines $ax + by + c_i = 0$,

i = 1, 2 is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

9. Condition of concurrency for three straight lines $L_i \equiv a_i x + b_i y + c_i = 0$, i = 1, 2, 3 is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

10. Equation of bisectors of angles between two lines :

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

11. Family of straight lines :
The general equation of family of straight line will be written in one parameter
The equation of straight line which passes through point of intersection of two given lines L_1 and L_2 can be taken as $L_1 + \lambda L_2 = 0$

12. Homogeneous equation : If $y = m_1x$ and $y = m_2x$ be the two equations
represented by $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$

13. General equation of second degree :
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of

straight line if $\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

If $y = m_1x + c$ & $y = m_2x + c$ represents two straight lines

then $m_1 + m_2 = \dfrac{-2h}{b}$, $m_1m_2 = \dfrac{a}{b}$.

14. Angle between pair of straight lines :
The angle between the lines represented by
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ or $ax^2 + 2hxy + by^2 = 0$

is $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{(a + b)}\right|$

(i) The two lines given by $ax^2 + 2hxy + by^2 = 0$ are
(a) Parallel and coincident iff $h^2 - ab = 0$
(b) Perpendicular iff $a + b = 0$

(ii) The two line given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are
(a) Parallel if $h^2 - ab = 0$ & $af^2 = bg^2$
(b) Perpendicular iff $a + b = 0$
(c) Coincident iff $g^2 - ac = 0$

13. Combined equation of angle bisector of the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

CIRCLE

1. General equation of a circle : $x^2 + y^2 + 2gx + 2fy + c = 0$ where g, f and c are constants

(i) Centre of the cirle is $(-g, -f)$

i.e. $\left(-\frac{1}{2}\text{coeff. of x}, \frac{-1}{2}\text{coeff. of y}\right)$

(ii) Radius is $\sqrt{g^2 + f^2 - c}$

2. Central (Centre radius) form of a circle :

(i) $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is circle centre and r is the radius.

(ii) $x^2 + y^2 = r^2$, where (0, 0) origin is circle centre and r is the radius.

3. Diameter form : If (x_1, y_1) and (x_2, y_2) are end pts. of a diameter of a circle, then its equation is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

4. Parametric equations :

(i) The parametric equations of the circle $x^2 + y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$,

where point $\theta \equiv (r\cos\theta, r\sin\theta)$

(ii) The parametric equations of the circle $(x - h)^2 + (y - k)^2 = r^2$ are $x = h + r\cos\theta$, $y = k + r\sin\theta$

(iii) The parametric equations of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are

$x = -g + \sqrt{g^2 + f^2 - c}\cos\theta$, $y = -f + \sqrt{g^2 + f^2 - c}\sin\theta$

(iv) For circle $x^2 + y^2 = a^2$, equation of chord joining θ_1 & θ_2 is

$$x\cos\frac{\theta_1 + \theta_2}{2} + y\sin\frac{\theta_1 + \theta_2}{2} = r\cos\frac{\theta_1 - \theta_2}{2}.$$

5. Concentric circles : Two circles having same centre C(h, k) but different radii r_1 & r_2 respectively are called concentric circles.

6. Position of a point w.r.t. a circle : A point (x_1, y_1) lies outside, on or inside a circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as

$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ is +ve, zero or –ve

7. Chord length (length of intercept) = $2\sqrt{r^2 - p^2}$

8. Intercepts made on coordinate axes by the circle :

(i) x axis = $2\sqrt{g^2 - c}$

(ii) y axis = $2\sqrt{f^2 - c}$

9. Length of tangent = $\sqrt{S_1}$

10. Length of the intercept made by line : $y = mx + c$ with the circle $x^2 + y^2 = a^2$ is

$2\sqrt{\frac{a^2(1 + m^2) - c^2}{1 + m^2}}$ or $(1 + m^2)|x_1 - x_2|$

where $|x_1 - x_2|$ = difference of roots i.e. $\frac{\sqrt{D}}{a}$

11. Condition of Tangency : Circle $x^2 + y^2 = a^2$ will touch the line $y = mx + c$ if $c = \pm a\sqrt{1 + m^2}$

12. Equation of tangent, T = 0 :

(i) Equation of tangent to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

(ii) Equation of tangent to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is $xx_1 + yy_1 = a^2$

(iii) **In slope form :** From the condition of tangency for every value of m.

The line $y = mx \pm a\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = a^2$ and its point of contact is

$$\left(\frac{\pm am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$$

(iv) Equation of tangent at $(a \cos \theta, a \sin \theta)$ to the circle $x^2 + y^2 = a^2$ is $x \cos \theta + y \sin \theta = a$.

13. Equation of normal :

(i) Equation of normal to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at any point $P(x_1, y_1)$ is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

(ii) Equation of normal to the circle $x^2 + y^2 = a^2$ at any point (x_1, y_1) is $xy_1 - x_1y = 0$

14. Equation of pair of tangents $SS_1 = T^2$

15. The point of intersection of tangents drawn to the circle $x^2 + y^2 = r^2$ at point θ_1 & θ_2 is given as

$$\left(\frac{r\cos\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}, \frac{r\sin\frac{\theta_1+\theta_2}{2}}{\cos\frac{\theta_1-\theta_2}{2}}\right)$$

16. Equation of the chord of contact of the tangents drawn from point P outside the circle is T = 0

17. Equation of a chord whose middle pt. is given by $T = S_1$

18. Director circle : Equation of director circle for $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$. Director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the given circle.

19. Equation of polar of point (x_1, y_1) w.r.t. the circle S = 0 is T = 0

20. Coordinates of pole : Coordinates of pole of the line

$lx + my + n = 0$ w.r.t the circle $x^2 + y^2 = a^2$ are $\left(\frac{-a^2l}{n}, \frac{-a^2m}{n}\right)$

21. Family of Circles :

(i) $S + \lambda S' = 0$ represents a family of circles passing through the pts. of intersection of

$S = 0$ & $S' = 0$ if $\lambda \neq -1$

(ii) $S + \lambda L = 0$ represent a family of circles passing through the point of intersection of S = 0 & L = 0

(iii) Equation of circle which touches the given straight line L = 0 at the given point (x_1, y_1) is given as

$(x - x_1)^2 + (y - y_1)^2 + \lambda L = 0$.

(iv) Equation of circle passing through two points $A(x_1, y_1)$ & $B(x_2, y_2)$ is given as

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

22. Equation of Common Chord is $S - S_1 = 0$.

23. The angle θ of intersection of two circles with centres C_1 & C_2 and radii r_1 & r_2 is given by

$\cos\theta = \dfrac{r_1^2 + r_1^2 - d^2}{2r_1r_2}$, where $d = C_1C_2$

24. Position of two circles : Let two circles with centres C_1, C_2 and radii r_1, r_2 .

Then following cases arise as

(i) $C_1 C_2 > r_1 + r_2 \Rightarrow$ do not intersect or one outside the other, 4 common tangents.

(ii) $C_1 C_2 = r_1 + r_2 \Rightarrow$ Circles touch externally, 3 common tangents.

(iii) $|r_1 - r_2| < C_1 C_2 < r_1 + r_2 \Rightarrow$ Intersection at two real points, 2 common tangents.

(iv) $C_1 C_2 = |r_1 - r_2| \Rightarrow$ internal touch, 1 common tangent.

(v) $C_1 C_2 < |r_1 + r_2| \Rightarrow$ one inside the other, no tangent.

Note : Point of contact divides $C_1 C_2$ in the ratio $r_1 : r_2$ internally or externally as the case may be

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25. Equation of tangent at point of contact of circle is $S_1 - S_2 = 0$

26. Radical axis and radical centre :

(i) Equation of radical axis is $S - S_1 = 0$

(ii) The point of concurrency of the three radical axis of three circles taken in pairs is called radical centre of three circles.

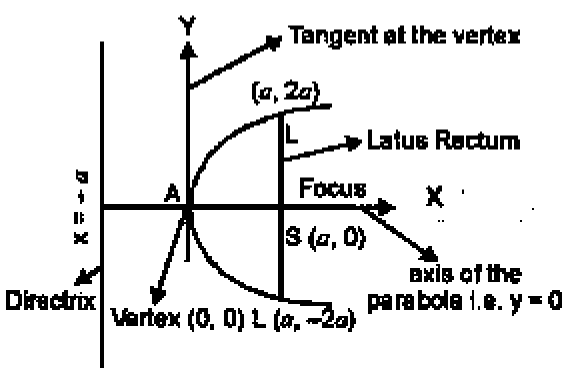
27. Orthogonality condition :

If two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ intersect each other orthogonally, then $2gg' + 2ff' = c + c'$.

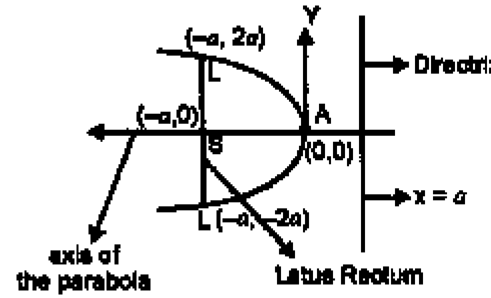
PARABOLA

1. Standard Parabola :

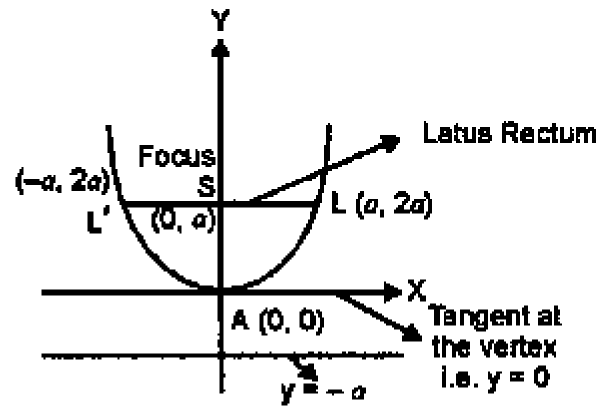
Imp. Terms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Vertex (v)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Focus (f)	(a, 0)	(–a, 0)	(0, a)	(0, –a)
Directrix (D)	x = –a	x = a	y = –a	y = a
Axis	y = 0	y = 0	x = 0	x = 0
L.R.	4a	4a	4a	4a
Focal distance	x + a	a – x	y + a	a – y
Parametric Coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric Equations	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = 2at^2$	$x = 2at$ $y = -at^2$



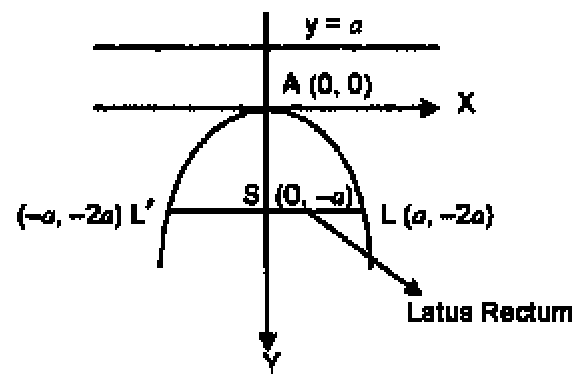
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

2. Special Form of Parabola

* Parabola which has vertex at (h, k), latus rectum ℓ and axis parallel to x-axis is
$(y - k)^2 = \ell(x - h)$

$\Rightarrow$ axis is $y = k$ and focus at $\left(h + \frac{\ell}{4}, k\right)$

* Parabola which has vertex at (h, k), latus rectum ℓ and axis parallel to y-axis is
$(x - h)^2 = \ell(y - k)$

$\Rightarrow$ axis is $x = h$ and focus at $\left(h, k + \frac{\ell}{4}\right)$

* Equation of the form $ax^2 + bx + c = y$ represents parabola.

i.e. $y - \frac{4ac - b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$, with vertex

$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$ and axes parallel to y-axis

Note : Parametric equation of parabola $(y - k)^2 = 4a(x - h)$ are $x = h + at^2$, $y = k + 2at$

3. Position of a point (x_1, y_1) and a line w.r.t. parabola $y^2 = 4ax$.

* The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$
according as $y_1^2 - 4ax_1 >, = \text{ or } < 0$
* The line $y = mx + c$ does not intersect, touches, intersect a parabola $y^2 = 4ax$ according as
$c > = < a/m$

Note : Condition of tangency for parabola $y^2 = 4ax$, we have $c = a/m$ and for other parabolas check disc. $D = 0$.

4. Equations of tangent in different forms :

(i) Point Form / Parametric form

Equations of tangent of all other standard parabolas at (x_1, y_1) / at t (parameter)

Equation of parabola	Tangent at (x_1, y_1)	Parametric coordinates 't'	Tangent of 't'
$y^2=4ax$	$yy_1=2a(x+x_1)$	$(at^2, 2at)$	$ty=x+at^2$
$y^2=-4ax$	$yy_1=-2a(x+x_1)$	$(-at^2, 2at)$	$ty=-x+at^2$
$x^2=4ay$	$xx_1=2a(y+y_1)$	$(2at, at^2)$	$tx=y+at^2$
$x^2=-4ay$	$xx_1=-2a(y+y_1)$	$(2at, -at^2)$	$tx=-y+at^2$

(ii) Slope form

Equations of tangent of all other parabolas in slope form

Equation of parabolas	Point of contact in terms of slope(m)	Equations of tangent in terms of slope (m)	Condition of Tangency
$y^2 = 4ax$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$	$c = \frac{a}{m}$
$y^2 = -4ax$	$\left(-\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx - \frac{a}{m}$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$(2am, am^2)$	$y = mx - am^2$	$c = -am^2$
$x^2 = -4ay$	$(-2am, -am^2)$	$y = mx + am^2$	$c = am^2$

5. Point of intersection of tangents at any two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is $(at_1t_2, a(t_1 + t_2))$ i.e. $(a(G.M.)^2, a(2A.M.))$

6. Combined equation of the pair of tangents drawn from a point to a parabola is $SS' = T^2$, where $S = y^2 - 4ax$, $S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$

7. Equations of normal in different forms

(i) Point Form / Parametric form

Equations of normals of all other standard parabolas at (x_1, y_1) / at t (parameter)

Eqⁿ. of parabola	Normal at (x_1, y_1)	Point 't'	Normals at 't'
$y^2 = 4ax$	$y-y_1 = \frac{-y_1}{2a}(x-x_1)$	$(at^2, 2at)$	$y+tx = 2at+at^3$
$y^2 = -4ax$	$y-y_1 = \frac{y_1}{2a}(x-x_1)$	$(-at^2, 2at)$	$y-tx = 2at+at^3$
$x^2 = 4ay$	$y-y_1 = -\frac{2a}{x_1}(x-x_1)$	$(2at, at^2)$	$x+ty = 2at+at^3$
$x^2 = -4ay$	$y-y_1 = \frac{2a}{x_1}(x-x_1)$	$(2at, -at^2)$	$x-ty = 2at+at^3$

(ii) Slope form

Equations of normal, point of contact, and condition of normality in terms of slope (m)

Eqⁿ. of parabola	Point of contact	Equations of normal	Condition of Normality
$y^2 = 4ax$	$(am^2, -2am)$	$y = mx-2am-am^3$	$c = -2am-am^3$
$y^2 = -4ax$	$(-am^2, 2am)$	$y = mx+2am+am^3$	$c = am+am^3$
$x^2 = 4ay$	$\left(-\frac{2a}{m}, \frac{a}{m^2}\right)$	$y = mx+2a+\frac{a}{m^2}$	$c = 2a+\frac{a}{m^2}$
$x^2 = -4ay$	$\left(\frac{2a}{m}, -\frac{a}{m^2}\right)$	$y = mx-2a-\frac{a}{m^2}$	$c = -2a-\frac{a}{m^2}$

Note :

(i) In circle normal is radius itself.

(ii) Sum of ordinates (y coordinate) of foot of normals through a point is zero.

(iii) The centroid of the triangle formed by taking the foot of normals as a vertices of concurrent normals of $y^2 = 4ax$ lies on x-axis.

8. Condition for three normals from a point (h, 0) on x-axis to parabola $y^2 = 4ax$

(i) We get 3 normals if $h > 2a$

(ii) We get one normal if $h \le 2a$.

(iii) If point lies on x-axis, then one normal will be x-axis itself.

9. (i) If normal of $y^2 = 4ax$ at t_1 meet the parabola again at t_2 then $t_2 = -t_1 - \frac{2}{t_1}$

(ii) The normals to $y^2 = 4ax$ at t_1 and t_2 intersect each other at the same parabola at t_3, then $t_1t_2 = 2$ and $t_3 = -t_1 - t_2$

10. (i) Equation of focal chord of parabola $y^2 = 4ax$ at t_1 is

$$y = \frac{2t_1}{t_1^2 - 1}(x - a)$$

If focal chord of $y^2 = 4ax$ cut (intersect) at t_1 and t_2 then $t_1t_2 = -1$ (t_1 must not be zero)

(ii) Angle formed by focal chord at vertex of parabola is

$$\tan\theta = \frac{2}{3}|t_2 - t_1|$$

(iii) Intersecting point of normals at t_1 and t_2 on the parabola $y^2 = 4ax$ is

$$(2a + a(t_1^2 + t_2^2 + t_1t_2),\ -at_1t_2(t_1 + t_2))$$

11. Equation of chord of parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is given by $T = S_1$

12. The locus of the mid point of a system of parallel chords of a parabola is called its diameter. Its equation is $y = \frac{2a}{m}$.

13. Equation of polar at the point (x_1, y_1) with respect to parabola $y^2 = 4ax$ is same as chord of contact and is given by

$$T = 0 \text{ i.e. } yy_1 = 2a(x + x_1)$$

Coordinates of pole of the line $\ell x + my + n = 0$ w.r.t. the parabola $y^2 = 4ax$ is $\left(\frac{n}{\ell}, \frac{-2am}{\ell}\right)$

14. **Diameter** : It is locus of mid point of set of parallel chords and equation is given by $T = S_1$

15. **Important results for Tangent :**

(i) Angle made by focal radius of a point will be twice the angle made by tangent of the point with axis of parabola

(ii) The locus of foot of perpendicular drop from focus to any tangent will be tangent at vertex.

(iii) If tangents drawn at ends point of a focal chord are mutually perpendicular then their point of intersection will lie on directrix.

(iv) Any light ray travelling parallel to axis of the parabola will pass through focus after reflection through parabola.

(v) Angle included between focal radius of a point and perpendicular from a point to directrix will be bisected of tangent at that point also the external angle will be bisected by normal.

(vi) Intercepted portion of a tangent between the point of tangency and directrix will make right angle at focus.

(vii) Circle drawn on any focal radius as diameter will touch tangent at vertex.

(viii) Circle drawn on any focal chord as diameter will touch directrix.

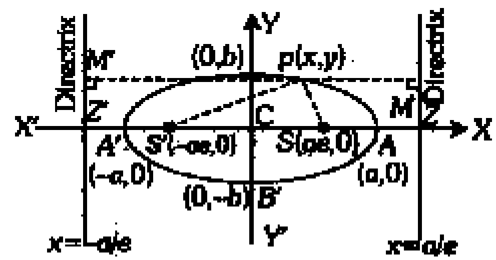
ELLIPSE

1. Standard Ellipse (e < 1)

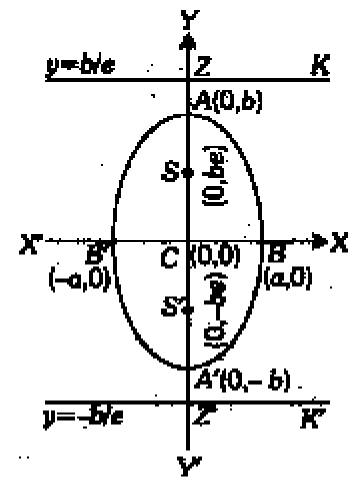
Ellipse / Imp. terms	$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$	
	For a > b	**For b > a**
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Length of major axis	2a	2b
Length of minor axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Relation in a, b and e	$b^2 = a^2(1 - e^2)$	$a^2 = b^2(1 - e^2)$
Length of latus rectum	$2b^2/a$	$2a^2/b$
Ends of latus rectum	$\left(\pm ae, \pm \frac{b^2}{a}\right)$	$\left(\pm \frac{a^2}{b}, \pm be\right)$
Parametric coordinates	$(a \cos\phi, b \sin\phi)$	$(a \cos \phi, b \sin \phi)$ $0 \le \phi < 2\pi$
Focal radii	$SP = a - ex_1$ $S'P = a + ex_1$	$SP = b - ey_1$ $S'P = b + ey_1$
Sum of focal radii	SP + S'P =	2a 2b
Distance bt^n foci	2ae	2be
Distance bt^n directrices	2a/e	2b/e
Tangents at the vertices	$x = -a, x = a$	$y = b, y = -b$

Note : If P is any point on ellipse and length of perpendiculars from to minor axis and major axis are p_1 & p_2, then $|x_p| = p_1$, $|y_p| = p_2$

$$\Rightarrow \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1$$



a > b



b > a

2. Special form of ellipse :

If the centre of an ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

3. Auxillary Circle : The circle described by taking centre of an ellipse as centre and major axis as a diameter is called an auxillary circle of the ellipse.

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse then its auxillary circle is $x^2 + y^2 = a^2$.

Note : Ellipse is locus of a point which moves in such a way that it divides the normal of a point on diameter of a point of circle in fixed ratio.

4. Position of a point and a line w.r.t. an ellipse :

* The point lies outside, on or inside the ellipse if

$$S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > , = \text{ or } < 0$$

* The line y = mx + c does not intersect, touches, intersect, the ellipse if

$a^2m^2 + b^2 < = > c^2$

5. Equation of tangent in different forms :

(i) **Point form :** The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

(ii) **Slope form :** If the line y = mx + c touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c^2 = a^2m^2 + b^2$. Hence, the straight line $y = mx \pm \sqrt{a^2m^2 + b^2}$ always represents the tangents to the ellipse.

Point of contact :

Line $y = mx \pm \sqrt{a^2m^2 + b^2}$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } \left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}}, \frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right).$$

(iii) **Parametric form :** The equation of tangent at any point $(a \cos \phi, b \sin \phi)$ is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1.$$

6. Equation of pair of tangents from (x_1, y_1) to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $SS_1 = T^2$

7. Equation of normal in different forms :

(i) **Point form :** The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{a^2x}{x_1} - \frac{b^2x}{y_1} = a^2 - b^2.$$

(ii) **Parametric form :** The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \phi, b \sin \phi)$ is

$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2$.

(iii) **Slope form :** If m is the slope of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the equation of normal is $y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$.

The co-ordinates of the point of contact are

$$\left(\frac{\pm a^2}{\sqrt{a^2 + b^2m^2}}, \frac{\pm mb^2}{\sqrt{a^2 + b^2m^2}}\right).$$

Note : In general three normals can be drawn from a point (x_1, y_1) to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

8. Properties of tangents & normals :

(i) Product of length of perpendicular from either focii to any tangent to the ellipse will be equal to square of semi minor axis.

(ii) The locus of foot of perpendicular drawn from either focii to any tangent lies on auxillary circle.

(iii) The circle drawn on any focal radius as diameter will touch auxillary circle.

(iv) The protion of the tangent intercepted between the point and directrix makes right angle at corresponding focus.

(v) Sum of square of intercept made by auxillary circle on any two perpendicular tangents of an ellipse will be constant.

(vi) If a light ray originates from one of focii, then it will pass through the other focus after reflection from ellipse.

9. Equation of chord of contact of the tangents drawn from the external point (x_1, y_1) to an ellipse is given by

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0$ i.e. $T = 0$.

10. The equation of a chord of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid point is (x_1, y_1) is $T = S_1$.

11. Equation of chord joining the points $(a \cos \theta, b \sin \theta)$ and $(a \cos \phi, b \sin \phi)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a} \cos \frac{\theta + \phi}{2} + \frac{y}{b} \sin \frac{\theta + \phi}{2} = \cos \frac{\theta - \phi}{2}$$

(i) Relation between eccentric angles of focal chord

$$\Rightarrow \tan \frac{\theta_1}{2}, \tan \frac{\theta_2}{2} = \frac{\pm e - 1}{1 \pm e}$$

(ii) Sum of feet of eccentric angles is odd π.

i.e. $\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n + 1)\pi$.

12. Equation of polar of the point (x_1, y_1) w.r.t. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 0$ i.e. T = 0. The pole of the line $\ell x + my + n = 0$ w.r.t. the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\left(\frac{-a^2\ell}{n}, \frac{-b^2n}{n}\right)$.

13. Eccentric angles of the extremities of latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $\tan^{-1}\left(\pm\frac{b}{ae}\right)$.

14. (i) Equation of the diameter bisecting the chords of slope in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = -\frac{b^2}{a^2m}x$$

(ii) **Conjugate Diameters :** The straight lines $y = m_1x$, $y = m_2x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $m_1m_2 = -\frac{b^2}{a^2}$.

(iii) Properties of conjugate diameters :

(a) If CP and CQ be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$CP^2 + CQ^2 = a^2 + b^2$

(b) If θ and ϕ are the eccentric angles of the extremities of two conjugate diameters, then

$$\theta - \phi = \pm\frac{\pi}{2}$$

(c) If CP, CQ be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and S, S' be two foci of the ellipse, then $SP.S'P = CQ^2$

(d) The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.

15. The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axis i.e. 4ab.

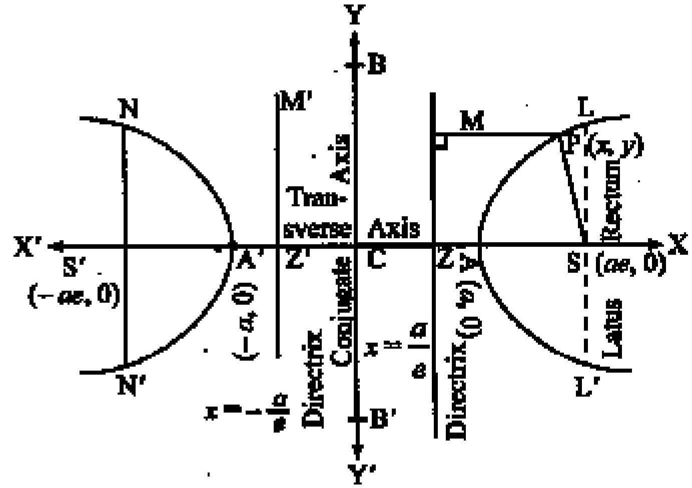
16. Length of subtangent and subnormal at $p(x_1, y_1)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2}{x_1} - x_1 \;\&\; (1 - e^2)\, x_1$$

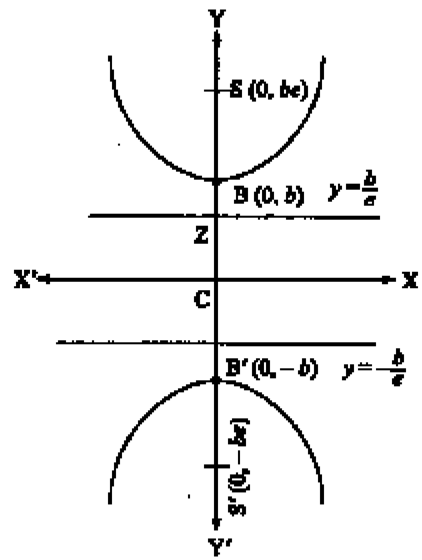
HYPERBOLA

1. Standard Hyperbola :

Hyperbola / Imp. terms	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	(0, 0)	(0, 0)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\frac{a^2+b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2+b^2}{b^2}\right)}$
Length of L.R.	$2b^2/a$	$2a^2/b$
Parametric co-ordinates	$(a \sec\phi,\ b \tan\phi)$ $0 \le \phi < 2\pi$	$(b \sec\phi,\ a \tan\phi)$ $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ $S'P = ex_1 + a$	$SP = ey_1 - b$ $S'P = ey_1 + b$
$S'P - SP$	2a	2b
Tangents at the vertices	$x = -a,\ x = a$	$y = -b,\ y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$



Hyperbola



Conjugate Hyperbola

2. Special form of hyperbola :
If the centre of hyperbola is (h, k) and axes are parallel to the co-ordinate axes, then its equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

3. Parametric equations of hyperbola :
The equations $x = a \sec \phi$ and $y = b \tan \phi$ are known as the parametric equations of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

4. Position of a point and a line w.r.t. a hyperbola :
The point (x_1, y_1) lies inside, on or outside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is +ve, zero or –ve.

The line $y = mx + c$ does not intersect, touches, intersect the hyperbola
according as $c^2 <, =, > a^2m^2 - b^2$.

5. Equations of tangents in different forms :
(a) **Point form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(b) **Parametric form :** The equation of tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \phi, b \tan \phi)$ is

$$\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1.$$

(c) **Slope form :** The equations of tangents of slope m to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2m^2 - b^2}$ and the co-ordinates of points of contacts are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right).$$

6. Equation of pair of tangents from (x_1, y_1) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by $SS_1 = T^2$

7. Equations of normals in different forms :
(a) **Point form :** The equation of normal to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

(b) **Parametric form :** The equation of normal at $(a \sec \theta, b \tan \theta)$ to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $ax \cos \theta + by \cot \theta = a^2 + b^2$

(c) **Slope form :** The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of the slope m of the normal is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$

(d) **Condition for normality :** If y = mx + c is the normal of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

then $c = \mp \frac{m(a^2+b^2)}{\sqrt{a^2 - b^2m^2}}$ or $c^2 = \frac{m(a^2+b^2)^2}{(a^2 - m^2b^2)}$, which is condition of normality.

(e) **Points of contact :** Co-ordinates of points of contact are $\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$.

8. The equation of director circle of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.

9. Equation of chord of contact of the tangents drawn from the external point (x_1, y_1) to the hyperbola is given by $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

10. The equation of chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ whose mid point is (x_1, y_1) is $T = S_1$.

11. Equation of chord joining the points P(a sec ϕ_1, b tan ϕ_1) and Q(a sec ϕ_2, b tan ϕ_2) is

$$\frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1+\phi_2}{2}\right).$$

12. Equation of polar of the point (x_1, y_1) w.r.t. the hyperbola is given by T = 0.

The pole of the line $\ell x + my + n = 0$ w.r.t.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\left(-\frac{a^2\ell}{n}, \frac{b^2m}{n}\right)$

13. The equation of a diameter of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2m}x$.

14. The diameters $y = m_1x$ and $y = m_2x$ are conjugate if $m_1m_2 = \frac{b^2}{a^2}$

15. Asymptotes of a hyperbola :

* The equations of asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$.

Asymptote to a curve touches the curve at infinity.

* The asymptote of a hyperbola passes through the centre of the hyperbola.

* The combined equation of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.
* The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $2\tan^{-1}\frac{y^2}{b^2}$ or $2\sec^{-1} e$.
* A hyperbola and its conjugate hyperbola have the same asymptotes.
* The bisector of the angles between the asymptotes are the coordinate axes.
* Equation of hyperbola – Equation of asymptotes = Equation of asymptotes – Equation of conjugate hyperbola = constant.

16. Rectangular or Equilateral Hyperbola :

* A hyperbola for which a = b is said to be rectangular hyperbola, its equation is $x^2 - y^2 = a^2$
* $xy = c^2$ represents a rectangular hyperbola with asymptotes $x = 0$, $y = 0$.
* Eccentricity of rectangular hyperbola is $\sqrt{2}$ and angle between asymptotes of rectangular hyperbola is 90º.
* Parametric equation of the hyperbola $xy = c^2$ are

  $x = ct$, $y = \frac{c}{t}$, where t is a parameter.
* Equation of chord joining t_1, t_2 on $xy = c^2$ is

  $x + y\, t_1t_2 = c(t_1 + t_2)$

* Equation of tangent at (x_1, y_1) to $xy = c^2$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

  Equation of tangent at t is $x + yt^2 = 2ct$
* Equation of normal at (x_1, y_1) to $xy = c^2$ is

  $xx_1 - yy_1 = x_1^2 - y_1^2$
* Equation of normal at t on $xy = c^2$ is

  $xt^3 - yt - ct^4 + c = 0$.

  (This results shows that four normal can be drawn from a point to the hyperbola $xy = c^2$)
* If a triangle is inscribed in a rectangular hyperbola then its orthocentre lies on the hyperbola.
* Equation of chord of the hyperbola $xy = c^2$ whose middle point is given is $T = S_1$
* Point of intersection of tangents at t_1 & t_2 to the hyperbola $xy = c^2$ is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$

MEASURES OF CENTRAL TENDENCY AND DISPERSION

1. Arithmetic mean :

(i) For ungrouped data (individual series) $\overline{x}$ =

$$\frac{x_1 + x_2 + \ldots\ldots + x_n}{n(\text{no. of terms})} = \frac{\sum_{i=1}^{n} x_i}{n}$$

(ii) For grouped data (continuous series)

(a) Direct method $\overline{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$, where x_i, i = 1 .... n

be n observations and f_i be their corresponding frequencies

(b) short cut method : $\overline{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$,

where A = assumed mean, $d_i = x_i - A$ = deviation for each term

2. Properties of A.M.

(i) In a statistical data, the sum of the deviation of items from A.M. is always zero.

(ii) If each of the n given observation be doubled, then their mean is doubled

(iii) If $\overline{x}$ is the mean of $x_1, x_2, \ldots\ldots x_n$. The mean of $ax_1, ax_2 \ldots\ldots ax_n$ is $a\overline{x}$ where a is any number different from zero.

(iv) Arithmetic mean is independent of origin i.e. it is x effected by any change in origin.

3. Geometric Mean :

(i) For ungrouped data

$$\text{G.M.} = (x_1\, x_2\, x_3 \ldots\ldots x_n)^{1/n}$$

or $$\text{G.M.} = \text{antilog}\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)$$

(ii) For grouped data

$$\text{G.M.} = \left(x_1^{f_1} x_2^{f_2} \ldots . x_n^{f_n}\right)^{\frac{1}{N}}, \text{ where } N = \sum_{i=1}^{n} f_i$$

$$= \text{antilog}\left(\frac{\sum_{i=1}^{n} f_i \log x_i}{\sum_{i=1}^{n} f_i}\right)$$

4. Harmonic Mean - Harmonic Mean is reciprocal of arithmetic mean of reciprocals.

(i) For ungrouped data H.M. = $\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$

(ii) For grouped data H.M. = $\frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$

5. Relation between A.M., G.M and H.M.

A.M. $\geq$ G.M. $\geq$ H.M.

Equality holds only when all the observations in the series are same.

6. Median :

(a) Individual series (ungrouped data) : If data is raw, arrange in ascending or descending order and n be the no. of observations.

If n is odd, Median = Value of $\left(\frac{n+1}{2}\right)^{th}$ observation

If n is even, Median = $\frac{1}{2}$ [Value of $\left(\frac{n}{2}\right)^{th}$ + value of $\left(\frac{n}{2}+1\right)^{th}$] observation.

(b) Discrete series : First find cumulative frequencies of the variables arranged in ascending or descending order and

Median = $\left(\frac{n+1}{2}\right)^{th}$ observation, where n is cumulative frequency.

(c) Continuous distribution (grouped data)

(i) For series in ascending order

$$\text{Median} = \ell + \frac{\left(\frac{N}{2}-C\right)}{f} \times i$$

Where ℓ = Lower limit of the median class.
f = Frequency of the median class.
N = Sum of all frequencies.
i = The width of the median class
C = Cumulative frequency of the class preceding to median class.

(ii) For series in descending order

$$\text{Median} = u - \frac{\left(\frac{N}{2}-C\right)}{f} \times i$$

where u = upper limit of median class.

7. Mode :

(i) For individual series : In the case of individual series, the value which is repeated maximum number of times is the mode of the series.

(ii) For discrete frequency distribution series : In the case of discrete frequency distribution, mode is the value of the variate corresponding to the maximum frequency.

(iii) For continuous frequency distribution : first find the model class i.e. the class which has maximum frequency.

For continuous series

$$\text{Mode} = \ell_1 + \left[\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right] \times i$$

Where ℓ_1 = Lower limit of the model class.
f_1 = Frequency of the model class.
f_0 = Frequency of the class preceding model class.
f_2 = Frequency of the class succeeding model class.
i = Size of the model class.

8. Relation between Mean, Mode & Median :

(i) In symmetrical distribution : Mean = Mode = Median

(ii) In Moderately symmetrical distribution : Mode = 3 Median – 2 Mean

Measure of Dispersion :

The degree to which numerical data tend to spread about an average value is called variation or dispersion.

Popular methods of measure of dispersion.

1. **Mean deviation :** The arithmetic average of deviations from the mean, median or mode is known as mean deviation.

(a) Individual series (ungrouped data)

$$\text{Mean deviation} = \frac{\Sigma|x - S|}{n}$$

Where n = number of terms, S = deviation of variate from mean mode, median.

(b) Continuous series (grouped data).

$$\text{Mean deviation} = \frac{\Sigma f|x - s|}{\Sigma f} = \frac{\Sigma f|x - s|}{N}$$

Note : Mean deviation is the least when measured from the median.

2. **Standard Deviation :**

S.D. (σ) is the square root of the arithmetic mean of the squares of the deviations of the terms from their A.M.

(a) For individual series (ungrouped data)

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{N}}$$ where $\bar{x}$ = Arithmetic mean of the series

N = Total frequency

(b) For continuous series (grouped data)

(i) Direct method $\sigma = \sqrt{\frac{\Sigma f_i (x_i - \bar{x})^2}{N}}$

Where $\bar{x}$ = Arithmetic mean of series

x_i = Mid value of the class

f_i = Frequency of the corresponding x_i

N = Σf = Total frequency

(ii) Short cut method

$$\sigma = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \quad \text{or} \quad \sigma = \sqrt{\frac{\Sigma d^2}{N} - \left(\frac{\Sigma d}{N}\right)^2}$$

Where d = x – A = Derivation from assumed mean A

f = Frequency of item (term)

N = Σf = Total frequency.

Variance – Square of standard direction

i.e. variance = $(S.D.)^2 = (\sigma)^2$

Coefficient of variance = Coefficient of S.D. × 100

$$= \frac{\sigma}{x} \times 100$$

MATRICES AND DETERMINANTS

MATRICES :

1. **Matrix -** A system or set of elements arranged in a rectangular form of array is called a matrix.

2. **Order of matrix :** If a matrix A has m rows & n columns then A is of order $m \times n$.
The number of rows is written first and then number of columns. Horizontal line is row & vertical line is column

3. **Types of matrices :** A matrix $A = (a_{ij})_{m\times n}$
A matrix $A = (a_{ij})_{mxn}$ over the field of complex numbers is said to be

Name	Properties
A row matrix	if m = 1
A column matrix	if n = 1
A rectangular matrix	if $m \neq n$
A square matrix	if m = n
A null or zero matrix	if $a_{ij} = 0\ \forall$ i j. It is denoted by O.
A diagonal matrix	if m = n and $a_{ij} = 0$ for $i \neq j$.
A scalar matrix	if m = n and $a_{ij} = 0$ for $i \neq j$ = k for i = j i.e. $a_{11} = a_{22}$ .... $= a_{nn} = k$ (cons.)
Identity or unit matrix	if m = n and $a_{ij} = 0$ for $i \neq j$ = 1 for i = j
Upper Triangular matrix	if m = n and $a_{ij} = 0$ for i > j
Lower Triangular matrix	if m = n and $a_{ij} = 0$ for i < j
Symmetric matrix	if m = n and $a_{ij} = a_{ji}$ for all i, j or $A^T = A$
Skew symmetric matrix	if m = n and $a_{ij} = -a_{ji}\ \forall$ i, j or $A^T = -A$

4. **Trace of a matrix :** Sum of the elements in the principal diagonal is called the trace of a matrix.

trace (A ± B) = trace A ± trace B

trace kA = k trace A

trace A = trace A^T

trace $I_n = n$ when I_n is identity matrix.

trace $O_n = O$ O_n is null matrix.

trace AB ≠ trace A trace B.

5. **Addition & subtraction of matrices :** If A and B are two matrices each of order same, then A + B (or A – B) is defined and is obtained by adding (or subtracting) each element of B from corresponding element of A

6. **Multiplication of a matrix by a scalar :**

$KA = K(a_{ij})_{m\times n} = (Ka)_{m\times n}$ where K is constant.

Properties :

(i) K(A + B) = KA + KB

(ii) $(K_1 K_2)A = K_1(K_2 A) = K_2(K_1A)$

(iii) $(K_1 + K_2)A = K_1A + K_2A$

7. **Multiplication of Matrices :** Two matrices A & B can be multiplied only if the number of columns in A is same as the number of rows in B.

Properties :

(i) In general matrix multiplication is not commutative i.e. AB ≠ BA.

(ii) A(BC) = (AB)C [Associative law]

(iii) A.(B + C) = AB + AC [Distributive law]

(iv) If AB = AC $\not\Rightarrow$ B = C

(v) If AB = 0, then it is not necessary A = 0 or B = 0

(vi) AI = A = IA

(vii) Matrix multiplication is commutative for +ve integral i.e. $A^{m+1} = A^m A = AA^m$

8. Transpose of a matrix :

A' or A^T is obtained by interchanging rows into columns or columns into rows

Properties :

(i) $(A^T)^T = A$

(ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(AB)^T = B^T A^T$

(iv) $(KA)^T = KA^T$

(v) $I^T = I$

9. Some special cases of square matrices : A square matrix is called

(i) Orthogonal matrix : if $AA^T = I_n = A^TA$

(ii) Idempotent matrix : if $A^2 = A$

(iii) Involutory matrix : if $A^2 = I$ or $A^{-1} = A$

(iv) Nilpotent matrix : if $\exists$ $p \in N$ such that $A^p = 0$

(v) Hermitian matrix : if $A^\theta = A$ i.e. $a_{ij} = \overline{a}_{ji}$

(vi) Skew - Hermitian matrix : if $A = -A^\theta$

DETERMINANT :

1. Minor & cofactor : If $A = (a_{ij})_{3\times3}$, then minor of a_{11} is

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ and so.}$$

cofactor of an element a_{ij} is denoted by C_{ij} or F_{ij} and is equal to $(-1)^{i+j} M_{ij}$

or $C_{ij} = M_{ij}$, if i = j

$= -M_{ij}$, if $i \neq j$

Note : $|A| = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$

and $a_{11}F_{21} + a_{12}F_{22} + a_{13}F_{23} = 0$

2. Determinant : if A is a square matrix then determinant of matrix is denoted by det A or |A|.

expansion of determinant of order 3 × 3

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\text{or} \quad = -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

Properties :

(i) $|A^T| = |A|$

(ii) By interchanging two rows (or columns), value of determinant differ by –ve sign.

(iii) If two rows (or columns) are identical then |A| = 0

(iv) $|KA| = K^n \det A$, A is matrix of order n × n

(v) If same multiple of elements of any row (or column) of a determinant are added to the corresponding elements of any other row (or column), then the value of the new determinant remain unchanged.

(vi) **Determinant of :**

(a) A nilpotent matrix is 0.

(b) An orthogonal matrix is 1 or − 1

(c) A unitary matrix is of modulus unity.

(d) A Hermitian matrix is purely real.

(e) An identity matrix is one i.e. $|I_n| = 1$, where I_n is a unit matrix of order n.

(f) A zero matrix is zero i.e. $|0_n| = 0$, where 0_n is a zero matrix of order n

(g) A diagonal matrix = product of its diagonal elements.

(h) Skew symmetric matrix of odd order is zero.

3. Multiplication of two determinants :

Multiplication of two second order determinants is defined as follows.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1\ell_1 + b_1\ell_2 & a_1m_1 + b_1m_2 \\ a_2\ell_1 + b_2\ell_2 & a_2m_1 + b_2m_2 \end{vmatrix}$$

If order is different then for their multiplication, express them firstly in the same order.

MATRICES AND DETERMINANTS :

1. Adjoint of a matrix :

adj A = $(C_{ij})^T$, where C_{ij} is cofactor of a_{ij}

Properties :

(i) $A(\text{adj } A) = (\text{adj}A)\, A = |A|I_n$

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $(\text{adj}AB) = (\text{adj}B)\,(\text{adj}A)$

(iii) $(\text{adj } A^T) = (\text{adj}A)^T$

(iv) $\text{adj}(\text{adj}A) = |A|^{n-2}$

(v) $(\text{adj } KA) = K^{n-1}(\text{adj } A)$

2. Inverse of a matrix :

(i) A^{-1} exists if A is non singular i.e. $|A| \neq 0$

(ii) $A^{-1} = \dfrac{\text{adj}A}{|A|}$, $|A| \neq 0$

(iii) $A^{-1}A = I_n = A\,A^{-1}$

(iv) $(A^T)^{-1} = (A^{-1})^T$

(v) $(A^{-1})^{-1} = A$

(vi) $|A^{-1}| = |A|^{-1} = \dfrac{1}{|A|}$

(vii) If A & B are invertible square matrices then $(AB)^{-1} = B^{-1}A^{-1}$

3. Rank of a matrix :

A non zero matrix A is said to have rank r, if

(i) Every square sub matrix of order (r + 1) or more is singular

(ii) There exists at least one square submatrix of order r which is non singular.

4. Homogeneous & non homogeneous system of linear equations :

A system of equations Ax = B is called a homogeneous system if B = 0. If B ≠ 0, then it is called non homogeneous system equations.

5. (a) Solution of non homogeneous system of linear equations :

(i) **Cramer's rule :** Determinant method

The non homogeneous system Ax = B, B ≠ 0 of n equations in n variables is -

Consistent (with unique solution) if |A| ≠ 0 and for each i = 1, 2, ... n,

$x_i = \frac{\det A_i}{\det A}$, where A_i is the matrix obtained from A by replacing i^{th} column with B.

Inconsistent (with no solution) if |A| = 0 and at least one of the det (A_i) is non zero.

Consistent (With infinite many solution), if |A| = 0 and all det (A_i) are zero.

(ii) **Matrix method :**

The non homogeneous system Ax = B, B ≠ 0 of n equations in n variables is -

Consistent (with unique solution) if |A| ≠ 0 i.e. if A is non singular, $x = A^{-1}B$.

Inconsistent (with no solution), if |A| = 0 and (adj A) B is a non null matrix.

Consistent (with infinitely many solutions), if |A| = 0 and (adj A) B is a null matrix.

(b) Solution of homogeneous system of linear equations :

The homogeneous system Ax = B, B = 0 of n equations in n variables is

(i) Consistent (with unique solution) if |A| ≠ 0 and for each i = 1, 2, ... n

$x_i = 0$ is called trivial solution.

(ii) Consistent (with infinitely many solution), if |A| = 0

(a) $|A| = |A_i| = 0$ (for determinant method)

(b) |A| = 0, (adj A) B = 0 (for matrix method)

NOTE : A homogeneous system of equations is never inconsistent.

FUNCTION

1. Modulus function :

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$$

Properties :

(i) $|x| \neq \pm x$

(ii) $|xy| = |x||y|$

(iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$

(iv) $|x + y| \leq |x| + |y|$

(v) $|x - y| \geq |x| - |y|$ or $\leq |x| + |y|$

(vi) $||a| - |b|| \leq |a - b|$ for equality $a.b \geq 0$.

(vii) If $a > 0$

$|x| = a \Rightarrow x = \pm a$

$|x| = -a \Rightarrow$ no solution

$|x| > a \Rightarrow x < -a$ or $x > a$

$|x| \leq a \Rightarrow -a \leq x \leq a$

$|x| < -a \Rightarrow$ No solution.

$|x| > -a \Rightarrow x \in R$

2. Logarithmic Function :

(i) $\log_b a$ to be defined $a > 0, b > 0, b \neq 1$

(ii) $\log_a b = c \Rightarrow b = a^c$

(iii) $\log_a b > c$

$\Rightarrow b > a^c, \quad a > 1$

or $b < a^c, \quad 0 < a < 1$

(iv) $\log_a b > \log_a c$

$\Rightarrow b > c, \quad$ if $a > 1$

or $b < c, \quad$ if $0 < a < 1$

Properties :

(i) $\log_a 1 = 0$

(ii) $\log_a a = 1$

(iii) $a^{\log_a b} = b$

if $k > 0, \quad k = b^{\log_b k}$

(iv) $\log_a b_1 + \log_a b_2 + ... + \log_a b_n = \log_a (b_1 b_2 ....b_n)$

(v) $\log_a \left(\dfrac{b}{c}\right) = \log_a b - \log_a c$

(vi) Base change formulae

$$\log_a b = \frac{\log_c b}{\log_c a} \quad \text{or} \quad \log_a b = \frac{1}{\log_b a}$$

(vii) $\log_{a^m} b^n = \dfrac{n}{m} \log_a b$

(viii) $\log_a \left(\dfrac{1}{b}\right) = -\log_a b = \log_{1/a} b$

(ix) $\log_{1/a} \left(\dfrac{b}{c}\right) = \log_a \left(\dfrac{c}{b}\right)$

(x) $a^{\log_b c} = c^{\log_b a}$

3. Greatest Integer function :

$f(x) = [x]$, where [.]denotes greatest integer function equal or less than x.

i.e., defined as $[4.2] = 4$, $[-4.2] = -5$

Period of $[x] = 1$

Properties :

(i) $x - 1 < [x] \le x$

(ii) $[x + I] = [x] + I$

$[x + y] \ne [x] + [y]$

(iii) $[x] + [-x] = 0, x \in I$

$= -1, x \notin I$

(iv) $[x] = I$, where I is an integer $x \in [I, I + 1)$

(v) $[x] \ge I, x \in [I, \infty)$

(vi) $[x] \le I, x \in (-\infty, I + 1]$

(vii) $[x] > I, [x] \ge I + 1, x \in [I + 1, \infty)$

(viii) $[x] < I, [x] \le I - 1, x \in (-\infty, I)$

4. Fractional part function :

$f(x) = \{x\}$ = difference between number & its integral part $= x - [x]$.

Properties :

(i) $\{x\}, x \in [0, 1)$

(ii) $\{x + I\} = \{x\}$

$\{x + y\} \ne \{x\} + \{y\}$

(iii) $\{x\} + \{-x\} = 0, \quad x \in I$

$= 1, x \notin I$

(iv) $[\{x\}] = 0, \{\{x\}\} = \{x\}, \{[x]\} = 0$

5. Signum function :

$$f(x) = \text{sgn}(x) = \begin{cases} -1, & x \in R^- \\ 0, & x = 0 \\ 1, & x \in R^+ \end{cases}$$

or $f(x) = \dfrac{|x|}{x}, \quad x \ne 0$

$= 0, \quad x = 0$

6. Definition :

Let A and B be two given sets and if each element $a \in A$ is associated with a unique element $b \in B$ under a rule f, then this relation (mapping) is called a function.

Graphically - no vertical line should intersect the graph of the function more than once.

Here set A is called domain and set of all f images of the elements of A is called range.

i.e., Domain = All possible values of x for which f(x) exists.

Range = For all values of x, all possible values of f(x).

Table : Domain and Range of some standard functions -

Functions	**Domain**	**Range**
Polynomial function	R	R
Identity function x	R	R
Constant function K	R	(K)
Reciprocal function $\frac{1}{x}$	R_0	R_0
x^2, $\|x\|$ (modulus function)	R	$R^+ \cup \{x\}$
x^3, $x\|x\|$	R	R
Signum function $\frac{\|x\|}{x}$	R	{-1, 0, 1}
$x + \|x\|$	R	$R^+ \cup \{x\}$
$x - \|x\|$	R	$R^- \cup \{x\}$
[x] (greatest integer function)	R	1
x - {x}	R	[0, 1]
$\sqrt{x}$	$[0, \infty)$	$[0, \infty]$
a^x (exponential function)	R	R^+
log x (logarithmic function)	R^+	R

Trigonometric Functions	Domain	Range
sin x	R	[-1, 1]
cos x	R	[-1, 1]
tan x	R- $\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$	R
cot x	R- $\{0, \pm\pi, \pm 2\pi, \ldots\}$	R
sec x	R - $\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$	R - (-1,1)
cosec x	R- $\{0, \pm\pi, \pm 2\pi\}$	R - (-1,1)

Inverse Trigo Functions	Domain	Range
$\sin^{-1} x$	(-1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	[-1,1]	$[0, \pi]$
$\tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	R -(-1,1)	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	R - (-1,1)	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$- {0}

7. Kinds of functions :

(i) One-one (injection) function - $f : A \to B$ is one-one if

$f(a) = f(b) \Rightarrow a = b$

or $a \neq b$

$\Rightarrow f(a) \neq f(b)$, $a, b \in A$

Graphically-no horizontal line intersects with the graph of the function more than once.

(ii) Onto function (surjection) - $f : A \to B$ is onto if

$R(f) = B$ i.e. if to each $y \in B \ \exists\ x \in A$ s.t. $f(x) = y$

(iii) Many one function : $f : A \to B$ is a many one function if there exist $x, y \in A$ s.t. $x \neq y$

but $f(x) = f(y)$

Graphically - atleast one horizontal line intersects with the graph of the function more than once.

(iv) Into function : f is said to be into function if $R(f) < B$

(v) One-one-onto function (Bijective) - A function which is both one-one and onto is called bijective function.

8. Inverse function : f^{-1} exists iff f is one-one & onto both

$f^{-1} : B \to A$, $f^{-1}(b) = a \quad \Rightarrow f(a) = b$

9. Transformation of curves :

(i) Replacing x by (x – a) entire graph will be shifted parallel to x-axis with |a| units.

If a is +ve it moves towards right.

a is –ve it moves toward left.

Similarly if y is replace by (y – a), the graph will be shifted parallel to y-axis,

upward if a is +ve

downward if a is –ve.

(ii) Replacing x by –x, take reflection of entire curve is y-axis.

Similarly if y is replaced by –y then take reflection of entire curve in x-axis.

(iii) Replacing x by |x|, remove the portion of the curve corresponding to –ve x (on left hand side of y-axis) and take reflection of right hand side on LHS.

(iv) Replace f(x) by |f(x)|, if on L.H.S. y is present and mode is taken on R.H.S. then portion of the curve below x-axis will be reflected above x-axis.

(v) Replace x by ax (a > 0), then divide all the value on x-axis by a.

Similarly if y is replaced by ay (a > 0) then divide all the values of y-axis by a.

10. Even and odd function : A function is said to be

(i) Even function if f(–x) = f(x) and

(ii) Odd function if f(–x) = –f(x).

11. Properties of even & odd function :

(a) The graph of an even function is always symmetric about y-axis.

(b) The graph of an odd function is always symmetric about origin.

(c) Product of two even or odd function is an even function.

(d) Sum & difference of two even (odd) function is an even (odd) function.

(e) Product of an even or odd function is an odd function.

(f) Sum of even and odd function is neither even nor odd function.

(g) Zero function i.e. f(x) = 0 is the only function which is even and odd both.

(h) If f(x) is odd (even) function then f'(x) is even (odd) function provided f(x) is differentiable on R.

(i) A given function can be expressed as sum of even & odd function.

i.e. $f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

= even function + odd function.

12. Increasing function : A function f(x) is an increasing function in the domain D if the value of the function does not decrease by increasing the value of x.

13. Decreasing function : A function f(x) is a decreasing function in the domain D if the value of function does not increase by increasing the value of x.

14. Periodic function: Function f(x) will be periodic if a +ve real number T exist such that

$f(x + T) = f(x), \quad \forall\ x \in \text{Domain}.$

There may be infinitely many such T which satisfy the above equality. Such a least +ve no. T is called period of f(x).

(i) If a function f(x) has period T, then

Period of f(xn + a) = T/n and

Period of (x/n + a) = nT

(ii) If the period of f(x) is T_1 & g(x) has T_2 then the period of f(x) ± g(x) will be L.C.M. of T_1 & T_2 provided it satisfies definition of periodic function.

(iii) If period of f(x) & g(x) are same T, then the period of af(x) + bg(x) will also be T.

Function	Period
sin x, cos x	2π
sec x, cosec x	
tan x, cot x	π
sin (x/3)	6π
tan 4x	$\pi/4$
cos $2\pi x$	1
|cos x|	π
$\sin^4 x + \cos^4 x$	$\pi/2$
$2\cos\left(\frac{x-\pi}{3}\right)$	6π
sin3 x + $\cos^3 x$	$2\pi/3$
$\sin^3 x + \cos^4 x$	2π
$\frac{\sin x}{\sin 5x}$	2π
$\tan^2 x - \cot^2 x$	π
x − [x]	1
[x]	1

NON PERIODIC FUNCTIONS :

$\sqrt{x}$, x^2, x^3, 5

$\cos x^2$

$x + \sin x$

$x \cos x$

$\cos\sqrt{x}$

15. Composite function :

If $f : X \to Y$ and $g : Y \to Z$ are two function, then the composite function of f and g, $gof : X \to Z$ will be defined as $gof(x) = g(f(x)), \ \forall\ x \in X$

In general gof $\neq$ fog

If both f and g are bijective function, then so is gof.

LIMIT

1. **Limit of a function :** $\lim_{x \to a} f(x) = \ell$ (finite quantity)

2. **Existence of limit :** $\lim_{x \to a} f(x)$ exists iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \ell$

3. **Indeterminate forms :** $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \infty \times 0, \infty^0, 0^\infty, 1^\infty$

4. **Theorems on limits :**

(i) $\lim_{x \to a} (k\, f(x)) = k \lim_{x \to a} f(x)$, k is a constant.

(ii) $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \text{Lim}_{x \to a}\, g(x)$

(iii) $\lim_{x \to a} f(x).g(x) = \lim_{x \to a} f(x).\ \text{Lim}_{x \to a}\, g(x)$

(iv) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

(v) $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$, provided value of g(x) function f(x) is continuous.

(vi) $\lim_{x \to a} [f(x) + k] = \lim_{x \to a} f(x) + k$

(vii) $\lim_{x \to a} \log(f(x)) = \log\left(\lim_{x \to a} f(x)\right)$

(viii) $\lim_{x \to a} (f(x))^{g(x)} = \left[\lim_{x \to a} f(x)\right]^{\lim_{x \to a} g(x)}$

5. **Limit of the greatest integer function :**
Let c be any real number

Case I : If c is not an integer, then $\lim_{x \to c} [x] = [c]$

Case II: If c is an integer, then $\lim_{x \to c^-} [x] = c - 1$, $\lim_{x \to c^+} [x] = c$

and $\lim_{x \to c} [x]$ = does not exist

6. **Methods of evaluation of limits :**

(i) **Factorisation method :** If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of $\frac{0}{0}$ form then factorize num. & devo. separately and cancel the common factor which is participating in making $\frac{0}{0}$ form.

(ii) **Rationalization method :** If we have fractional powers on the expression in num, deno or in both, we rationalize the factor and simplify.

(iii) **When $x \to \infty$:** Divide num. & deno. by the highest power of x present in the expression and then after removing the indeterminate form, replace $\frac{1}{x}, \frac{1}{x^2}$,.. by 0.

(iv) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(v) **By using standard results (limits) :**

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin x}$

(b) $\lim_{x \to 0} \frac{\tan x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan x}$

(c) $\lim_{x \to 0} \sin x = 0$

(d) $\lim_{x \to 0} \cos x = \lim_{x \to 0} \frac{1}{\cos x} = 1$

(e) $\lim_{x \to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$

(f) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\sin^{-1} x}$

(g) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1 = \lim_{x \to 0} \frac{x}{\tan^{-1} x}$

(h) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(i) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(j) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(k) $\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a}$

(l) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$

(m) $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$

(n) $\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = 1$

(o) $\lim_{x \to 0} (1 + x)^{1/x} = e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

(p) $\lim_{x \to 0} (1 + ax)^{1/x} = e^a = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$

(vi) **By substitution :**

(a) If $x \to a$, then we can substitute
$x = a + t \Rightarrow t = x - a$
If $x \to a$, $t \to 0$.

(b) When $x \to -\infty$ substitute $x = -t \Rightarrow t \to \infty$

(c) When $x \to \infty$ substitute $t = \frac{1}{x} \Rightarrow t \to 0^+$

(vii) **By using some expansion :**

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$

$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ......$

$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - .....$

$e^{x \ln a} = a^x = 1 + x\log_e a + \frac{(x\log_e a)^2}{2!} + \frac{(x\log_e a)^3}{3!} + ...$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ......$

$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + ...$

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + ...$

7. Sandwich Theorem : In the neighbour hood of x = a
$f(x) < g(x) < h(x)$

$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \ell$, then $\lim_{x \to a} g(x) = \ell$.

$\Rightarrow \quad \ell < \lim_{x \to a} g(x) < \ell$.

DIFFERENTIATION

1. SOME STANDARD DIFFERENTIATION :

Function	Derivative	Function	Derivative
A cons. (k)	0	x^n	nx^{n-1}
$\log_a x$	$\frac{1}{x\log_e a}$	$\log_e x$	$\frac{1}{x}$
a^x	$a^x \log_e a$	e^x	e^x
sin x	cos x	cos x	−sin x
tan x	$\sec^2 x$	cot x	$-\text{cosec}^2 x$
cosec x	−cosec x cot x	sec x	sec x tan x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, -1<x<1$	$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}, -1<x<1$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{1-x^2}}, \lvert x\rvert>1$	$\text{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{1-x^2}}, -1\lvert x\rvert>\lvert$
$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in R$	$\cot^{-1} x$	$-\frac{1}{1+x^2}, x\in R$
[x]	$0, x \notin I$	$\lvert x\rvert$	$\frac{x}{\lvert x\rvert}, x\neq 0$

NOTE : $\frac{d}{dx}[x]$ does not exist at any integral Point.

2. FUNDAMENTAL RULES FOR DIFFERENTIATION :

(i) $\frac{d}{dx} f(x) = 0$ if and only if $f(x) =$ constant

(ii) $\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}f(x)$, where c is a constant.

(iii) $\frac{d}{dx}\left(f(x)\pm g(x)\right) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

(iv) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, where u & v are functions of x. (Product rule)

or $\frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$.

(v) If $\frac{d}{dx} f(x) = \phi(x)$, then $\frac{d}{dx} f(ax + b)$

$= a\,\phi(ax + b)$

(vi) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (quotient rule)

(vii) If $y = f(u)$, $u = g(x)$ [chain rule or differential co-efficient of a function of a function]

then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Illy If $y = f(u)$, $u = g(v)$, $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

i.e if $y = u^n \Rightarrow \frac{dy}{dx} = nu^{n-1} \frac{du}{dx}$

OR

(viii) Differentiation of composite functions

Suppose a function is given in form of fog(x) or f[g(x)], then differentiate applying chain rule

i.e., $\frac{d}{dx} f[g(x)] = f'g(x) \cdot g'(x)$

(ix) $\frac{d}{dx}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\frac{du}{dx}$, $u \neq 0$

(x) $\frac{d}{dx}|u| = \frac{u}{|u|}\frac{du}{dx}$, $u \neq 0$

(xi) **Logarithmic Differentiation :** If a function is in the form $(f(x))^{g(x)}$ or $\frac{f_1(x)\ f_2(x)....}{g_1(x)\,g_2(x)....}$ We first take log on both sides and then differentiate.

(a) $\log_e(mn) = \log_e m + \log_e n$

(b) $\log_e \frac{m}{n} = \log m - \log_e n$

(c) $\log_e (m)^n = n\log_e m$ (d) $\log_n m \ \log_m n = 1$

(e) $\log_{a^n} x^m = \frac{m}{n}\log_a x$ (f) $a^{\log_a x} = x$

(g) $\log_e e = 1$ (h) $\log_n m = \frac{\log_e m}{\log_e n}$

(xii) **Differentiation of implicit function :** If f (x, y) = 0, differentiate w.r.t. x and collect the terms containing $\frac{dy}{dx}$ at one side and find $\frac{dy}{dx}$.

[The relation f(x, y) = 0 in which y is not expressible explicitly in terms of x are called implicit functions]

(xiii) **Differentiation of parametric functions :** If x = f(t) and y = g(t), where t is a parameter, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

(xiv) **Differentiation of a function w.r.t. another function :** Let y = f(x) and z = g(x), then differentiation of y w.r.t. z is

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$$

(xv) **Differentiation of inverse Trigonometric functions using Trigonometrical Transformation :** To solve the problems involving inverse trigonometric functions first try for a suitable substitution to simplify it and then differentiate. If no such substitution is found then differentiate directly by using trigonometrical formula frequently.

3. Important Trigonometrical Formula :

(i) $\sin 2x = 2\sin x.\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x} = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

(viii) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$

(vi) $\cos 3x = 4\cos^3 x - 3\cos x$

(ix) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

(x) $\sin^{-1} x + \cos^{-1} x = \pi/2$

(xi) $\sec^{-1} x + \mathrm{cosec}^{-1} x = \pi/2$

(xii) $\tan^{-1} x + \cot^{-1} x = \pi/2$

(xiii) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\dfrac{x \pm y}{1 \mp xy}\right)$

(xiv) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right)$

(xv) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\ \sqrt{1-y^2}\right)$

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(xvi) $\sin^{-1} \sin(x) = x$, for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

$\cos^{-1}(\cos x) = x$, for $0 \le x \le \pi$

$\tan^{-1}(\tan x) = x$, for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(xvii) $\sin^{-1}(-x) = -\sin^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$,

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

(xviii) $\sin^{-1}\left(\dfrac{1}{x}\right) = \mathrm{cosec}^{-1} x$, $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1} x$,

$\tan^{-1}\left(\dfrac{1}{x}\right) = \cot^{-1} x$, $\cot^{-1}\left(\dfrac{1}{x}\right) = \tan^{-1} x$,

$\sec^{-1}\left(\dfrac{1}{x}\right) = \cos^{-1} x$, $\mathrm{cosec}^{-1}\left(\dfrac{1}{x}\right) = \sin^{-1} x$

(xix) $\sin^{-1}(\cos\theta) = \sin^{-1}\left(\sin\left(\dfrac{\pi}{2}-\theta\right)\right) = \dfrac{\pi}{2} - \theta$

$\cos^{-1}(\sin\theta) = \cos^{-1}\left(\cos\left(\dfrac{\pi}{2}-\theta\right)\right) = \dfrac{\pi}{2} - \theta$

$\tan^{-1}(\cot\theta) = \tan^{-1}\left(\tan\left(\dfrac{\pi}{2}-\theta\right)\right) = \dfrac{\pi}{2} - \theta$

4. Some Useful Substitutions :

Part A

Expression	Substitution	Formula	Result
$3x - 4x^3$	$x = \sin\theta$	$3\sin\theta - 4\sin^3\theta$	$\sin 3\theta$
$4x^3 - 3x$	$x = \cos\theta$	$4\cos^3\theta - 3\cos\theta$	$\cos 3\theta$
$\frac{3x - x^3}{1 - 3x^2}$	$x = \tan\theta$	$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	$\tan 3\theta$
$\frac{2x}{1 + x^2}$	$x = \tan\theta$	$\frac{2\tan\theta}{1 + \tan^2\theta}$	$\sin 2\theta$
$\frac{2x}{1 - x^2}$	$x = \tan\theta$	$\frac{2\tan\theta}{1 - \tan^2\theta}$	$\tan 2\theta$
$1 - 2x^2$	$x = \sin\theta$	$1 - 2\sin^2\theta$	$\cos 2\theta$
$2x^2 - 1$	$x = \cos\theta$	$2\cos^2\theta - 1$	$\cos 2\theta$
$1 - x^2$	$x = \sin\theta$	$1 - \sin^2\theta$	$\cos^2\theta$
	$x = \cos\theta$	$1 - \cos^2\theta$	$\sin^2\theta$
$x^2 - 1$	$x = \sec\theta$	$\sec^2\theta - 1$	$\tan^2\theta$
	$x = \text{cosec}\,\theta$	$\text{cosec}^2\theta - 1$	$\cot^2\theta$
$1 + x^2$	$x = \tan\theta$	$1 + \tan^2\theta$	$\sec^2\theta$
	$x = \cot\theta$	$1 + \cot^2\theta$	$\text{cosec}^2\theta$

Part B

Expression	Substitution
$a^2 + x^2$	$x = a\tan\theta$ or $x = a\cot\theta$
$\frac{a+x}{a-x}$ or $\frac{a-x}{a+x}$	$x = a\tan\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $x = a\,\text{cosec}\,\theta$
$\sqrt{\frac{a^2+x^2}{a^2-x^2}}$ or $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos\theta$

5. Successive differentiations or higher order derivatives :

(a) If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$ is called the first derivative of y w.r.t. x

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(f'(x))$$

is called the second derivative of y w.r.t. x lllly $\frac{d^3y}{dx^3} = \frac{d^2}{dx^2}(f'(x))$ etc......

Thus, This process can be continued and we can obtain derivatives of higher order

Note : To obtain higher order derivative of parametric functions we use chain rule

i.e. if $x = 2t$, $y = t^2$

$$\Rightarrow \frac{dy}{dx} = t$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(t) = 1.\frac{dt}{dx} = \frac{1}{t}$$

(b) If $y = (ax + b)^m$ $m \notin I$, then

$y_n = m(m-1)(m-2) \ldots\ldots (m-n+1)(ax + b)^{m-n}.a^n$

(c) If $m \in I$, then

$y_m = m!\ a^m$ and $y_{m+1} = 0$

(d) If $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n n!}{(ax+b)^{n+1}} a^n$

(e) If $y = \log(ax + b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n} a^n$

(f) If $y = \sin(ax + b)$, then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$

If $y = \cos(ax + b)$, then $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$

6. n^{th} Derivatives of Some Functions :

(i) $\frac{d^n}{dx^n}(x^n) = n!$

(ii) $\frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right)$

(iii) $\frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{\pi n}{2}\right)$

(iv) $\frac{d^n}{dx^n}(e^{mx}) = m^n e^{mx}$

(v) $\frac{d^n}{dx^n}(\log x) = (-1)^{n-1}(n-1)!\ x^{-n}$

NOTE : If $u = g(x)$ is such that $g'(x) = K$ (constant)

then $\frac{d^n}{dx^n} f(g(x)) = K^n \left[\frac{d^n}{du^n} f(u)\right]_{u=g(x)}$

7. Differentiation of Infinite Series : method is illustrated with the help of example

if $y = x^{x^{x \ldots \infty}}$ then function becomes $y = x^y$ now taking log on both sides

i.e $\log y = y \log x$, differentiating both sides w.r.t. x

we get $\frac{1}{y}\frac{dy}{dx} = y\frac{1}{x} + \log x \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{\left(\frac{1}{y} - \log x\right)} = \frac{y^2}{x(1 - y\log x)}$$

8. L-hospital rule :

if as $x \to a$ f(x) & g(x) either both $\to 0$ or both $\to \infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

(a) it can be applied only on 0/0 or ∞/∞ form

(b) Numerator & denominator are differentiated separately not $\frac{u}{v}$ formulae.

(c) If R.H.S. exist or d'not exist because value $\to \infty$, then L.H rule can be applied.

But if value fluctuate on R.H.S. then L.H. rule can't be applied.

If it is applied continuously then at each step 0/0 or ∞/∞ should be checked.

9. Differentiation of Determinant :

$$\Delta = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix} = |C_1 \ C_2 \ C_3|$$

$$\Delta' = \begin{vmatrix} R'_1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R'_3 \end{vmatrix}$$

$$= |C'_1 \ C_2 \ C_3| + |C_1 \ C'_2 \ C_3| + |C_1 \ C_2 \ C'_3|$$

APPLICATION OF DERIVATIVES

TANGENT AND NORMAL :

1. Geometrically f'(a) represents the slope of the tangent to the curve y = f(x) at the point (a, f(a))

2. If the tangent makes an angle ψ (say) with +ve x direction then

$$f'(x) = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \tan\psi = \text{slope of the tangent.}$$

3. If the tangent is parallel to x-axis, $\psi = 0$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 0.$$

4. If the tangent is perpendicular to x-axis, $\psi = \frac{\pi}{2}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} \to \infty$$

5. If the tangent line makes equal angle with the axes, then

$$\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \pm 1.$$

6. Equation of the tangent to the curve y = f(x) at a point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$$

7. **Length of intercepts made on axes by the tangent :**

$$x - \text{intercept} = x_1 - \left\{\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}\right\}$$

$$y - \text{intercept} = y_1 - x_1 \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}$$

8. **Length of perpendicular from origin to the tangent :**

$$= \left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2_{(x_1,y_1)}}}\right|$$

9. Slope of the normal $= -\dfrac{1}{\text{Slope of the tangent}}$

$$= -\left(\frac{dx}{dy}\right)_{(x_1,\, y_1)}$$

10. If normal makes an angle of ϕ with +ve direction of x-axis, then $\frac{dy}{dx} = -\cot\phi$.

11. If the normal is parallel to x-axis $\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = 0$.

12. If the normal is perpendicular to x-axis $\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = 0$.

13. If normal is equally inclined from both the axes or cuts equal intercept then $\left(\frac{dy}{dx}\right) = \pm 1$.

14. The equation of the normal to the curve $y = f(x)$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}}(x - x_1)$$

15. Length of intercept made on axes by the normal :

$$x - \text{intercept} = x_1 + y_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

$$y - \text{intercept} = y_1 + x_1\left(\frac{dx}{dy}\right)_{(x_1,y_1)}$$

16. Length of perpendicular from origin to normal :

$$= \left|\frac{x_1 + y_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$$

17. Angle of intersection of the two curves :

$$\tan\theta = \pm\frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1 - \left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2}$$

where $\left(\frac{dy}{dx}\right)_1$ is the slope of first curve & $\left(\frac{dy}{dx}\right)_2$ of second. If both curves intersect orthogonally then $\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = -1$

18. Length of tangent, normal, subtangent & subnormal :

$$\text{Length of tangent} = \left|\frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}\right|$$

$$\text{Length of normal} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\text{Length of sub-tangent} = \frac{y}{dy/dx}$$

$$\text{Length of sub-normal} = y\frac{dy}{dx}$$

MONOTONICITY, MAXIMA & MINIMA :

1. A function is said to be monotonic function in a domain if it is either monotonic increasing or monotonic decreasing in that domain

2. At a point function f(x) is monotonic increasing if $f'(a) > 0$

At a point function f(x) is monotonic decreasing if $f'(a) < 0$

3. In an interval [a, b], a function f(x) is

Monotonic increasing if $f'(x) \geq 0$

Monotonic decreasing if $f'(x) \leq 0$

constant if $f'(x) = 0 \ \forall x \in (a, b)$

Strictly increasing if $f'(x) > 0$

Strictly decreasing if $f'(x) < 0$

4. Maximum & Minimum Points :

Maxima : A function f(x) is said to be maximum at x = a, if there exists a very small +ve number h, such that

$f(x) < f(a), \ \forall x \in (a - h, a + h), x \neq a.$

Minima : A function f(x) is said to be minimum at x = b, if there exists a very small +ve number h, such that

$f(x) > f(b), \ \forall x \in (b - h, b + h), x \neq b.$

Remark :

(a) The maximum & minimum points are also known as extreme points.

(b) A function may have more than one maximum & minimum points.

5. Conditions for Maxima & Minima of a function :

(i) **Necessary condition :** A point x = a is an extreme point of a function f(x) if $f'(a) = 0$, provided $f'(a)$ exists.

(ii) **Sufficient condition :**

(a) The value of the function f(x) at x = a is maximum if $f'(a) = 0$ and $f''(a) < 0$.

(b) The value of the function f(x) at x = a is minimum if $f'(a) = 0$ and $f''(a) > 0$.

6. Working rule for finding local maxima & Local Minima :

(i) Find the differential coefficient of f(x) w.r.to x, i.e. $f'(x)$ and equate it to zero.

(ii) Solve the equation $f'(x) = 0$ and let its real roots (critical points) be a, b, c ......

(iii) Now differentiate $f'(x)$ w.r.to x and substitute the critical points in it and get the sign of $f''(x)$ for each critical point.

(iv) If $f''(a) < 0$, then the value of f(x) is maximum at x = 0 and if $f''(a) > 0$, then the value of f(x) is minimum at x = a. Similarly by getting the sign of $f''(x)$ for other critical points (b, c, ...) we can find the points of maxima and minima.

7. Absolute (Greatest and Least) values of a function in a given interval :

(i) A minimum value of a function f(x) in an interval [a, b] is not necessarily its greatest value in that interval. Similarly a minimum value may not be the least value of the function.

(ii) If a function f(x) is defined in an interval [a, b], then greatest or least values of this function occurs either at x = a or x = b or at those values of x for which $f'(x) = 0$.

Thus greatest value of f(x) in interval [a, b]

$= \max [f(a), f(b), f(c), f(d)]$

Least value of f(x) in interval [a, b]

$= \min. [f(a), f(b), f(c), f(d)]$

Where x = c, x = d are those points for which $f'(x) = 0$.

8. Some Geometrical Results :

In Usual Notations	Results
Area of equilateral	$\frac{\sqrt{3}}{4}$ (side)^2.
and its perimeter	3 (side)
Area of square	(side)^2
Perimeter	4(side)
Area of rectangle	$\ell \times b$
Perimeter	$2(\ell \times b)$
Area of trapezium	$\frac{1}{2}$(sum of parallel sides) × (distance between them)
Area of circle	πr^2
Perimeter	$2\pi r$
Volume of sphere	$\frac{4}{3}\pi r^3$
Surface area of sphere	$4\pi r^2$
Volume of cone	$\frac{1}{3}\pi r^2 h$
Surface area of cone	$\pi r\ell$
Volume of cylinder	$\pi r^2 h$
Curved surface area	$2\pi rh$
Total surface area	$2\pi r(h + r)$
Volume of cuboid	$\ell \times b \times h$
Surface area of cuboid	$2(\ell b + bh + h\ell)$
Area of four walls	$2(\ell \times b)\ h$
Volume of cube	ℓ^3
Surface area of cube	$6\ell^2$
Area of four walls of cube	$4\ell^2$

ROLLE'S THEOREM & LAGRANGES THEOREM:

1. **Rolle's Theorem :** If f(x) is such that
   (a) It is continuous on [a, b]
   (b) It is differentiable on (a, b) and
   (c) f(a) = f(b), then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

2. **Mean value theorem [Lagrange's theorem] :**
   (i) If f(x) is such that
      (a) It is continuous on [a, b]
      (b) It is differentiable on (a, b), then
      there exists at least one $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

   (ii) If for c in lagrange's theorem (a < c < b) we can say that $c = a + \theta h$ where $0 < \theta < 1$ and $h = b - a$ the theorem can be written as
   $f(a + h) = f(a) + h f'(a + \theta h),\ 0 < \theta < 1,\ h = b - a$

INDEFINITE INTEGRATION

1. (i) If $\frac{d}{dx}F(x) = f(x)$, then $\int f(x)dx = F(x) + c$

Here $\int\{\ \}dx$ is the notation of integration, f(x) is the integrand, c is any real no. (integrating constant)

(ii) $\frac{d}{dx}\int f(x)dx = f(x)$

(iii) $\int f'(x)dx = f(x) + c,\ c \in R$

(iv) $\int k\ f(x)dx = k\int f(x)\ dx$

(v) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

2. FUNDAMENTAL FORMULAE :

Function	Integration
$\int x^n dx$	$\frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\int (ax+b)^n dx$	$\frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1} + c,\ n \neq -1$
$\int \frac{1}{x}dx$	$\log\lvert x\rvert + c$
$\int \frac{1}{ax+b}dx$	$\frac{1}{a}(\log\lvert ax + b\rvert) + c$
$\int e^x dx$	$e^x + c$
$\int a^x dx$	$\frac{a^x}{\log_e a} + c$
$\int \sin x\ dx$	$-\cos x + c$

Function	Integration
$\int \cos x\ dx$	$\sin x + c$
$\int \sec^2 x\ dx$	$\tan x + c$
$\int \operatorname{cosec}^2 x\ dx$	$-\cot x + c$
$\int \sec x \tan x\ dx$	$\sec x + c$
$\int \operatorname{cosec} x \cot x\, dx$	$-\operatorname{cosec} x + c$
$\int \tan x\ dx$	$-\log\lvert\cos x\rvert + c = \log\lvert\sec x\rvert + c$
$\int \cot x\ dx$	$\log\lvert\sin x\rvert + c = -\log\lvert\operatorname{cosec} x\rvert + c$
$\int \sec x\ dx$	$\log\lvert\sec x + \tan x\rvert + c = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$
$\int \operatorname{cosec} x\ dx$	$\log\lvert\operatorname{cosec} x - \cot x\rvert + c = \log\tan\frac{x}{2} + c$
$\int \frac{dx}{\sqrt{1-x^2}}$	$\sin^{-1} x + c = -\cos^{-1} x + c$
$\int \frac{dx}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$
$\int \frac{dx}{1+x^2}$	$\tan^{-1} x + c = -\cot^{-1} x + c$
$\int \frac{dx}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c = \frac{-1}{a}\cot^{-1}\frac{x}{a} + c$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1} x + c = -\operatorname{cosec}^{-1} x + c$
$\int \frac{dx}{\lvert x\rvert\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a} + c = \frac{-1}{a}\operatorname{cosec}^{-1}\frac{x}{a} + c$

3. INTEGRATION BY SUBSTITUTION :

By suitable substitution, the variable x in $\int f(x)dx$ is changed into another variable t so that the integrand f(x) is changed into F(t) which is some standard integral. Some following suggestions will prove useful.

Function	Substitution	Integration
$\int f(ax+b)dx$	$ax + b = t$	$\frac{1}{a}F(ax + b) + c$
$\int f(x)\, f'(x)dx$	$f(x) = t$	$\frac{(f(x))^2}{2} + c$
$\int f(\phi(x))\, \phi(x)dx$	$\phi(x) = t$	$\int f(t)dt$
$\int \frac{f'(x)}{f(x)}dx$	$f(x) = t$	$\log\lvert f(x)\rvert + c$
$\int (f(x))^n f'(x)dx$	$f(x) = t$	$\frac{(f(x))^{n+1}}{n+1} + c,\ n \neq -1$
$\int \frac{f'(x)}{\sqrt{f(x)}}dx$	$f(x) = t$	$2[f(x)]^{1/2} + c$

SOME RECOMMENDED SUBSTITUTION :

Function	Substitution
$\sqrt{a^2-x^2},\ \frac{1}{\sqrt{a^2-x^2}},\ a^2 - x^2$	$x = a \sin\theta$ or $a \cos\theta$
$\sqrt{x^2+a^2},\ \frac{1}{\sqrt{x^2+a^2}},\ x^2 + a^2$	$x = a \tan\theta$ or $x = a \sinh\theta$
$\sqrt{x^2-a^2},\ \frac{1}{\sqrt{x^2-a^2}},\ x^2 - a^2$	$x = a \sec\theta$ or $x = a \cosh\theta$
$\sqrt{\frac{x}{a+x}},\ \sqrt{\frac{a+x}{x}},$	
$\sqrt{x(a+x)},\ \frac{1}{\sqrt{x(a+x)}}$	$x = a \tan^2\theta$
$\sqrt{\frac{x}{a-x}},\ \sqrt{\frac{a-x}{x}},$	
$\sqrt{x(a-x)},\ \frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2\theta$
$\sqrt{\frac{x}{x-a}},\ \sqrt{\frac{x-a}{x}},$	
$\sqrt{x(x-a)},\ \frac{1}{\sqrt{x(a-x)}}$	$x = a \sec^2\theta$
$\sqrt{\frac{a-x}{a+x}},\ \sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}},\ \sqrt{(x-\alpha)(\beta-x)},\ (\beta > \alpha)$	$x = \alpha \cos^2\theta + \beta \sin^2\theta$

IMPORTANT RESULTS USING STANDARD SUBSTITUTIONS :

Function	Integration
$\int \frac{1}{x^2 - a^2}$	$\frac{1}{2a}\log\left\|\frac{x-a}{x+a}\right\| + c$
	$= \frac{-1}{a}\coth^{-1}\frac{x}{a} + c$ when $x > a$
$\int \frac{1}{a^2 - x^2}dx$	$\frac{1}{2a}\log\left\|\frac{a+x}{a-x}\right\| + c$
	$= \frac{1}{a}\tanh^{-1}\frac{x}{a} + c$, when $x < a$
$\int \frac{dx}{\sqrt{x^2 - a^2}}$	$\log\{\|x + \sqrt{x^2 - a^2}\|\} + c$
	$= \cosh^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{dx}{\sqrt{x^2 + a^2}}$	$\log\{\|x + \sqrt{x^2 + a^2}\|\} + c$
	$= \sinh^{-1}\left(\frac{x}{a}\right) + c$
$\int \sqrt{a^2 - x^2}\,dx$	$\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + c$
$\int \sqrt{x^2 - a^2}\,dx$	$\frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \log\{\|x + \sqrt{x^2 - a^2}\|\} + c$
$\int \sqrt{x^2 + a^2}\,dx$	$\frac{1}{2}x\sqrt{x^2 + a^2} + \frac{1}{2}a^2 \log\{\|x + \sqrt{x^2 + a^2}\|\} + c$

INTEGRATION OF FUNCTIONS USING ABOVE STANDARD RESULTS :

Function	Method
$\int \frac{1}{ax^2 + bx + c}dx$ or	**Express :** $ax^2 + bx + c =$
$\int \frac{1}{\sqrt{ax^2 + bx + c}}dx$ or	$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$
$\int \sqrt{(ax^2 + bx + c)}\,dx$	then use appropriate formula
$\int \frac{px + q}{ax^2 + bx + c}dx$ or	**Express :** $px + q$
$\int \frac{px + q}{\sqrt{ax^2 + bx + c}}dx$ or	$= \lambda\frac{d}{dx}(ax^2 + bx + c) + \mu$
$\int (px + q)\sqrt{(ax^2 + bx + c)}\,dx$	evaluate λ & μ by equating coefficient of x and constant, the integral reduces to known form
$\int \frac{P(x)}{ax^2 + bx + c}dx$, where P(x) is a polynomial of degree 2 or more	Apply division rule and express it in form $Q(x) + \frac{R(x)}{ax^2 + bx + c}$ The integral reduces to known form

$\int \frac{1}{a\sin^2 x + b\cos^2 x + c}\,dx$ or $\int \frac{1}{(a\sin x + b\cos x)^2}\,dx$	Divide numerator & denominator by $\cos^2 x$, then put $\tan x = t$ & solve.
$\int \frac{dx}{a\sin x + b\cos x + c}\,dx$	Replace $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ then put $\tan x/2 = t$ and replace $1 + \tan^2 x/2 = \sec^2 x/2$
$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$	**Express :** num. = λ(deno.) + $\mu \frac{d}{dx}$(deno.) Evaluate λ & μ. Thus integral reduces to known form.
$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$	**Express :** Num. = λ(deno.) + $\mu \frac{d}{dx}$(deno.) + ν Evaluate λ, μ, ν. Thus integral reduces to known form.
$\int \frac{x^2 \pm a^2}{x^4 + kx^2 + a^4}\,dx$	Divide numerator & denominator by x^2 and put $\left(x \pm \frac{a^2}{x}\right) = t$, the integral becomes one of standard forms.

$\int \frac{x^2}{x^4 + kx^2 + a^4}\,dx$	Divide numerator & denominator by 2 and then add & sub. a^2. Thus the form reduces as above.
$\int \frac{dx}{x^4 + kx^2 + a^2}$	Divide num & deno. by $2a^2$ and then add & sub x^2. Thus the form reduces to the known form.

4. INTEGRATION BY PARTS :

when integrand involves more than one type of functions the formula of integration by parts is used to integrate the product of the functions i.e.

(i) $\int u.\upsilon\,dx = u.\int \upsilon\,dx - \int\left[\frac{du}{dx}\left(\int \upsilon\,dx\right)\right]dx$

or $\int (\text{1st fun.}) . (\text{2nd fun.})\,dx$

$= (\text{1st fun}) \int \text{2nd fun.}dx - \int\left[\left(\frac{d}{dx}\text{1st fun.}\right)\left(\int \text{2nd fun.}dx\right)\right]dx$

(ii) **Rule to choose the first function :** first fun. should be choosen in the following order of preference (ILATE). [The fun. on the left is normally chosen as first function]

I – Inverse trigonometric function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function

(iii) (a) $\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + c$

(b) $\int e^{mx}[mf(x)+f'(x)]dx = e^{mx} f(x) + c$

(c) $\int e^{mx}\left[f(x)+\frac{f'(x)}{m}\right]dx = \frac{e^{mx}f(x)}{m} + c.$

(iv) $\int [xf'(x)+f(x)]dx = x\ f(x) + c.$

NOTE : Breaking (iii) & (iv) integral into two integrals. Integrate one integral by parts and keeping other integral as it is by doing so we get the result (integral).

(v) $\int e^{ax}\sin bx\,dx$ and $\int e^{ax}\sin(bx+c)dx$

$= \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + k$ and

$\frac{e^{ax}}{a^2+b^2}[a \sin (bx + c) - b \cos(bx + c)] + k_1$

(vi) $\int e^{ax}\cos bx\,dx$ and $\int e^{ax}\cos(bx+c)\,dx$

$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + k$

and $\frac{e^{ax}}{a^2+b^2}[a\cos(bx+c)+b\sin(bx+c)] + k_1.$

5. INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS USING PARTIAL FRACTION :

Every Rational fun. may be represented in the form $\frac{P(x)}{Q(x)}$, where P(x), Q(x) are polynomials.

If degree of numerator is less than that of denominator, the rational fun. is said to be proper other wise it is improper. If deg (num.) ≥ deg(deno.) apply division rule

i.e. $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, for integrating $\frac{r(x)}{g(x)}$, resolve the fraction into partial factors. The following table illustrate the method.

Types of proper rational functions	Types of partial fractions
$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$, a, b, c are distinct	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where $x^2 + bx + c$ can not be factorised	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
$\frac{px^3+qx^2+rx+s}{(x^2+ax+b)(x^2+cx+d)}$, where $x^2 + ax + b$, $x^2 + cx + d$ can not be factorised	$\frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{x^2+cx+d}$

6. INTEGRATION OF IRRATIONAL ALGEBRAIC FUNCTIONS :

(i) If integrand is a function of x & $(ax + b)^{1/n}$ then put $(ax + b) = t^n$

(ii) If integrand is a function of x, $(ax + b)^{1/n}$ and $(ax + b)^{1/m}$ then put $(ax + b) = t^p$ where p = (L.C.M. of m & n).

(iii) To evaluate $\int \frac{dx}{\text{linear}\sqrt{\text{linear}}}$ put $\sqrt{\text{linear}} = t$

(iv) To evaluate $\int \frac{dx}{\text{quad.}\sqrt{\text{linear}}}$ put linear $= t^2$

(v) To evaluate $\int \frac{dx}{\text{linear.}\sqrt{\text{quadratic}}}$ put linear = 1/t

or $\int \frac{dx}{(\text{linear})^2.\sqrt{\text{quadratic}}}$

or $\int \frac{x\,dx}{(\text{linear})^2.\sqrt{\text{quadratic}}}$

(vi) To evaluate $\int \frac{dx}{\text{pure quad.}\sqrt{\text{pure quad}}}$ put $\sqrt{\text{pure quad}} = t$

(vii) To evaluate $\int \frac{dx}{\text{pure quad.}\sqrt{\text{pure quad}}}$ put $x = \frac{1}{t}$ and then is the resulting integral, put $\sqrt{\text{pure quad}} = u$

(viii) To evaluate $\int \frac{dx}{\text{quad.}\sqrt{\text{quad}}}$ or $\int \frac{\text{linear}}{\text{quad.}\sqrt{\text{quad}}}\,dx$

and if the quadratic not under the square root can be resolved into real linear factors, then resolve $\frac{1}{\text{quadratic}}$ or $\left(\frac{\text{linear}}{\text{quadratic}}\right)$ into partial fractions and split the integral into two, each of which is of the form : $\int \frac{dx}{\text{linear}\sqrt{\text{quad.}}}$

7. INTEGRATION USING TRIGONOMETRICAL IDENTITIES :

(A) To evaluate trigonometric functions transform the function into standard integrals using trigonometric identities as

(i) $\sin^2 mx = \frac{1 - \cos 2mx}{2}$

(ii) $\cos^2 mx = \frac{1 + \cos 2mx}{2}$

(iii) $\sin mx = 2\sin\frac{mx}{2}\cos\frac{mx}{2}$

(iv) $\sin^3 mx = \frac{3\sin mx - \sin 3mx}{4}$

(v) $\cos^3 mx = \frac{3\cos mx + \cos 3mx}{4}$

(vi) $\tan^2 mx = \sec^2 mx - 1$

(vii) $\cot^2 mx = \operatorname{cosec}^2 mx - 1$

(viii) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(ix) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

(x) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

(B) $\int \sin^m x \cos^m x dx$.

(i) if m is odd put cos x = t
(ii) if m is even put sin x = t
(iii) if m & n both odd put sin x or cos x as t
(iv) if m & n both even use the formula of $\sin^2 x$ & $\cos^2 x$
(v) if m & n rational no. & $\frac{m+n-2}{2}$ is –ve integer put tan x = t

8. INTEGRATION BY SUCCESSIVE REDUCTION (REDUCTION FORMULA) :

Function	Integration
$\int x^n e^{ax} dx$, $n \in N$	$\frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$ where $I_{n-1} = \int x^{n-1} e^{ax} dx$
$\int x^n \sin x\, dx$	$-x^n \cos x + nx^{n-1} \sin x - n(n-1) I_{n-2}$
$\int \sin^n x\, dx$	$-\frac{\sin^{n-1} \cos x}{n} + \frac{n-1}{n} I_{n-2}$
$\int \cos^n x\, dx$	$\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$
$\int \tan^n x\, dx$	$\frac{(\tan x)^{n-1}}{n-1} - I_{n-2}$
$\int \cot^n x\, dx$	$-\frac{(\cot x)^{n-1}}{n-1} - I_{n-2}$

$\int \sec^n x\, dx$	$\frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$
$\int \operatorname{cosec}^n x\, dx$	$-\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$
$\int \sin^m x \cos^n x\, dx$	$\frac{\cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,\, n-2}}{(m+n)}$ $-\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n}$

NOTE : These formulae are specifically useful when m & n are both even nos.

DEFINITE INTEGRATION

1. Definite Integration :

If $\int f(x)\,dx = F(x) + c$, then

$\int_a^b f(x)\,dx = F(x) + c\Big|_a^b = F(b) - F(a)$ is called definite integral of f(x) w.r.t. x from x = a to x = b Here a is called lower limit and b is called upper limit.

Remarks :

* To evaluate definite integral of f(x). First obtain the indefinite integral of f(x) and then apply the upper and lower limit.
* For integration by parts in definite integral we use following rule.

$$\int_a^b uv\,dx = \left\{u\int v.dx\right\}_a^b - \int_a^b \left(\frac{du}{dx}.\int v.dx\right)dx$$

* When we use method of substitution. We note that while changing the independent variable in a definite integral, the limits of integration must also we changed accordingly.

PROPERTIES OF DEFINITE INTEGRAL :

I. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

II. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

III. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where a < c < b

This property is mainly used for modulus function, greatest integer function & breakable function

IV. $\int_b^a f(x)\,dx = \int_b^a f(a+b-x)\,dx$ or $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

V. $\int_{-a}^a f(x)\,dx = \int_0^a [f(x)+f(-x)]\,dx$

$$= \begin{cases} 2\int_0^a f(x)\,dx & \text{,if } f(x) \text{ is an even function} \\ 0 & \text{,if } f(x) \text{is an odd function} \end{cases}$$

VI. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{,if } f(2a-x) = f(x) \\ 0 & \text{,if } f(2a-x) = -f(x) \end{cases}$

VII. If f(x) is a periodic function with period T, Then

$$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$$

and further if $a \in R^+$, then

$$\int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)\,dx, \quad \int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^T f(x)\,dx,$$

$$\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^b f(x)\,dx$$

VIII. If m and M are the smallest and greatest values of a function f(x) on an interval [a, b], then

$$m(b-a) < \int_a^b f(x)\,dx < M(b-a)$$

IX. $\left|\int_a^b f(x)\,dx\right| < \int_a^b |f(x)\,dx|$

X. If $f(x) < g(x)$ on [a, b], then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

2. Differentiation Under Integral Sign :

Leibnitz's Rule :

(i) If f(x) is continuous and u(x), v(x) are differentiable functions in the interval [a, b], then, $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt =$

$$f\{v(x)\}\frac{d}{dx}\{v(x)\} - f\{u(x)\}\frac{d}{dx}\{u(x)\}.$$

(ii) If the function $\phi(x)$ and $\psi(x)$ are defined on [a, b] and differentiable at a point $x \in (a, b)$, and f(x, t) is continuous, then,

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\,dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x,t)\,dt + \left\{\frac{d\psi(x)}{dx}\right\} f(x, \psi(x)) - \left\{\frac{d\phi(x)}{dx}\right\} f(x, \phi(x)).$$

3. Reduction Formulae :

(i) $\int_a^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx$

$$= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{2}{3}\cdot 1, & \text{if n is odd} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if n is even} \end{cases}$$

(ii) For integration $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ follow the following steps

(a) If m is odd put cos x = t

(b) If n is odd put sin x = t

(c) If m and n are even use $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$ and then use

$$\int_0^{\pi/2} \sin^n x\,dx \quad \text{or} \quad \int_0^{\pi/2} \cos^n x\,dx$$

(iii) $\int_0^{\infty} e^{-ax} \cos bx \ dx = \frac{a}{a^2+b^2}$

(iv) $\int_0^{\infty} e^{-ax} \sin bx \ dx = \frac{b}{a^2+b^2}$

(v) $\int_0^{\infty} e^{-ax} x^n dx = \frac{n!}{a^n+1}$

(vi) $\int_0^{\pi/2} \sin^n x \cos^m x \ dx$

$$= \begin{bmatrix} \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{2}{3+n}\cdot\frac{1}{1+n} & ;\ \text{if m is odd and n may be even or odd} \\ \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3} & ;\ \text{if m is enen and n is odd} \\ \frac{m-1}{m+n}\cdot\frac{m-3}{m+n-2}\cdots\frac{1}{2+n}\cdot\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & ;\ \text{if m is even and n is even} \end{bmatrix}$$

These formulae can be expressed as a single formula :

$$\int_0^{\pi/2} \sin^m x \cos^n x \ dx$$

$$= \frac{[(m-1)\ (m-3)\ldots.]\ [(n-1)\ (n-3)\ \ldots..]}{(m-n)\ (m+n-2)\ \ldots.}$$

to be multiplied by $\frac{\pi}{2}$ when m and n are both even integers.

4. Summation of series by Definite integral or limit as a sum :

(i) $\int_a^b f(x)\ dx = \lim_{h\to 0} h[f(a) + f(a + h) + f(a + 2h) + \ldots.. + f(a + (n - 1)h]$

where nh = b – a.

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\ dx$

[i.e. exp. the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$ replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and the limit of the sum is $\int_0^1 f(x)\ dx$]

5. Key Results :

* $\int_0^{\pi/2} \log \sin x\ dx = \int_0^{\pi/2} \log \cos x\ dx = \frac{-\pi}{2}\log 2$

* $\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)} dx = \int_0^{\pi/2} \frac{f(\cos x)}{f(\sin x)+f(\cos x)} dx$

$$= \int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x)+f(\cot x)}dx = \int_0^{\pi/2} \frac{f(\sec x)}{f(\sec x)+f(\operatorname{cosec} x)}dx$$

$$= \int \frac{f(\operatorname{cosec} x)}{f(\operatorname{cosec} x)+f(\sec x)}dx = \int_0^{\pi/2} \frac{f(\cot x)}{f(\tan x)+f(\cot x)}dx = \pi/4.$$

* $\int_0^{\pi/2} \sin mx \sin nx \, dx = \int_0^{\pi/2} \cos mx \,.\, \cos nx \, dx$

$$= \begin{cases} 0 & \text{if } m, n \text{ are different } + \text{ve integers} \\ \frac{\pi}{2} & \text{if } m = n \end{cases}$$

* $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{\pi}{4}a^2$

* $\int_0^a \frac{1}{\sqrt{a^2 - x^2}}dx = \frac{\pi}{2}$

* $\int_0^a \frac{x}{\sqrt{a^2 - x^2}}dx = a$

* $\int_0^a \frac{x^2}{\sqrt{a^2 - x^2}}dx = \frac{\pi a^2}{6} + \frac{\sqrt{3}a^2}{8}$

* $\int_0^a \frac{x\,dx}{(a^2+x^2)^{3/2}}dx = \frac{1}{\sqrt{2}a^2}.$

* $\int_0^a x^2\sqrt{a^2 - x^2}\,dx = \frac{\pi a^4}{16}$

* $\int_0^a x^2\sqrt{\frac{a^2 - x^2}{a+x}}\,dx = a^3\left(\frac{\pi}{4} - \frac{2}{3}\right)$ if $a > 0$

* $\int_0^{2a} \sqrt{2ax - x^2}\,dx = \frac{\pi a^2}{2}$

* If $n \in N$, then $\int_0^a (a^2 - x^2)^n dx = \frac{2.4.6......(2n)}{3.5.7.....(2n+1)}a^{2n+1}$

* If $a < b$ then

(i) $\int_a^b \frac{dx}{\sqrt{x-a}\,\sqrt{b-x}} = \pi$

(ii) $\int_0^a \sqrt{\frac{x-a}{a+x}}\,dx = \frac{\pi(b-a)}{2}$

(iii) $\int_0^a \sqrt{(x-a)(b-x)}\,dx = \frac{\pi}{2}(b-a)^2$

(iv) $\int_a^b \frac{dx}{x\sqrt{(x-a)(b-x)}} = \frac{\pi}{\sqrt{ab}}^{\,2}$ ab > 0

* If a > 0 then

(i) $\int_0^a \frac{\sqrt{a+x}}{a-x}\,dx = \frac{a}{2}(\pi+2)$

(ii) $\int_0^a \frac{\sqrt{a-x}}{a+x}\,dx = \frac{a}{2}(\pi-2)$

(iii) $\int_0^a \frac{a+x}{\sqrt{a-x}}\,dx = \frac{10a\sqrt{a}}{3}$

(iv) $\int_0^a \sqrt{\frac{a+x}{a-x}}\,dx = \left(\frac{\pi}{2}+1\right)a$

* If a > 0, n ∈ N, then

(i) $\int_0^\infty \sqrt{x}\,e^{-ax}\,dx = \frac{\sqrt{\pi}}{2a\sqrt{a}}$ a > 0

(ii) $\int_0^\infty e^{-r^2x^2}dx = \frac{\sqrt{\pi}}{2r}$ (r > 0)

(iii) $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \log_e(b/a)$ (a, b > 0)

* If f(x) is continuous on [a, b] then there exists a point c ∈ (a, b) s.t $\int_a^b f(x)dx$ = f(c) [b − a]. The no. $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ is called the mean value of the fun. f(x) on the interval [a, b]. The above result is called the first mean value theorem for integrals.

* $\int_0^{2k}(x-[x])dx = k$, where k ∈ I,

∵ x − [x] is a periodic function with period 1.

* If f(x) is a periodic fun. with period T, then

$\int_a^{a+T} f(x)\,dx$ is independent of a.

* $\int_0^{\pi/4}\log(1+\tan x)dx = \frac{\pi}{8}\log 2$

DIFFERENTIAL EQUATIONS

1. Order of a differential equation : The order of a differential equation is the order of the highest derivative occurring in it.

2. Degree of a differential equation : The degree of a differential equation is the degree of the highest order derivative occurring in it when the derivatives are made free from the radical sign.

Eg. (i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 5y = 0$

(ii) $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(iii) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(1+\frac{dy}{dx}\right)^2 + 5y = 0$

order of (i) 2 (ii) 1 & (iii) 3,
degree of (i) 1 (ii) 2 & (iii) 2

3. SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE :

(A) Differential equation of the form $\frac{dy}{dx} = f(x)$ or

$\frac{dy}{dx} = f(y)$

Integrate both sides i.e. $\int dy = \int f(x)\, dx$

or $\int \frac{dy}{f(y)} = \int dx$ to get its solution.

(B) **Variable Separable Form :** Differential equation of the form $\frac{dy}{dx} = f(x)\, g(y)$

This can be integrated as

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + c$$

(C) **Homogeneous Equations :** It is a differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, where f(x, y) and g(x, y) are homogeneous functions of x and y of the same degree. A function f(x, y) is said to be homogeneous of degree n if it can be written as $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$.

Such an equation can be solved by putting y = vx or x = vy. After substituting y = vx or x = vy. The given equation will have variables separable in v and x.

(D) **Equations Reducible to Homogeneous form and variable separable form**

* Form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$ .... (1)

where $\frac{a}{A} \neq \frac{b}{B}$

This is non Homogeneous

Put x = X + h and y = Y + k in (1)

$\therefore \frac{dy}{dx} = \frac{dY}{dX}$ Put ah + bk + c = 0, Ah + Bk + C = 0, find h, k

Then $\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$. This is homogeneous.

Solve it and then put X = x − h, Y = y − k we shall get the solution.

* Form $\frac{dy}{dx} = \frac{ax + by + c}{Ax + By + C}$ ..... (1),

where $\frac{a}{A} = \frac{b}{B} = k$ say

$\therefore \frac{dy}{dx} = \frac{k(Ax + By) + c}{Ax + By + C}$

Put $Ax + By = z \Rightarrow A + B\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dz}{dx} = A + B\frac{kz + c}{z + c}$

This is variable separable form and can be solved.

* Form $\frac{dy}{dx} = f(ax + by + c)$

Put $ax + by = z \Rightarrow a + b\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \frac{dz}{dx} = a + b\, f(z)$

This is variable separable form and can be solved.

(E) **Linear equation :**

* **In y :** $\frac{dy}{dx} + Py = Q$, where P, Q are function of x alone or constant.

its solution $ye^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

where $e^{\int P\,dx}$ is called the integrating factor (I.F.) of the equation.

* **In x :** $\frac{dx}{dy} + Rx = S$, where R, S are functions of y alone or constant.

its solution $x e^{\int R\,dy} = \int S.e^{\int R.dy}\, dy + c$

where $e^{\int R.dy}$ is called the integrating factor (I.F.) of the equation.

(F) **Equation reducible to linear form :**

* Differential equation of the form $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x or constant is called Bernoulli's equation. On dividing through out by y^n, we get

$y^{-n}\frac{dy}{dx} + py^{-n+1} = Q$

Put $y^{-n+1} = z$

$\Rightarrow$ The given equation will be linear in z and can be solved in the usual manner.

Note : In general solution of differential equation we can take integrating constant c as $\tan^{-1} c$, e^c, log c etc. according to our convenience.

VECTORS

1. Types of vectors :

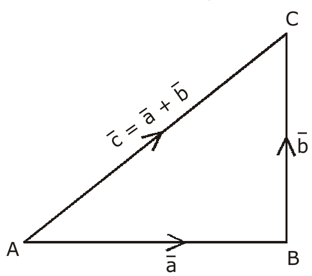
(a) Zero or null vector : A vector whose magnitude is zero is called zero or null vector.

(b) Unit vector : $\hat{a} = \dfrac{\vec{a}}{|a|} = \dfrac{\text{Vector a}}{\text{Magnitude of a}}$

(c) Equal vector : Two vectors $\bar{a}$ and $\bar{b}$ are said to be equal if |a| = |b| and they have the same direction.

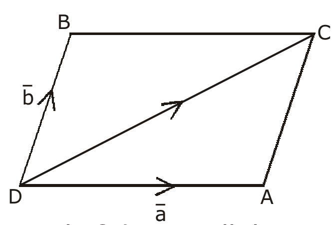
2. Triangle law of addition : $\overline{AB} + \overline{BC} = \overline{AC}$

$$\bar{c} = \bar{a} + \bar{b}$$



3. Parallelogram law of addition : $\overline{OA} + \overline{OB} = \overline{OC}$

$$\bar{a} + \bar{b} = \bar{c}$$



where OC is a diagonal of the parallelogram OABC

4. Vectors in terms of position vectors of end points -

$\overline{AB} = \overline{OB} - \overline{OA}$ = Position vector of B – position vector of A

i.e. any vector = p.v. of terminal pt – p.v. of initial pt.

5. Multiplication of a vector by a scalar :

If $\vec{a}$ is a vector and m is a scalar, then $m\vec{a}$ is a vector and magnitude of $m\vec{a} = m|a|$

and if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

then $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$

6. Distance between two points :

Distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

= Magnitude of $\vec{AB}$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

7. Position vector of a dividing point :

(i) If $A(\bar{a})$ & $B(\vec{b})$ be two distinct pts, the p.v. $\bar{c}$ of the point C dividing [AB] in ratio $m_1 : m_2$ is given by

$$\vec{c} = \frac{m_1\vec{b} + m_2\vec{a}}{m_1 + m_2}$$

(ii) p.v. of the mid point of [AB] is $\frac{1}{2}$ [p.v. of A + p.v. of B]

(iii) If point C divides AB in the ratio $m_1 : m_2$ externally,

then p.v. of C is $\bar{c} = \dfrac{m_1\bar{b} - m_2\bar{a}}{m_1 - m_2}$

(iv) p.v. of centriod of triangle formed by the points A($\bar{a}$), B($\vec{b}$) and C ($\vec{c}$) is $\frac{\bar{a}+\bar{b}+\bar{c}}{3}$

(v) p.v. of the incentre of the triangle formed by the points A($\vec{\alpha}$), B($\vec{\beta}$) and C($\vec{\gamma}$) is

$\frac{a\bar{\alpha}+b\bar{\beta}+c\bar{\gamma}}{a+b+c}$ where a = |BC|, b = |CA|, c = |AB|

8. Some results :

(i) If D, E, F are the mid points of sides BC, CA & AB respectively, then $\overline{AD} + \overline{BE} + \overline{CF} = \bar{0}$

(ii) If G is the centriod of ΔABC, then $\overline{GA} + \overline{GB} + \overline{GC} = \bar{0}$

(iii) If O is the circumcentre of a ΔABC, then

$\overline{OA} + \overline{OB} + \overline{OC} = 3\,\overline{OG} = \overline{OH}$ where G is centriod and H is orthocentre of ΔABC.

(iv) If H is orthocentre of ΔABC, then

$\overline{HA} + \overline{HB} + \overline{HC} = 3\overline{HG} = \overline{OH}$

9. Collinearity of three points :

(i) Three points A, B and C are collinear if $\overline{AB} = \lambda\overline{AC}$ for some non zero scalar λ.

(ii) The necessary and sufficient condition for three points with p.v. $\bar{a}$, $\bar{b}$, $\bar{c}$ to be collinear is that there exist three scalars l, m, n all non zero such that

$l\bar{a} + m\bar{b} + n\bar{c} = \bar{0}$, $l + m + n = 0$

10. Coplanar and non coplanar vector :

(i) If $\bar{a}$, $\bar{b}$, $\bar{c}$ be three non coplanar non zero vector then $x\bar{a} + y\bar{b} + z\bar{c} = 0$

$\Rightarrow$ x = 0, y = 0, z = 0

(ii) If $\bar{a}$, $\bar{b}$, $\bar{c}$ be three coplanar vectors, then a vector $\bar{c}$ can be expressed uniquely as linear combination of remaining two vectors i.e. $\bar{c} = \lambda\bar{a} + \mu\bar{b}$

(iii) Any vector $\bar{r}$ can be expressed uniquely as inner combination of three non coplanar & non zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ i.e. $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$

11. Products of vectors :

(I) Scalar or dot product of two vectors :

(i) $\bar{a} \cdot \bar{b} = |a|\,|b| \cos\theta$

(ii) Projection of $\bar{a}$ in the direction of $\bar{b} = \frac{\bar{a}.\bar{b}}{|\bar{b}|}$

& Projection of $\bar{b}$ in the direction of $\bar{a} = \frac{\bar{a}.\bar{b}}{|\bar{a}|}$

(iii) Component of $\bar{r}$ on $\bar{a} = \left(\frac{\bar{r}.\bar{a}}{|\bar{a}|^2}\right)\bar{a}$

Component of $\bar{r} \perp$ to $\bar{a} = \bar{r} - \left(\frac{\bar{r}.\bar{a}}{|\bar{a}|^2}\right)\bar{a}$

(iv) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(v) $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(vi) If $\bar{a}$ and $\bar{b}$ are like vectors, then $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|$ and

If $\bar{a}$ and $\bar{b}$ are unlike vectors, then $\bar{a}.\bar{b} = -|\bar{a}||\bar{b}|$

(vii) $\bar{a}, \bar{b}$ are $\perp \Leftrightarrow \bar{a}.\bar{b} = 0$

(viii) $(\bar{a}.\bar{b}).\bar{b}$ is not defined

(ix) $(\bar{a} \pm \bar{b})^2 = a^2 \pm 2\bar{a}.\bar{b} + b^2$

(x) $|\bar{a} + \bar{b}| = |\bar{a}| + |\bar{b}| \quad \Rightarrow \bar{a} \parallel \bar{b}$

(xi) $|\bar{a} + \bar{b}|^2 = |a|^2 + |b|^2 \quad \Rightarrow \bar{a} \perp \bar{b}$

(xii) $|\bar{a} + \bar{b}| = |\bar{a} - \bar{b}| \quad \Rightarrow \bar{a} \perp \bar{b}$

(xiii) work done by the force :

work done = $\bar{F}.\bar{d}$, where $\bar{F}$ is force vector and $\bar{d}$ is displacement vector.

(II) Vector or cross product of two vectors :

(i) $\bar{a} \times \bar{b} = |a|\,|b| \sin\theta\, \hat{n}$

(ii) if $\bar{a}, \bar{b}$ are parallel $\Leftrightarrow \bar{a} \times \bar{b} = \bar{0}$

(iii) $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

(iv) $\hat{n} = \pm \dfrac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$

(v) let $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(vi) $\bar{a} \times \bar{a} = 0$

(vii) $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$

(viii) $\bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$

(ix) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \bar{0}$, $\hat{i} \times \hat{j} = \hat{k}$,

$\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

(x) Area of triangle :

(a) $\dfrac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

(b) If $\bar{a}, \bar{b}, \bar{c}$ are p.v. of vertices of ΔABC,

then = $\dfrac{1}{2}\,|(\bar{a} \times \bar{b}) + (\bar{b} \times \bar{c}) + (\bar{c} \times \bar{a})|$

(xi) Area of parallelogram :

(a) If $\bar{a}$ & $\bar{b}$ are two adjacent sides of a parallelogram, then area = $|\bar{a} \times \bar{b}|$

(b) If $\bar{a}$ and $\bar{b}$ are two diagonals of a parallelogram, then area = $\dfrac{1}{2}\,|\bar{a} \times \bar{b}|$

(xii) Moment of Force :
Moment of the force F acting at a point A about O is
Moment of force = $\overline{OA} \times \bar{F} = \bar{r} \times \bar{F}$

(xiii) Lagrange's identity : $|\bar{a} \times \bar{b}|^2 = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} \\ \bar{a}.\bar{b} & \bar{b}.\bar{b} \end{vmatrix}$

(III) Scalar triple product :

(i) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$(\vec{a} \times \vec{b}).\vec{c} = [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and $[\vec{a}\,\vec{b}\,\vec{c}]$ = volume of the parallelopiped whose coterminus edges are formed by $\vec{a}, \vec{b}, \vec{c}$

(ii) $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$,

but $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{b}\,\vec{a}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$ etc.

(iii) $[\vec{a}\,\vec{b}\,\vec{c}] = 0$ if any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear or equal.

(iv) $(\vec{a} \times \vec{b}).\vec{c} = \vec{a}.(\vec{b} \times \vec{c})$ etc.

(v) $[\hat{i}\,\hat{j}\,\hat{k}] = 1$

(vi) If λ is a scalar, then $[\lambda\vec{a}\,\vec{b}\,\vec{c}] = \lambda[\vec{a}\,\vec{b}\,\vec{c}]$

(vii) $[\vec{a} + \vec{d}\;\;\vec{b}\;\;\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{d}\,\vec{b}\,\vec{c}]$

(viii) $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$

(ix) Volume of tetrahedron ABCD is $\frac{1}{6}|\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD}|$

(x) Four points with p.v. $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ will be coplanar if

$[\vec{d}\,\vec{b}\,\vec{c}] + [\vec{d}\,\vec{c}\,\vec{a}] + [\vec{d}\,\vec{a}\,\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]$

(xi) Four points A, B, C, D are coplanar if

$[\overrightarrow{AB}\;\;\overrightarrow{AC}\;\;\overrightarrow{AD}] = 0$

(xii) (a) $[\vec{a} + \vec{b}\;\;\vec{b} + \vec{c}\;\;\vec{c} + \vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

(b) $[\vec{a} - \vec{b}\;\;\vec{b} - \vec{c}\;\;\vec{c} - \vec{a}] = 0$

(c) $[\vec{a} \times \vec{b}\;\;\vec{b} \times \vec{c}\;\;\vec{c} \times \vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

(d) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then so are $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$ and $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{a}$ and $\vec{a} - \vec{b}$, $\vec{b} - \vec{c}$, $\vec{c} - \vec{a}$ are also coplanar.

(IV) Vector triple Product :

If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are known as vector triple product and is defined as

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

and $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Clearly in general $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ but $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ if and only if $\vec{a}, \vec{b}$ & $\vec{c}$ are collinear

12. Application of Vector in Geometry :

(i) Direction cosines of $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ are $\frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$.

(ii) Incentre formula : The position vector of the incentre of Δ ABC is $\frac{a\vec{a} + b\vec{b} + c\vec{c}}{a + b + c}$.

(iii) Orthocentre formula : The position vector of the orthocentre of Δ ABC is $\frac{\vec{a}\tan A + \vec{b}\tan B + \vec{c}\tan C}{\tan A + \tan B + \tan C}$

(iv) Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

(v) The vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(vi) Shortest distance between two parallel lines : Let ℓ_1 and ℓ_2 be two lines whose equations are ℓ_1 : $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and ℓ_2 : $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ respectively.

Then, shortest distance

$$PQ = \left|\frac{(b_1 \times b_2) \cdot (a_2 - a_1)}{|b_1 \times b_2|}\right| = \left|\frac{[b_1\ b_2\ (a_2 - a_1)]}{|b_1 \times b_2|}\right|$$

shortest distance between two parallel lines : The shortest distance between the parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is given by $d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$.

If the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ intersect, then the shortest distance between them is zero.

Therefore, $[b_1\, b_2\, (a_2 - a_1)] = 0$

$$\Rightarrow \quad [(\vec{a}_2 - \vec{a}_1)\vec{b}_1\vec{b}_2] = 0 \Rightarrow (\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0.$$

(vii) Vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is

$\vec{r} \cdot \hat{n} = d$.

If $\vec{n}$ is not a unit vector, then to reduce the equation $\vec{r} \cdot \vec{n} = d$ to normal form we divide both sides by $|\vec{n}|$ to obtain $\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{d}{|\vec{n}|}$ or $\vec{r} \cdot \hat{n} = \frac{d}{|\vec{n}|}$.

(viii) The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ or $[\vec{r}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]$, where λ and μ are scalars.

(ix) Vector equation of a plane passing through a point $\vec{a}\vec{b}\vec{c}$ is $\vec{r} = (1 - s - t)\vec{a} + s\vec{b}t + \vec{c}$

or $\vec{r}.(\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = [\vec{a}\vec{b}\vec{c}]$.

(x) The equation of any plane through the intersection of planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is

$\vec{r} \cdot (\vec{n}_1 + \lambda n_2) = d_1 + \lambda d_2$, where λ is an arbitrary constant.

(xi) The perpendicular distance of a point having position vector $\vec{a}$ from the plane $\vec{r}.\vec{n} = d$ is given by

$$p = \frac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}.$$

(xii) An angle θ between the planes $\vec{r}_1.\vec{n}_1 = d_1$ and $\vec{r}_2.\vec{n}_2 = d_2$ is given by $\cos\theta = \pm\frac{n_1 \cdot n_2}{|n_1||n_2|}$.

(xiii) The equation of the planes bisecting the angles between the planes $\vec{r}_1.\vec{n}_1 = d_1$ and $\vec{r}_2.\vec{n}_2 = d_2$ are $\frac{|\vec{r}.\vec{n}_1 - d_1|}{|\vec{n}_1|} = \frac{|\vec{r}.\vec{n}_2 - d_2|}{|\vec{n}_2|}$

(xiv) The plane $\vec{r}.\vec{n} = d$ touches the sphere $|\vec{r} - \vec{a}| = R$, if $\frac{|\vec{a}.\vec{n} - d|}{|\vec{n}|} = R$.

(xv) If the position vectors of the extremities of a diameter of a sphere are $\vec{a}$ and $\vec{b}$, then its equation is

$(\vec{r} - \vec{a}).(\vec{r} - \vec{b}) = 0$ or $|\vec{r}|^2 - \vec{r}.(\vec{a} - \vec{b}) + \vec{a}.\vec{b} = 0$.

THREE DIMENSIONAL GEOMETRY

1. Points in Space :

(i) Origin is (0, 0, 0)

(ii) Equation of x-axis is y = 0, z = 0

(iii) Equation of y-axis is z = 0, x = 0

(iv) Equation of z-axis is x = 0, y = 0

(v) Equation of YOZ plane is x = 0

(vi) Equation of ZOX plane is y = 0

(vii) Equation of XOY plane is z = 0

2. Distance formula :

(i) Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(ii) Distance between origin (0, 0, 0) & point (x, y, z)

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}$$

(iii) Distance of a point p(x, y, z) from coordinate axes OX, OY, OZ is given by

$$\sqrt{y^2 + z^2}, \ \sqrt{z^2 + x^2} \ \text{and} \ \sqrt{x^2 + y^2}$$

3. Section formula :

The coordinates of a point which divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio m : n

* Internally are $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$

* Externally are $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

* Coordinates of the centroid of a triangle are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

* Coordinates of centroid of a tetrahedron

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Note :

* Area of triangle is given by $\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

Where $\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$ and so.

* Condition of collinearity $\frac{x_1 - x_2}{x_2 - x_3} = \frac{y_1 - y_2}{y_2 - y_3} = \frac{z_1 - z_2}{z_2 - z_3}$

* Volume of tetrahedron $= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

4. Direction cosines and direction ratios of a line :

* If α, β, γ are the angles which a directed line segment makes with the +ve direction of the coordinate axes, then $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ are called direction cosines of the line and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ i.e. $\ell^2 + m^2 + n^2 = 1$, where $0 \le \alpha, \beta, \gamma \le \pi$

* If ℓ, m, n are direction cosines of a line and a, b, c are proportional to ℓ, m, n respectively, then a, b, c are called direction ratios of the line and

$$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{\sqrt{\ell^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}.$$

* Direction cosines of x-axis are 1, 0, 0, similarly direction cosines of y-axis and z-axis are respectively 0, 1, 0 and 0, 0, 1.
* If ℓ, m, n are d.c.s of a line OP and (x, y, z) are coordinates of P then x = ℓr, y = mr and z = nr where r = OP.
* Direction cosines of PQ = r, where P is (x_1, y_1, z_1) and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}$$

* If a, b, c are direction no. of a line, then $a^2 + b^2 + c^2$ need not to be equal to 1.

Note : Direction cosines of a line are unique but the direction ratios of line are not unique.

If $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ be two points and L be a line with d.c.'s ℓ, m, n, then projection of [PQ] on

$L = \ell(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

5. Straight line in space :

* Equation of a straight line passing through a fixed point and having d.r.'s a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$ (is the symmetrical form)

* Equation of a line passing through two points is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

* The angle θ between the lines whose d.c.'s are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 is given by
  $\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$.

  The lines are || if $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ and

  The lines are $\perp$ if $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$
* The angle θ between the lines whose d.r.s are a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

  The lines are || if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ and

  The lines are $\perp$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
* Length of the projection of PQ upon AB with d.c., ℓ, m, n

  $= (x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n$, where $p(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.
* Two straight lines in space (not in same plane) which are neither parallel nor intersecting are called skew lines.
* Shortest distance between two skew lines,

$$\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$$ and

$$\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$ is given

$$\text{s.d.} = \pm \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1\ell_2 - \ell_1n_2)^2 + (\ell_1m_2 - m_1\ell_2)^2}}$$

* Two straight lines are coplanar if they are intersecting or parallel

condition $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

6. Plane : A plane is a surface such that if two points are taken in it, straight line joining them lies wholly in the surface.

* Ax + By + Cz + D = 0 represents a plane whose normal has d.c.s proportional to A, B, C.
* Equation of plane through origin is given by Ax + By + Cz = 0.
* Equation of plane passing through a point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, where A, B, C are d.r.'s of a normal to the plane.
* Equation of plane through the intersection of two planes

  $P \equiv a_1x + b_1y + c_1z + d_1 = 0$ and

  $Q \equiv a_2x + b_2y + c_2z + d_2 = 0$ is $P + \lambda Q = 0$.
* Equation of plane which cuts off intercepts a, b, c respectively on the axes x, y and z is

  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

* Normal form of the equation of plane is $\ell x + my + nz = p$, where ℓ, m, n are the d.c.'s of the normal to the plane and p is the length of perpendicular from the origin.
* ax + by + cz + k = 0 represents a plane || to the plane ax + by + cz + d = 0 and $\perp$ to the line

  $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.
* Equation of plane through three non collinear points is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

or $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

* The angle between the two planes is given by

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

where θ is the angle between the normals.

plane are $\perp$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

plane are || if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = 0$.

* If AP be the $\perp$ from A to the given plane, then it is || to the normal, so that its equation is

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c} = r \text{ (say)}$$

Any point P on it is $(ar + \alpha, br + \beta, cr + \gamma)$

* Length of the $\perp$ from $P(x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is given by

$$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

* Distance between two parallel planes $(ax + by + cz + d_1 = 0, ax + by + cz + d_2 = 0)$ is

given by $\left| \frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}} \right|$

* Two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ lie on the same or different sides of the plane $ax + by + cz + d = 0$, according as the expression $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same or different sign.

* Bisector of the angles between the planes

$a_1x + b_1y + c_1z + d_1 = 0$
and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

if $a_1a_2 + b_1b_2 + c_1c_2$ is –ve then origin lies in the acute angle between the planes provided d_1 and d_2 are of same sign.

7. Line and Plane :

If $ax + by + cz + d = 0$ represents a plane and

$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ represents a straight line, then

* The line is $\perp$ to the plane if $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$

* The line is || to the plane if $a\ell + bm + cn = 0$.

* The line lies in the plane if
$a\ell + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d = 0$

* The angle θ between the line and the plane is given by

$$\sin\theta = \frac{a\ell + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{\ell^2 + m^2 + n^2}}$$

* General equation of the plane containing the line

$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$. where
$A\ell + Bm + Cn = 0$.

* Length of the perpendicular from a point (x_1, y_1, z_1)

to the line $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is given by

$$p^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - [\ell(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)]^2$$