1. T- Ratios of various angles and their Signs in four quadrants

- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\cos^2\theta + \sin^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$

- **Values of T -functions of some Particular Angles**

$\theta \to$	(0)	$\left(\frac{\pi}{6}\right)$	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{3}\right)$	$\left(\frac{\pi}{2}\right)$	(π)	$\left(\frac{3\pi}{2}\right)$	(2π)
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

II	I
only $\sin\theta$ and $\text{cosec}\,\theta$ are + ve	All are + ve
only $\tan\theta$ and $\cot\theta$ are - ve	only $\cos\theta$ and $\sec\theta$ are - ve
III	IV

- **Values of T -functions in terms of other T -functions**

Quadrant	I	II	III	IV	IV	I	II	III	IV
Angle	$(2\pi + x)$	$(\pi - x)$	$(\pi + x)$	$(2\pi - x)$	$(-x)$	$\left(\frac{\pi}{2} - x\right)$	$\left(\frac{\pi}{2} + x\right)$	$\left(\frac{3\pi}{2} - x\right)$	$\left(\frac{3\pi}{2} + x\right)$
sin	$\sin x$	$\sin x$	$-\sin x$	$-\sin x$	$-\sin x$	$\cos x$	$\cos x$	$-\cos x$	$-\cos x$
cos	$\cos x$	$-\cos x$	$-\cos x$	$\cos x$	$\cos x$	$\sin x$	$-\sin x$	$-\sin x$	$\sin x$
tan	$\tan x$	$-\tan x$	$\tan x$	$-\tan x$	$-\tan x$	$\cot x$	$-\cot x$	$\cot x$	$-\cot x$
cosec	$\text{cosec}\,x$	$\text{cosec}\,x$	$-\text{cosec}\,x$	$-\text{cosec}\,x$	$-\text{cosec}\,x$	$\sec x$	$\sec x$	$-\sec x$	$-\sec x$
sec	$\sec x$	$-\sec x$	$-\sec x$	$\sec x$	$\sec x$	$\text{cosec}\,x$	$-\text{cosec}\,x$	$-\text{cosec}\,x$	$\text{cosec}\,x$
cot	$\cot x$	$-\cot x$	$\cot x$	$-\cot x$	$-\cot x$	$\tan x$	$-\tan x$	$\tan x$	$-\tan x$

2. Range of T - Ratios:

- $-1 \le \sin\theta \le 1$
- $-1 \le \cos\theta \le 1$
- $-\infty < \tan\theta < \infty$
- $|\text{cosec}\,\theta| \ge 1$
- $|\sec\theta| \ge 1$
- $-\infty < \cot\theta < \infty$

3. Period of T - ratios

- All T-ratios are periodic functions.
- Period $\tan\theta$ of and $\cot\theta$ is π
- Period of $\sin\theta, \cos\theta, \text{cosec}\,\theta$ and $\sec\theta$ is 2π

4. Sum and Difference formula:

- $\cos(x+y) = \cos x\cos y - \sin x\sin y$
- $\sin(x+y) = \sin x\cos y + \cos x\sin y$
- $\cos(x-y) = \cos x\cos y + \sin x\sin y$
- $\sin(x-y) = \sin x\cos y - \cos x\sin y$

- $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$,
- $\cot(x+y)=\dfrac{\cot x\cot y-1}{\cot y+\cot x}$
- $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$
- $\cot(x-y)=\dfrac{\cot x\cot y+1}{\cot y-\cot x}$
- $\sin 2x=2\sin x\cos x$
- $\cos 2x=\cos^2 x-\sin^2 x=2\cos^2 x-1=1-2\sin^2 x$
- $\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$
- $\cos x=\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2}=2\cos^2\dfrac{x}{2}-1=1-2\sin^2\dfrac{x}{2}$
- $1+\cos 2x=2\cos^2 x$
- $1-\cos 2x=2\sin^2 x$
- $1+\cos x=2\cos^2\dfrac{x}{2}$
- $1-\cos x=2\sin^2\dfrac{x}{2}$
- $\dfrac{1-\cos x}{\sin x}=\tan\dfrac{x}{2}$
- $\dfrac{1+\cos x}{\sin x}=\cot\dfrac{x}{2}$
- $\sin x=\sqrt{\dfrac{1-\cos 2x}{2}}$
- $\cos x=\sqrt{\dfrac{1+\cos 2x}{2}}$
- $\tan x=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$
- $\sin\dfrac{x}{2}=\sqrt{\dfrac{1-\cos x}{2}}$
- $\cos\dfrac{x}{2}=\sqrt{\dfrac{1+\cos x}{2}}$
- $\tan\dfrac{x}{2}=\sqrt{\dfrac{1-\cos x}{1+\cos x}}$
- $\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$
- $\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$
- $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$
- $\sin x=\dfrac{2\tan x/2}{1+\tan^2 x/2}$
- $\cos x=\dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}$
- $\tan x=\dfrac{2\tan x/2}{1-\tan^2 x/2}$
- $\sin 3x=3\sin x-4\sin^3 x$
- $\cos 3x=4\cos^3 x-3\cos x$
- $\tan 3x=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$
- $\sin(x+y)\sin(x-y)=\sin^2 x-\sin^2 y=\cos^2 y-\cos^2 x$
- $\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y=\cos^2 y-\sin^2 x$

5. Product Into Sum or Difference Formulae:

- $2\sin x\cos y=\sin(x+y)+\sin(x-y)$
- $2\cos x\sin y=\sin(x+y)-\sin(x-y)$
- $2\cos x\cos y=\cos(x+y)+\cos(x-y)$
- $2\sin x\sin y=\cos(x-y)-\cos(x+y)$

6. Sum and Difference Into Product Formulae :

- $\sin x+\sin y=2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$
- $\sin x-\sin y=2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$
- $\cos x+\cos y=2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$
- $\cos x-\cos y=-2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

7. Trigonometric Ratio of Some Important Angles :

- $\sin 18° = \frac{\sqrt{5}-1}{4}$
- $\tan 22\frac{1}{2}° = \sqrt{2}-1$
- $\tan 15° = 2-\sqrt{3}$
- $\cos 36° = \frac{\sqrt{5}+1}{4}$
- $\cot 22\frac{1°}{2} = \sqrt{2}+1$
- $\cot 15° = 2+\sqrt{3}$

8. Maximum and Minimum Values (of $a\cos\theta + b\sin\theta$):

Let $a = r\cos\alpha$ and $b = r\sin\alpha$

Then, $a\cos\theta + b\sin\theta = r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = r\cos(\alpha - \theta)$ where $r = \sqrt{a^2+b^2}$

But $-1 \le \cos(\alpha - \theta) \le 1$ $\therefore -r \le a\cos\theta + b\sin\theta \le r$

So the maximum value is $\left(\sqrt{a^2+b^2}\right)$ and minimum value is $\left(-\sqrt{a^2+b^2}\right)$

9. Some other useful Results :

- $\sin(A+B+C) = \sum \sin A\cos B\cos C - \prod \sin A$
- $\sin A \sin(60° - A)\sin(60° + A) = \frac{1}{4}\sin 3A$
- $\cos(A+B+C) = \prod \cos A - \sum \cos A \sin B \sin C$
- $\cos A\cos(60° - A)\cos(60° + A) = \frac{1}{4}\cos 3A$
- $\tan(A+B+C) = \frac{\sum \tan A - \prod \tan A}{1-\sum \tan A \tan B}$
- $\tan A \tan(60° - A)\tan(60° + A) = \tan 3A$
- $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2 = 1+\sin A$
- $\tan A - \cot A = -2\cot 2A$
- $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2 = 1-\sin A$
- $\tan A + \cot A = 2\operatorname{cosec} 2A$
- $\cos A+\cos B+\cos C+\cos(A+B+C) = 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$
- $\sin A+\sin B+\sin C+\sin(A+B+C) = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$
- $\cos A\cos 2A\cos 2^2A.....\cos 2^{n-1}A = \frac{\sin 2^n A}{2^n \sin A}$
- $\sin a+\sin(a+d)+\sin(a+2d)+.........\sin[a+(n-1)d] = \frac{\sin\frac{nd}{2}}{\sin\frac{d}{2}}\sin\left(\frac{2a+(n-1)d}{2}\right)$
- $\cos a+\cos(a+d)+\cos(a+2d)+.........\cos[a+(n-1)d] = \frac{\sin\frac{nd}{2}}{\sin\frac{d}{2}}\cos\left(\frac{2a+(n-1)d}{2}\right)$

1. Solution of Trigonometric Equations

- **Principal solutions** : The solutions of a trigonometric equation for which $0 \le x < 2\pi$ are called principal solutions.
- **General solution** : A solution of a trigonometric equation, generalised by means of periodicity, is known as the general solution.

2. Some Examples of Principal Solutions

- $\sin\theta = 0 \Rightarrow \theta = \{0, \pi\}$
- $\cos\theta = 0 \Rightarrow \theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$
- $\tan\theta = 0 \Rightarrow \theta = \{0, \pi\}$
- $\sin\theta = \frac{1}{2} \Rightarrow \theta = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$
- $\cos\theta = \frac{1}{2} \Rightarrow \theta = \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
- $\tan\theta = \sqrt{3} \Rightarrow \theta = \left\{\frac{\pi}{3}, \frac{4\pi}{3}\right\}$
- $\sin\theta = -\frac{1}{2} \Rightarrow \theta = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
- $\cos\theta = -\frac{1}{2} \Rightarrow \theta = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$
- $\tan\theta = -\sqrt{3} \Rightarrow \theta = \left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$
- $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$
- $\cos\theta = 1 \Rightarrow \theta = 0$
- $\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2}$
- $\cos\theta = -1 \Rightarrow \theta = \pi$
- $\sin^2\theta = \frac{1}{4} \Rightarrow \theta = \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
- $\cos^2\theta = \frac{1}{4} \Rightarrow \theta = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$
- $\tan^2\theta = 1 \Rightarrow \theta = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

3. General Solution of Some Important Equations

- $\sin\theta = 0 \Rightarrow \theta = n\pi$ where $n \in I$
- $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\pi/2$ where $n \in I$
- $\tan\theta = 0 \Rightarrow \theta = n\pi$ where $n \in I$
- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $n \in I$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $n \in I$
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $n \in I$
- $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$ where $n \in I$
- $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$ where $n \in I$
- $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$ where $n \in I$
- $\sin\theta = 1 \Rightarrow \theta = 2n\pi + \pi/2$ where $n \in I$
- $\sin\theta = -1 \Rightarrow \theta = 2n\pi - \pi/2$ where $n \in I$
- $\cos\theta = 1 \Rightarrow \theta = 2n\pi,$ where $n \in I$
- $\cos\theta = -1 \Rightarrow \theta = 2n\pi + \pi,$ where $n \in I$

4. General solution of $a\cos\theta + b\sin\theta = c$ where $|c| \le \sqrt{a^2 + b^2}$

- Put $a = r\cos\alpha$ and $b = r\sin\alpha$, where $r = \sqrt{a^2 + b^2}$

  Then the equation becomes $r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = c$

  $\Rightarrow \cos(\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}} = \cos\beta$ (say) $\Rightarrow \theta - \alpha = 2n\pi \pm \beta$

  $\therefore \theta = 2n\pi \pm \beta + \alpha$ (where $\tan\alpha = b/a$) is the general solution.

- Alternatively, putting $a = r\sin\alpha$ and $b = r\cos\alpha$ where $r = \sqrt{a^2+b^2}$, we get

$$\sin(\theta+\alpha) = \frac{c}{\sqrt{a^2+b^2}} = \sin\gamma \quad \text{(say)} \qquad \Rightarrow \theta+\alpha = n\pi + (-1)^n\gamma$$

$\Rightarrow \theta = n\pi + (-1)^n\gamma - \alpha$ (where $\tan\alpha = a/b$) is the general solution.

- Both the methods give the same set of values of θ

5. Inverse Circular Functions:

- The mathematical definition of a function from set A to set B is that to each element $a \in A$ there exists a unique element $b \in B$.
- In direct trigonometric function, we are given the angle and we calculate the trigonometric ratio (sine, cosine, etc.)
- To many values of the angle, the value of trigonometric ratio is same. e.g. $\tan\theta = 1$ for $\theta = \frac{\pi}{4}, 5\frac{\pi}{4}, 9\frac{\pi}{4}$, etc.
- Direct trigonometric function quite obviously follow the definition of a function (they are many-one functions.)
- Inverse trigonometry deals with obtaining the angle, given the value of trigonometry ratio.
- In inverse trignometry, if we say that to a certain value of the trigonometric ratio, there corresponds many values of the angle, it violates the definition of function (it becomes a one -many relation).
- Hence, some restrictions have been imposed on the angles, and these are based on the principle values of the angles.
- The inverse of sine function is defined as $\sin^{-1}x = \theta$ where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
- e.g., $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ only although $\sin\frac{5\pi}{6}, \sin\frac{13\pi}{6}$, etc. are also equal to $\frac{1}{2}$.
- Similarly, $\sin^{-1}\left(-\sqrt{3}/2\right) = -\frac{\pi}{3}$ only.

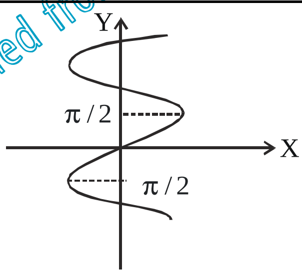
6. Graphs of Inverse Trignometric Functions

- $y = \sin^{-1}x$

Domain : $x \in [-1,1]$

Range: Principal value of y is

$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

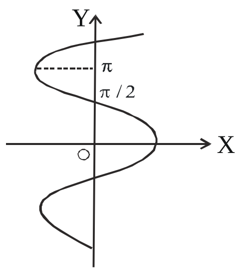


- $y = \cos^{-1}x$

Domain: $x \in [-1,1]$

Range: Principal value of y is

$y \in [0,\pi]$

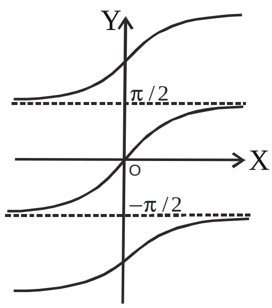


- $y = \tan^{-1}x$

Domain: $x \in R$

Range: Principal value of is

$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

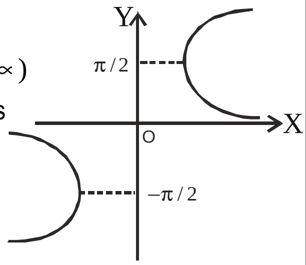


- $y = \text{cosec}^{-1}x$

Domain: $x \in (-\infty, -1] \cup [1, \infty)$

Range : Principal value of y is

$y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

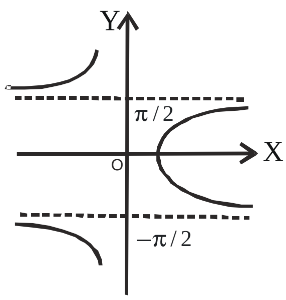


- $y = \sec^{-1}x$

Domain: $x \in (-\infty, -1] \cup [1, \infty)$

Range: Principal value of y is

$y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

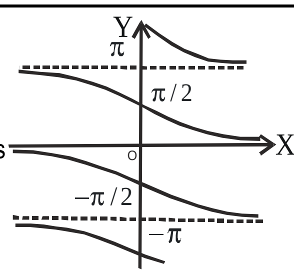


- $y = \cot^{-1}x$

Domain: $x \in R$

Range: Principal value of y is

$y \in (0,\pi)$



7. Important Results

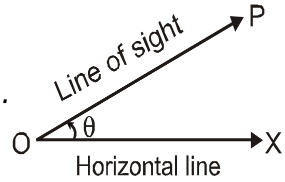
- $\sin\left(\sin^{-1}x\right)=x$
- $\cos\left(\cos^{-1}x\right)=x$
- $\tan\left(\tan^{-1}x\right)=x$
- $\sin^{-1}(\sin\theta)=\beta$ where β lies in principal range. e.g. $\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$ whereas $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)\neq\frac{2\pi}{3}$
- $\sin^{-1}(-x)=-\sin^{-1}x$
- $\cos^{-1}(-x)=\pi-\cos^{-1}x$
- $\tan^{-1}(-x)=-\tan^{-1}x$
- $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x$
- $\sec^{-1}(-x)=\pi-\sec^{-1}x$
- $\cot^{-1}(-x)=\pi-\cot^{-1}x$
- $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy<1$

  $=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy>1$
- $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$
- $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$
- $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}$
- $\sin^{-1}x=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$
- $\cos^{-1}x=\tan^{-1}\frac{\sqrt{1-x^2}}{x}$
- $\cot^{-1}x=\tan^{-1}\frac{1}{x}$
- $\sec^{-1}x=\cos^{-1}\frac{1}{x}$
- $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}$
- $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$
- $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}$
- $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$
- $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$, $|x|<1$
- $2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}$, $|x|\leq 1$
- $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$, $x\geq 0$
- $\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]$, $-\frac{\pi}{2}\leq\sin^{-1}x-\sin^{-1}y\leq\frac{\pi}{2}$
- $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}-y\sqrt{1-x^2}\right]$, $-\frac{\pi}{2}\leq\sin^{-1}x-\sin^{-1}y\leq\frac{\pi}{2}$
- $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]$, $0\leq\cos^{-1}x+\cos^{-1}y\leq\pi$
- $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left[xy+\sqrt{1-x^2}\sqrt{1-y^2}\right]$, $0\leq\cos^{-1}x-\cos^{-1}y\leq\pi$

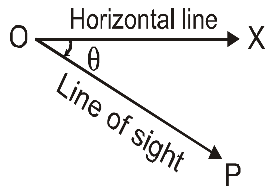
1. **Introduction**

- Let 'O' be the observer's eye and OX be the horizontal line through O.

- If the object P is at a higher level than O, then angle $POX(=\theta)$ is called the ***angle of elevation*** .



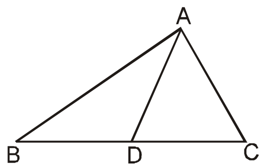
- If the object P is at a lower level than O, then angle POX is called the ***angle of depression***.



2. **Some results that will be useful in solving problems**

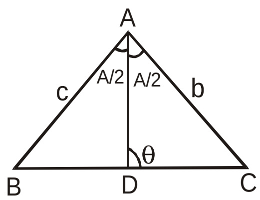
In a triangle ABC,

- If AD is median, then $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$



- If AD is the angle bisector of $\angle BAC$ .then

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$



- If a line is perpendicular to a plane, then it is perpendicular to every line lying in that plane .

1. Arithmetic Progression (AP)

- In an Arithmetic Progression (AP), the consecutive terms increase/decrease by a fixed quantity.
- n^{th} term, $T_n = a + (n-1)d$ where, $a =$ first term, $d =$ common difference, $n =$ number of terms
- Sum of n terms, $S_n = \frac{n}{2}(a+\ell) = \frac{n}{2}[2a+(n-1)d]$ where, $\ell =$ last term
- **Sum of first and last terms of an AP is equal to the sum of two terms which are equidistant from the first and the last terms.**
- AM between two numbers a and b, $A = \frac{a+b}{2}$
- n AMs between two numbers a and b denoted by $A_1, A_2, A_3 ....A_n$ form an AP given by $a, A_1, A_2, A_3 ....A_n, b$
- The sum of n AM's between a and b is equal to $n\left(\frac{a+b}{2}\right) = nA$
- Arithmetic mean of n positive numbers $a_1, a_2, a_3 ....a_n$ is $A = \frac{a_1 + a_2 + ...a_n}{n}$
- If a, b, c are in AP, then ak, bk, ck are also in AP

  $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are also in AP

  $a \pm k, b \pm k, c \pm k$ are also in AP
- For solving problems, 3 terms in AP are taken as $a-d, a, a+d$

  4 terms in AP are taken as $a-3d, a-d, a+d, a+3d$
- Common difference when general term is given, $d = T_n - T_{n-1}$
- n^{th} term when general sum of n terms is given, $T_n = S_n - S_{n-1}$

2. Geometric Progression (GP)

- In a Geometric Progression (GP), the consecutive terms increase / decrease by a fixed ratio.
- n^{th} term, $T_n = ar^{n-1}$ where, $a =$ first term, $r =$ common ratio, $n =$ number of terms
- Sum of n terms, $S_n = \frac{a(r^n - 1)}{r-1} = \frac{a(1-r^n)}{1-r}$
- If $|r| < 1$, the sum of infinite terms, $S_\infty = \frac{a}{1-r}$
- **Product of first and last terms of a GP is equal to the product of two terms which are equidistant from the first and the last terms.**
- GM between two numbers a and b, $G = \sqrt{ab}$
- n GMs between numbers a and b denoted by $G_1, G_2, G_3, ...G_n$ form a GP given by $a, G_1, G_2, G_3, ...G_n, b$
- The product of GM's between a and b is equal to $\left(\sqrt{ab}\right)^{1/n} = (ab)^{n/2}$

- Geometric mean of n positive numbers $a_1, a_2, ...a_n$ is $G = (a_1, a_2, ...a_n)^{1/n}$
- If a, b, c are in GP, then ak, bk, ck are also in GP $(k \neq 0)$

  $\frac{a}{k}, \frac{b}{k}, \frac{c}{k},$ are also in GP $(k \neq 0)$
- For solving problems, 3 terms are taken as $ar, a, \frac{a}{r}$

  4 terms are taken as $ar^3, ar, \frac{a}{r}, \frac{a}{r^3}$
- If $a_1, a_2, ...a_n$ is a G.P. with common ratio r, then

  $\log a_1, \log a_2, ...\log a_n$ is an A.P. with common difference equal to $\log r$

3. Harmonic Progression (HP)

- In a Harmonic Progression (HP), the reciprocals of two consecutive terms increase /decrease by a fixed quantity.
- $a_1, a_2, a_3, -------a_n$ are said to be in HP if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, -------\frac{1}{a_n}$ are in AP
- If H is HM of a and b, then $\frac{1}{a} - \frac{1}{H} = \frac{1}{H} - \frac{1}{b} \Rightarrow H = \frac{2ab}{a+b}$

4. Relation among A,G,H

- If a and b are two numbers, then

  $A = \frac{a+b}{2}$, $G = \sqrt{ab}$, $H = \frac{2ab}{a+b}$,
- $AH = G^2$
- G is Geometric Mean of A and H
- $A > G > H$

5. Summation of Natural Numbers

The following identities hold good for all $n \in N$.

- $\sum n = 1 + 2 + 3 + -----------n = \frac{n(n+1)}{2}$
- $\sum n^2 = 1^2 + 2^2 + 3^2 + -----------n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum n^2 = 1^2 + 2^2 + 3^2 + -----------n^2 = \sum n^3 = \left[\frac{n(n+1)}{2}\right]^2$
- If n^{th} term is given by $T_n = an^3 + bn^2 + cn + d,$

  then the sum of n terms is given by

  $S_n = \Sigma T_n = \Sigma(an^3 + bn^2 + cn + d) = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + d\Sigma 1$

  $= a\left[\frac{n(n+1)}{2}\right]^2 + \frac{bn(n+1)(2n+1)}{6} + \frac{cn(n+1)}{2} + dn$

6. Arithmetico - Geometric Series

- This series is a combination of AP and GP in the manner $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \ldots\ldots$

  (where, a = first term, d = common difference, r = common ratio)
- n^{th} term, $T_n = [a+(n-1)d]r^{n-1}$
- Sum of n terms, $S_n = \dfrac{a}{(1-r)} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]r^n}{(1-r)}$
- If $|r|<1$, then the sum of infinite terms, $S_\infty = \dfrac{a}{1-r} + \dfrac{dr}{(1-r)^2}$ if $|r|<1$

7. Method of difference in summation of series

- If n^{th} term of a series can be written as $t_n = f(n) - f(n+1)$,

  then, $S_n = t_1 + t_2 + \ldots..t_n$

  $= [f(1) - f(2)] + [f(2) - f(3)] + [f(3) - f(4)] + \ldots.[f(n) - f(n+1)]$

  $= f(1) - f(n+1)$
- e.g. consider the series $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + ------\dfrac{1}{n.(n+1)}$

  Here, $t_n = \dfrac{1}{n} - \dfrac{1}{n+1}$ where, $f(n) = \dfrac{1}{n}$

  $\therefore S_n = \left(\dfrac{1}{1} - \dfrac{1}{2}\right) + \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \ldots\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$

8. Summation of Trigonometric Series

If $a_1, a_2, \ldots..a_n$ are in AP with common difference 'd', then

- $\sin a_1 + \sin a_2 + \ldots.. + \sin a = \dfrac{\sin\left(\dfrac{a_1+a_n}{2}\right)\sin\left(\dfrac{nd}{2}\right)}{\sin\left(\dfrac{d}{2}\right)}$
- $\cos a_1 + \cos a_2 + \ldots.. + \cos a = \dfrac{\cos\left(\dfrac{a_1+a_n}{2}\right)\sin\left(\dfrac{nd}{2}\right)}{\sin\left(\dfrac{d}{2}\right)}$

QUADRATIC EQUATIONS AND INEQUATIONS

1. Standard Form

A quadratic equation in standard form is $ax^2 + bx + c = 0$ where, $a, b, c \in R$ and $a \neq 0$

- Roots of the equation are given by $\alpha, \beta = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Sum of the roots, $\alpha + \beta = -b/a$

  Product of the roots, $\alpha\beta = c/a$
- The equation $ax^2 + bx + c = 0$ can be expressed as $ax^2 + bx + c = a(x-\alpha)(x-\beta)$

  $= ax^2 + bx + c = a[x^2 - (\alpha+\beta)x + \alpha\beta] = ax^2 - a(\alpha+\beta)x + a\,\alpha\,\beta$

  Equating coeff. of x and constant terms on both sides, we have $b = -a(\alpha+\beta)$ and $c = a\,\alpha\beta$

  $\therefore\ \alpha + \beta = -b/a$ and $\alpha\beta = c/a$
- Difference of roots, $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \sqrt{\left(\dfrac{-b}{a}\right)^2 - 4\dfrac{c}{a}} = \dfrac{\sqrt{b^2 - 4ac}}{a}$

2. Nature of Roots

- $D = b^2 - 4ac$ is called the discriminant.
- If $D > 0$, roots are real and unequal

  $D = 0$, roots are real and equal

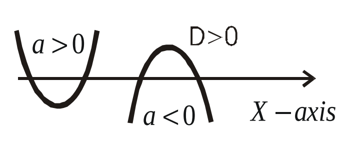
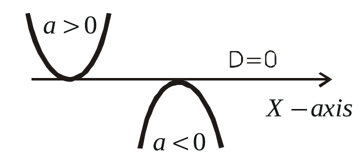
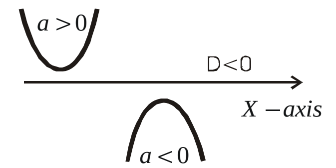
  $D < 0$, roots are complex and unequal
- **If the roots are complex, they always occur as conjugate pairs.**

3. Graphs of Quardratic Polynomial

Let $f(x) = ax^2 + bx + c$, $(a \neq 0)$ be a quadratic polynomial.

- **Shape**: The shape of quadratic polynomial is always a parabola.
- **Opening:** If $a > 0$, parabola opens upwards.

  If $a < 0$, parabola opens downwards.
- **Intersection with X-axis**

Condition	$D > 0$	$D = 0$	$D < 0$
Result	parabola intersects X-axis at two distinct points	parabola touches X-axis	parabola does not intersect X-axis
Graph			

- **Maximum and Minimum values of** $f(x)$

  V is called the vertex of parabola.

  The coordinates of V are $\left(\dfrac{-b}{2a}, \dfrac{-D}{4a}\right)$
- If $a > 0$ and $D < 0$, parabola opens upwards and does not intersect X-axis. Hence, $f(x) > 0$ for all real x.

  If $a < 0$ and $D < 0$, parabola opens downwards and does not intersect X-axis. Hence, $f(x) < 0$ for all real x.

4. Roots of a cubic equation

If α,β,γ are the roots of $ax^3+bx^2+cx+d=0$,

then, $\alpha+\beta+\gamma=-b/a$, $\alpha\beta+\alpha\gamma+\beta\gamma=c/a$, $\alpha\beta\gamma=-d/a$

5. Condition for Common Roots

Let there be two quadratic equations $ax^2+bx+c=0$ and $dx^2+ex+f=0$

- If **both roots are common**, the two equations must essentially be the same. Hence, $\frac{a}{d}=\frac{b}{e}=\frac{c}{f}$
- If **only one root is common**, and the common root is α, then $a\alpha^2+b\alpha+c=0$ and $d\alpha^2+e\alpha+f=0$

  Solving the two equations by Cramer's rule, we get $\frac{\alpha^2}{bf-ce}=\frac{-\alpha}{af-cd}=\frac{1}{ae-bd}$

  Hence, $\alpha=\frac{bf-ce}{cd-af}=\frac{cd-af}{ae-bd}$ or $(cd-af)^2=(bf-ce)(ae-bd)$ which is the condition for one common root.

6. Roots when sum of coefficients is zero

- If the sum of coefficients of a polynomial equation $f(x)=0$ is zero then 1 is a root of the equation $f(x)=0$
- In the equation $ax^2+bx+c=0$, if $a+b+c=0$, then the roots are 1 and c/a and if $a-b+c=0$, then the roots are -1 and $-c/a$.

7. Condition for roots in a given ratio

If the roots of the equation $ax^2+bx+c=0$ are in the ratio $m:n$, then the roots can be taken as mr and nr

Then, $mr+nr=-b/a$ and $mr\times nr=c/a$

$\Rightarrow$ $r=\frac{-b}{a(m+n)}$ and $r^2=\frac{c}{a\,mn}$ $\Rightarrow$ $\left[\frac{b}{a(m+n)}\right]^2=\frac{c}{mna}$

$\therefore$ $b^2mn=ac(m+n)^2$ which is the condition if the roots of the equation are in a given ratio $m:n$

8. Equation whose both roots bear a fixed pattern

If α,β are roots of $ax^2+bx+c=0$, then

- the equation whose roots are $-\alpha,-\beta$ is $a(-x)^2+b(-x)+c=0$
- the equation whose roots are $\frac{1}{\alpha},\frac{1}{\beta}$ is $a\left(\frac{1}{x}\right)^2+b\left(\frac{1}{x}\right)+c=0$
- the equation whose roots are $k\alpha$ and $k\beta$ is $a\left(\frac{x}{k}\right)^2+b\left(\frac{x}{k}\right)+c=0$
- the equation whose roots are $\alpha+k$ and $\beta+k$ is $a(x-k)^2+b(x-k)+c=0$

9. Remainder Theorem

If a polynomial $f(x)$ is divided by $x-\alpha$, the remainder obtained is $f(\alpha)$

10. Factor Theorem

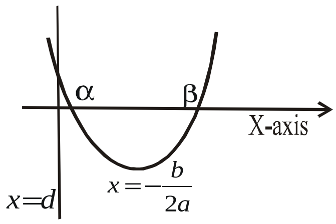
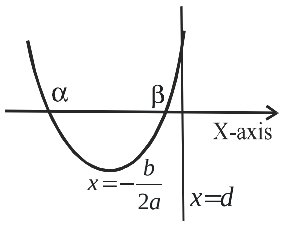
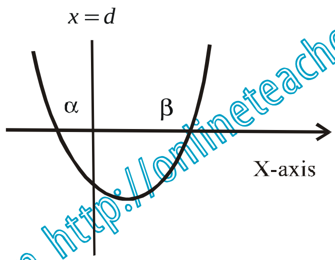
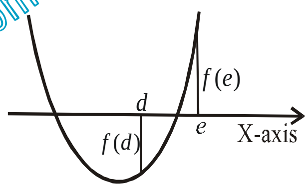
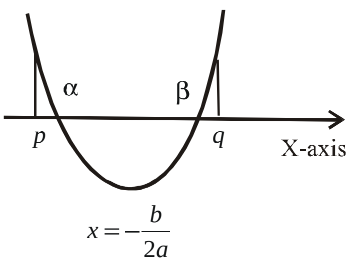
A polynomial $f(x)$ is divisible by $x-\alpha$ if $f(\alpha)=0$

If $f(\alpha)=0$ and $f'(\alpha)=0$, then $(x-\alpha)^2$ is the factor of $f(x)=0$ and α is known as repeated root of $f(x)=0$.

Also, $f(x)=ax^2+bx+c=a(x-\alpha)^2$ where, $\alpha=-b/2a$

11. Conditions for location of roots

Let $f(x) = ax^2 + bx + c$ where, $a > 0$. Then, the conditions for various requirements of the roots are given in the table below.

Requirement	Graph	Conditions
both roots α, β of $f(x) = 0$ to be greater than a specified number d		$b^2 - 4ac \ge 0,\ f(d) > 0,\ -\frac{b}{2a} > d$
both roots α, β of $f(x) = 0$ to be less than a specified number d		$b^2 - 4ac \ge 0,\ f(d) > 0,\ -\frac{b}{2a} < d$
number d to lie between the roots α and β of $f(x) = 0$		$f(d) < 0$
exactly one root of $f(x) = 0$ to lie in the interval (d, e)		$f(d).f(e) < 0$
both roots α and β of $f(x) = 0$ confined between numbers p and q		$b^2 - 4ac \ge 0,\ f(p) > 0,\ f(q) > 0,$ $p < -\frac{b}{2a} < q$

12. Descarte's rule of signs

The maximum number of positive real roots of a polynomial $f(x)$ is the number of changes of signs in $f(x)$ and the maximum number of negative real roots of $f(x)$ is the number of changes of signs in $f(-x)$.

13. Polynomial and Algebraic Inequations

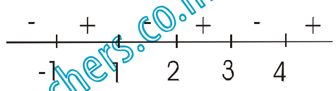
- Here, we solve the inequations of the type $f(x) < 0$, $f(x) \leq 0$, $f(x) > 0$, $f(x) \geq 0$, $P(x)\,Q(x) \leq 0$, $\frac{P(x)}{Q(x)} > 0$, $\frac{P(x)}{Q(x)} < 0$, etc. where, $f(x)$, $P(x)$ and $Q(x)$ are polynomials in x.
- To solve the above inequations, we follow the following steps also known as sign method.
  (1) Factorise $P(x)$ and $Q(x)$ into linear factors
  (2) Make coefficient of x positive in all factors
  (3) Plot the critical points on a number line n critical points will divide the number line in $n+1$ regions.
  (4) In the rightmost region, the expression bears positive sign and in other regions, the expression bears alternate negative and positive signs.
  (5) The region with appropriate sign matching with the expression is the desired domain.

Example: Solve for x in $\frac{(x+1)(x^2-3x+2)}{-x^2+7x-12} \leq 0$

Step- 1: Factorise expression into linear factors $\Rightarrow \frac{(x+1)(x-1)(x-2)}{(x-3)(4-x)} \leq 0$

Step-2: Make coefficient of x positive in all factors $\Rightarrow \frac{(x+1)(x-1)(x-2)}{(x-3)(x-4)} \geq 0$

Step-3: Plot critical points, i.e, $-1, 1, 2, 3, 4,$ on a number line



Step-4: Assign + ve and -ve values to the regions.

Step-5: Since, the expression at step $2 \geq 0$, the desired domain is $x \in [-1,1] \cup [2,3) \cup (4, \infty)$

14. Laws of inequality

- If $a > b$, then $a + c > b + c$
  $ac > bc$ provided $c > 0$
  $ac < bc$ provided $c < 0$
- If $a^x > a^y$, then $x > y$ provided $a > 1$
  $x < y$ provided $0 < a < 1$
- If $\log_a x > \log_a y$, then $x > y$ provided $a > 1$
  $x < y$ provided $0 < a < 1$
- $a^2 + b^2 + c^2 \geq ab + bc + ac$
- If $x > 0$, then $x + \frac{1}{x} \geq 2$
  $x < 0$, then $x + \frac{1}{x} \leq -2$
- If $a_1, a_2, ...a_n$ are positive real numbers,

  then, $(a_1 + a_2 + ...a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ...\frac{1}{a_n}\right) \geq n^2$

  e.g., $(a_1 + a_2 + a_3)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \geq 3^2 = 9$