Continuity and Differentiability

This chapter requires a good understanding of limits. The concepts of continuity and differentiability are more or less obvious extensions of the concept of limits.

Section - 1 INTRODUCTION TO CONTINUITY

We start with a very intuitive introduction to continuity. Consider the two graphs given in the figure below:

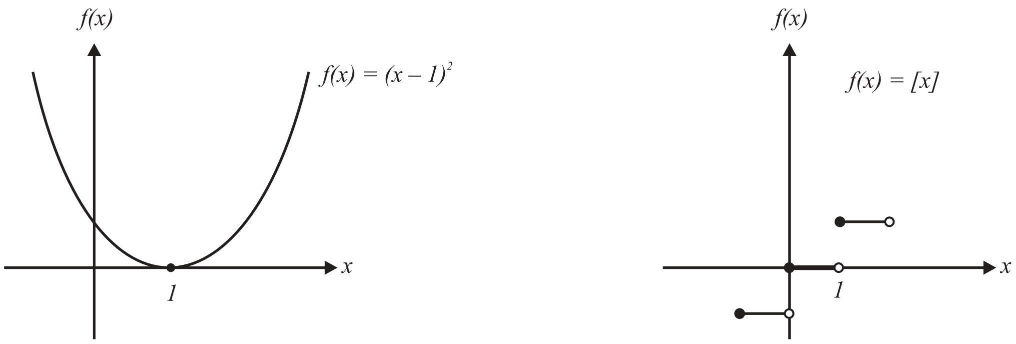

Fig - 1

Our purpose is to analyse the behaviour of these functions around the region $x = 1$.

The obvious visual difference between the two graphs around $x = 1$ is that whereas the first graph passes uninterrupted (without a break) through $x = 1$, the second function suffers a break at $x = 1$ (there is a jump).

This visual difference, put into mathematical language, gives us the concept and definition of continuity. Mathematically, we say that the function $f(x) = (x-1)^2$ is continuous at $x = 1$ while $f(x) = [x]$ is discontinuous at $x = 1$

For $f(x) = (x-1)^2$,

$$\text{LHL }(\text{at } x = 1) = \lim_{x \to 1^-} (x-1)^2 = 0$$

and

$$\text{RHL }(\text{at } x = 1) = \lim_{x \to 1^+} (x-1)^2 = 0$$

and

$f(1) = 0$

$\Rightarrow$ LHL = RHL = $f(1)$

For $f(x) = [x]$,

LHL $(\text{at } x = 1) = \lim_{x \to 1^-} [x] = 0$

and

RHL $(\text{at } x = 1) = \lim_{x \to 1^+} [x] = 1$

and

$f(1) = 1$

$\Rightarrow$ LHL $\neq$ RHL = $f(1)$

From the discussion above, try to see that for a function to be continuous at $x = a$, all the three quantities, namely, LHL, RHL and $f(a)$ should be equal. In any other scenario, the function becomes discontinuous.

Discontinuities therefore arise in the following cases:

(a) **One or more than one of the three quantities, LHL, RHL and *f* (*a*) is not defined**. Lets consider some examples:

(i) $f(x) = \dfrac{1}{x}$ around $x = 0$.

LHL $= -\infty$, RHL $= +\infty$, $f(0)$ is not defined. Therefore, $f(x) = \dfrac{1}{x}$ is discontinuous at $x = 0$ which is obvious from the graph:

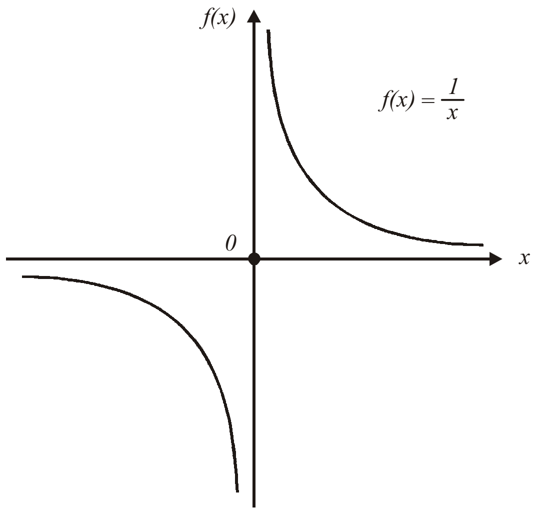

Fig - 2

(ii) $f(x) = \left\{ \dfrac{x^2 - 1}{x - 1} \text{ for } x \neq 1 \right\}$ around $x = 1$

LHL = RHL = 2 but $f(1)$ is not defined. Therefore, this function's graph has a hole at $x = 1$; it is discontinuous at $x = 1$:

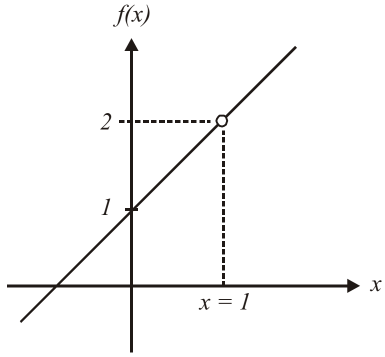

Fig - 3

(b) **All the three quantities are defined, but any pair of them is unequal** (or all three are unequal). Lets go over some examples again:

(i) $f(x) = [x]$ around any integer I

LHL = $I - 1$, RHL = I, $f(I) = I$

$\Rightarrow$ LHL $\neq$ RHL = $f(I)$ so this function is discontinuous at all integers as we already know.

(ii) $f(x) = \{x\}$ around any integer I

LHL = 1, RHL = 0, $f(I) = 0$

$\Rightarrow$ LHL $\neq$ RHL = $f(I)$ so this function is also discontinuous at all integers.

(iii) $f(x) = \begin{Bmatrix} 1, & x \notin \mathbb{Z} \\ 0, & x \in \mathbb{Z} \end{Bmatrix}$ around any integer I

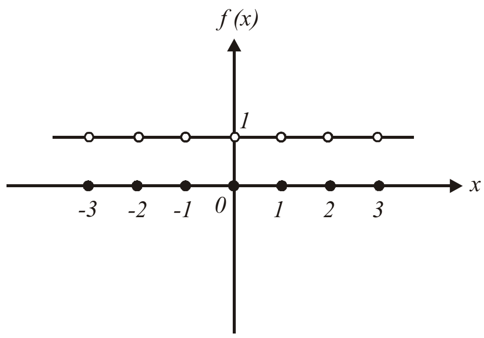

Fig - 4

From the figure, we notice that at any integer I, LHL = 1, RHL = 1, $f(I) = 0$

$\Rightarrow$ LHL = RHL $\neq f(I)$ so that this function is again discontinuous.

(iv) $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$ around $x = 0$

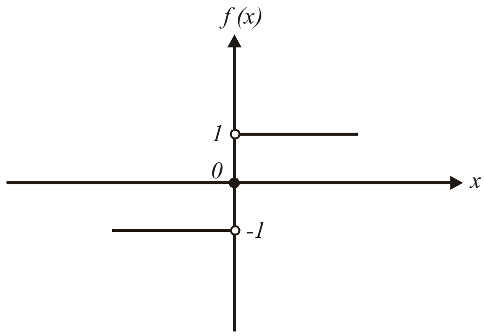

Fig - 5

At $x = 0$, we see that

LHL = –1, RHL =1, $f(0) = 0$

$\Rightarrow$ LHL $\neq$ RHL $\neq f(0)$ and this function is discontinuous.

To summarize, if we intend to evaluate the continuity of a function at $x = a$, which means that we want to determine whether $f(x)$ will be continuous at $x = a$ or not, we have to evaluate all the three quantities, LHL, RHL and $f(a)$. If these three quantities are finite and equal, $f(x)$ is continuous at $x = a$. In all other cases, it is discontinuous at $x = a$

$$\boxed{\text{LHL}(\text{at } x = a) = \text{RHL}(\text{at } x = a) = f(a): \text{ for continuity at } x = a}$$

Example – 1

Find the value of $f(1)$ if the function $f(x) = \frac{x^{m+1} - (m+1)x + m}{(x-1)^2}, x \neq 1$ is continuous at $x = 1$

Solution: We are going to encounter a lot of questions similar to the one above over the next few pages; we are given a definition of $f(x)$ and are asked to determine $f(a)$ so that $f(x)$ becomes continuous at $x = a$. We simply evaluate $\lim\limits_{x \to a} f(x)$ and if it exists, we let $f(a)$ equal to this limit. This ensures that the necessary and sufficient condition for continuity at $x = a$, i.e

$$\lim_{x \to a} f(x) = f(a)$$

is satisfied.

For this question,

$$\lim_{x\to1} f(x) = \lim_{x\to1}\frac{x^{m+1}-(m+1)x+m}{(x-1)^2}$$

$$= \lim_{x\to1}\frac{(x^{m+1}-x)-m(x-1)}{(x-1)^2}$$

$$= \lim_{x\to1}\frac{x(x^m-1)-m(x-1)}{(x-1)^2}$$

$$= \lim_{x\to1}\frac{x(x^{m-1}+x^{m-2}+...+1)-m}{(x-1)}$$

$$= \lim_{x\to1}\frac{(x^m-1)+(x^{m-1}-1)+...(x-1)}{(x-1)}$$

$$= m+(m-1)+...+1$$

$$= \frac{m(m+1)}{2}$$

Therefore

$$f(1) = \frac{m(m+1)}{2}$$ ◀

Example – 2

Find the value of $f\left(\frac{\pi}{4}\right)$ if the function

$$f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, \qquad x \neq \frac{\pi}{4}$$

is continuous at $x = \frac{\pi}{4}$

Solution: This question is very similar to the previous one:

$$\lim_{x\to\pi/4} f(x) = \lim_{x\to\pi/4}\frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$= \lim_{x\to\pi/4} \frac{\left(\sqrt{2}\cos x - 1\right)\cdot(\sin x)}{(\cos x - \sin x)}$$

$$= \lim_{x\to\pi/4} \frac{(\sqrt{2}\cos x - 1)}{(\cos x - \sin x)} \cdot \frac{(\sqrt{2}\cos x + 1)}{(\sqrt{2}\cos x + 1)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} \cdot \sin x$$

$$= \lim_{x\to\pi/4} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{(\cos x + \sin x)}{\left(\sqrt{2}\cos x + 1\right)} \cdot \sin x$$

$$= \lim_{x\to\pi/4} \cdot \frac{\cos x + \sin x}{\sqrt{2}\cos x + 1} \cdot \sin x$$

$$= \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{\sqrt{2}.\frac{1}{\sqrt{2}} + 1} . \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}$$

Therefore,

$$f\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

◀

Example – 3

Find the values of a and b if

$$f(x) = \begin{cases} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous at $x = 0$

Solution: We obviously require

$$\lim_{x\to a} f(x) = f(0) = 1$$

(Note that in all the three examples above, we have not found the LHL and RHL separately; we only determine the limit assuming it exists, or in other words assuming that LHL = RHL. We will soon encounter cases where LHL and RHL need to be separately determined)

We use the expansion series for sin x and cos x to get:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x + ax\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - b\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}{x^3}$$

Must be 0

$$= \lim_{x\to 0} \frac{\boxed{(1+a-b)}\, x - \left(\frac{a}{2!} - \frac{b}{3!}\right)x^3 + \left(\frac{a}{4!} - \frac{b}{5!}\right)x^5 \ldots}{x^3}$$

Now notice that $(1 + a - b)$ must be necessarily 0 otherwise $\lim_{x\to 0} \frac{(1+a-b)x}{x^3}$ will become infinite,

Hence,

$$1 + a - b = 0 \qquad \text{...(i)}$$

Also,

$$\lim_{x\to 0} f(x) = -\left(\frac{a}{2!} - \frac{b}{3!}\right) = 1$$

$$6a - 2b + 12 = 0 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get

$$a = \frac{-5}{2} \qquad b = \frac{-3}{2}$$

◀

Example – 4

Find the condition on $f(x)$ and $g(x)$ which makes the function

$$F(x) = \lim_{n\to\infty} \frac{f(x) + x^{2n} g(x)}{1 + x^{2n}} \text{ continuous everywhere.}$$

Solution: Note from the definition of $F(x)$ that we have a variable n present. Lets first try to make $F(x)$ independent of n.

The obvious way is to consider three separate cases for x:

$$|x| = 1, \quad |x| > 1, \quad |x| < 1$$

$\boxed{|x| = 1}$:

$$F(x) = \frac{f(x) + (1)\cdot g(x)}{1+1} = \frac{f(x) + g(x)}{2}$$

$\boxed{|x|>1}$:

$$F(x)=\lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$$

$$=\lim_{n\to\infty}\frac{x^{-2n}f(x)+g(x)}{x^{-2n}+1}$$

$$= g(x) \quad \left(\text{because } x^{-2n}\to 0 \text{ as } n\to\infty \text{ for } |x|>1\right)$$

$\boxed{|x|<1}$:

$$F(x)=\lim_{n\to\infty}\frac{f(x)+x^{2n}g(x)}{1+x^{2n}}$$

$$= f(x) \quad \left(\text{because } x^{2n}\to 0 \text{ as } n\to\infty \text{ for } |x|<1\right)$$

Therefore, we can now rewrite $F(x)$ independently of n in the following manner

$$F(x)=\begin{cases} f(x) & \text{when } |x|<1 \quad \text{or} \quad -1<x<1 \\ \dfrac{f(x)+g(x)}{2} & \text{when } |x|=1 \quad \text{or} \quad x=1,-1 \\ g(x) & \text{when } |x|>1 \quad \text{or} \quad x<-1, x>1 \end{cases}$$

$F(x)$ could be discontinuous at only 2 points, $x = 1$ or $x = -1$

To ensure continuity at these points

$$\text{LHL } (\text{at } x=-1)=\text{RHL}(\text{at } x=-1)=F(-1)$$

$$g(-1)=f(-1)=\frac{f(-1)+g(-1)}{2}$$

$$\Rightarrow g(-1)=f(-1)$$

$$\text{LHL } (\text{at } x=1)=\text{RHL}(\text{at } x=1)=F(1)$$

$$f(1)=g(1)=\frac{f(1)+g(1)}{2}$$

$$\Rightarrow f(1)=g(1)$$

Therefore, for continuity of $F(x)$,

$f(1) = g(1)$

$f(-1) = g(-1)$

Example – 5

If $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}}, & \frac{-\pi}{6}<x<0 \\ b, & x=0 \\ e^{\frac{\tan 2x}{\tan 3x}}, & 0<x<\frac{\pi}{6}\end{cases}$ is continuous at $x = 0$, find the values of a and b.

Solution: For continuity at $x = 0$,

LHL (at $x = 0$) = $f(0)$ = RHL (at $x = 0$)

$$\text{LHL}(\text{at } x=0)=\lim_{x\to 0^-} f(x)$$

$$=\lim_{x\to 0^-}(1+|\sin x|)^{\frac{a}{|\sin x|}}$$

$$=\lim_{y\to 0}(1+y)^{\frac{1}{y}.a}\quad\left\{\begin{array}{l}\text{where } |\sin x|=y;\\ \text{as } x\to 0^-, y\to 0\end{array}\right\}$$

$$=e^a$$

$$\text{RHL}(\text{at } x=0)=\lim_{x\to 0^+} f(x)$$

$$=\lim_{x\to 0^+} e^{\frac{\tan 2x}{\tan 3x}}$$

$$=\lim_{x\to 0^+} e^{\frac{2}{3}.\frac{\tan 2x}{2x}\frac{3x}{\tan 3x}}$$

$$=e^{2/3}$$

$$\Rightarrow e^a=b=e^{2/3}$$

$$\Rightarrow a=\frac{2}{3} \text{ and } b=e^{2/3}$$

◀

Example – 6

Let $f:\mathbb{R}\to\mathbb{R}$ be a function satisfying $f(x+y)=f(x)+f(y)$ for all $x, y\in\mathbb{R}$. If the function $f(x)$ is continuous at $x = 0$ show that it is continuous for all $x\in\mathbb{R}$

Solution: In questions that have such functional equations, we should try to substitute certain trial values for the variables to gain useful information; in this particular question, since we are given some condition about $f(x)$ at $x = 0$, we should try to find $f(0)$

Putting $x = y = 0$, we get

$$f(0) = 2f(0)$$

$$\Rightarrow f(0) = 0$$

Now, since $f(x)$ is continuous at $x = 0$,

$$\text{LHL} = \text{RHL} = f(0)$$

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

or equivalently

$$\lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = f(0) = 0$$

For continuity at an arbitrary value of x, the LHL, RHL and $f(x)$ should be equal.

$$\text{LHL}(\text{at } x) = \lim_{h \to 0} f(x-h)$$

$$= \lim_{h \to 0} \{f(x) + f(-h)\}$$

$$= \lim_{h \to 0} f(x) + \lim_{h \to 0} f(-h)$$

$$= f(x)$$

Similarly,

$$\text{RHL (at } x) = f(x)$$

Therefore, $f(x)$ is continuous for all values of x ◀

Example – 7

Let $f(x) = \begin{cases} \dfrac{\sin 2x + \sin x}{x} & x < 0 \\ a & x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}} & x > 0 \end{cases}$. Find the values of a and b so that $f(x)$ is continuous at $x = 0$.

Solution: $\text{LHL}(\text{at } x = 0) = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{\sin 2x + \sin x}{x}$$

$$= \lim_{x \to 0} \left\{ \frac{2.\sin 2x}{2x} + \frac{\sin}{x} \right\}$$

$$= 3$$

$f(0) = a$

$$\text{RHL}(\text{at } x=0) = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0} \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}$$

$$= \lim_{x\to 0} \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} \times \frac{\left(\sqrt{x+bx^2}+\sqrt{x}\right)}{\left(\sqrt{x+bx^2}+\sqrt{x}\right)}$$

$$= \lim_{x\to 0} \frac{bx^2}{bx^{3/2}\left(\sqrt{x+bx^2}+\sqrt{x}\right)}$$

$$= \lim_{x\to 0} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{x+bx^2}}$$

$$= \lim_{x\to 0} \frac{1}{1+\sqrt{1+bx}}$$

$$= \frac{1}{2}$$

Since LHL $\neq$ RHL, no such values of a and b exist that could make the function continuous at $x = 0$ ◀

Example – 8

Let $F(x) = \begin{cases} \left(1+x+\dfrac{f(x)}{x}\right)^{\frac{1}{x}}, & x \neq 0 \\ e^3, & x = 0 \end{cases}$ and $G(x) = \begin{cases} \left(1+\dfrac{f(x)}{x}\right)^{\frac{1}{x}}, & x \neq 0 \\ k, & x = 0 \end{cases}$

where $f(x)$ is some function of x. If $F(x)$ is continuous at $x = 0$, find the value of k so that $G(x)$ is also continuous at $x = 0$.

Solution: Since $F(x)$ is continuous at $x = 0$,

$$\lim_{x\to 0} F(x) = F(0) = e^3$$

$$\Rightarrow \lim_{x\to 0}\left(1+x+\frac{f(x)}{x}\right)^{\frac{1}{x}} = e^3$$

$$\Rightarrow \lim_{x\to 0} e^{\frac{1}{x}\cdot\left\{x+\frac{f(x)}{x}\right\}} = e^3$$

$$\Rightarrow \lim_{x\to 0}\left(1+\frac{f(x)}{x^2}\right) = 3$$

$$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2} = 2$$

For $G(x)$ to be continuous at $x = 0$, k should be equal to $\lim\limits_{x\to 0} G(x)$

$$\lim_{x\to 0} G(x) = \lim_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{1}{x}}$$

$$= e^{\lim\limits_{x\to 0}\frac{1}{x}\cdot\left(\frac{f(x)}{x}\right)}$$

$$= e^{\lim\limits_{x\to 0}\frac{f(x)}{x^2}}$$

$$= e^2$$

Therefore,

$$k = e^2$$

◀

Example – 9

Show that the function $f(x) = \begin{Bmatrix} \dfrac{e^{1/x}-1}{e^{1/x}+1} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{Bmatrix}$ is discontinuous at $x = 0$

Solution: $\text{LHL}(\text{at } x = 0) = \lim\limits_{h\to 0} f(-h)$

$$= \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1}$$

Now, as $h \to 0, \dfrac{-1}{h} \to -\infty$ so $e^{-1/h} \to 0$

Therefore , LHL = –1

Similarly,

$$\text{RHL} = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{h\to 0} \frac{1-e^{-1/h}}{1+e^{-1/h}}$$

$$=1$$

Since $\text{LHL} \neq \text{RHL} \neq f(0)$, $f(x)$ is discontinuous at $x = 0$ ◀

Example – 10

Let $f(x) = \begin{pmatrix} \dfrac{a^{2[x]+\{x\}}-1}{2[x]+\{x\}} & x \neq 0 \\ \ell n\, a & x = 0 \end{pmatrix}$. Evaluate the continuity of $f(x)$ at $x = 0$.

Solution: The LHL and RHL might differ due to the discontinuous nature of $[x]$ and $\{x\}$. Lets determine their value:

$$\text{LHL}(\text{at } x = 0) = \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} \frac{a^{2[-h]+\{-h\}}-1}{2[-h]+\{-h\}}$$

For $h > 0$, it is obvious that $[-h] = -1$ and $\{-h\} = 1-h$.

Therefore, $\text{LHL} = \lim_{h\to 0} \frac{a^{-2+1-h}-1}{-2+1-h}$

$$= \lim_{h\to 0} \frac{a^{-1-h}-1}{-1-h}$$

$$= \frac{a^{-1}-1}{-1}$$

$$= 1 - a^{-1}$$

Similarly,

$$\text{RHL} = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} \frac{a^{2[h]+\{h\}}-1}{2[h]+\{h\}}$$

$$= \lim_{h\to 0} \frac{a^{h}-1}{h}$$

$$= \ln a$$

Since, $\text{LHL} \neq \text{RHL}$; $f(x)$ is discontinuous at $x = 0$. ◀

Note:

(i) Some authors talk about removable and irremovable discontinuities. Let us discuss what this means:

Consider $f(x) = \dfrac{x^2-1}{x-1}, x \neq 1$

The LHL and RHL at $x = 1$ exist and both are equal to 2.

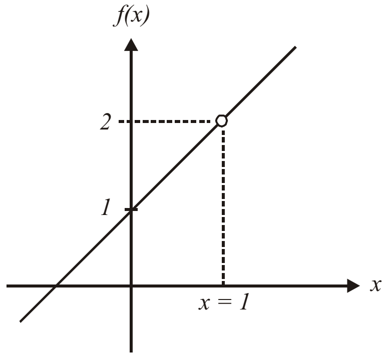

Fig - 6

There is a hole in the graph at $x = 1$ and therefore the function is discontinuous at $x = 1$

We can, if we want to, fill this hole by redefining the function in the following manner:

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 2 & x = 1 \end{cases}$$

The additional definition at $x = 1$ fills the hole and removes the discontinuity. Hence, such a discontinuity would be called a *removable discontinuity.*

By now, you should have realised that a discontinuity is removable only if the LHL and RHL are equal, since only then we can redefine $f(a)$ to make all the three quantities equal.

If the LHL and RHL are themselves non-equal, no redefinition of $f(a)$ could possibly make the function continuous and hence, such a discontinuity would be called irremovable. For example, $f(x) = [x]$ and $f(x) = \{x\}$ suffer from irremovable discontinuities at all integers.

(ii) If f and g are two continuous functions at a point $x = a$ (which is common to their domains), then $f \pm g$ and fg will also be continuous at $x = a$. Furthermore, if $g(a) \neq 0$, then $\dfrac{f}{g}$ will also be continuous at $x = a$.

(iii) If g is continuous at $x = a$ and f is continuous at $x = g(a)$, then $f(g(x))$ will be continuous at $x = a$.

(iv) Any polynomial function is continuous for all values of x.

(v) The functions $\sin x$, $\cos x$ and e^x (or a^x) are continuous for all values of x. $\ln x$ (or $\log_a x$) is continuous for all $x > 0$.

TRY YOURSELF - I

Q. 1 Find the values of a and $f(0)$ if $f(x)$ is continuous at $x = 0$, where

$$f(x) = \frac{\sin 2x + a\sin x}{x^3} \quad , x \neq 0$$

Q. 2 Find the value of a if $f(x)$ is continuous at $x = 0$, where

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & x < 0 \\ a & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & x > 0 \end{cases}$$

Q. 3 If the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\log(1+ax) - \log(1-bx)}{x} & x \neq 0 \\ k & x = 0 \end{cases}$$

is continuous at $x = 0$, find k.

Q. 4 Let $f(xy) = f(x)\,f(y)$ for every $x, y \in \mathbb{R}$. If $f(x)$ is continuous at any one point $x = a$, then prove that $f(x)$ is continuous for all $x \in \mathbb{R} \sim \{0\}$.

Q. 5 Find the values of a and b so that $f(x)$ is continuous at $x = 0$, where

$$f(x) = \begin{cases} 3 & x = 0 \\ \left(\dfrac{1+ax+bx^3}{x^2}\right)^{1/x} & x \neq 0 \end{cases}.$$

Q. 6 Discuss the continuity of $f(x) = [x] + [-x]$ at integer points.

Q. 7 Discuss the continuity of $f(x)$ in $[0, 2]$ where

$$f(x) = \begin{cases} \cos \pi x & x \leq 1 \\ |2x-3|[x] & x > 1 \end{cases}$$

Q. 8 If $f(x) = \begin{cases} -1 & , x < 0 \\ 0 & , x = 0 \\ 1 & , x > 0 \end{cases}$ and $g(x) = x(1-x^2)$, then discuss the continuity of $f(g(x))$, and $g(f(x))$.

Section - 2 DIFFERENTIABILITY

Having seen the concept and the physical (graphical) significance of continuity, we now turn our attention to the concept of differentiability.

We will again start with an intuitive, graphical introduction.

Consider the two graphs given in the figure below:

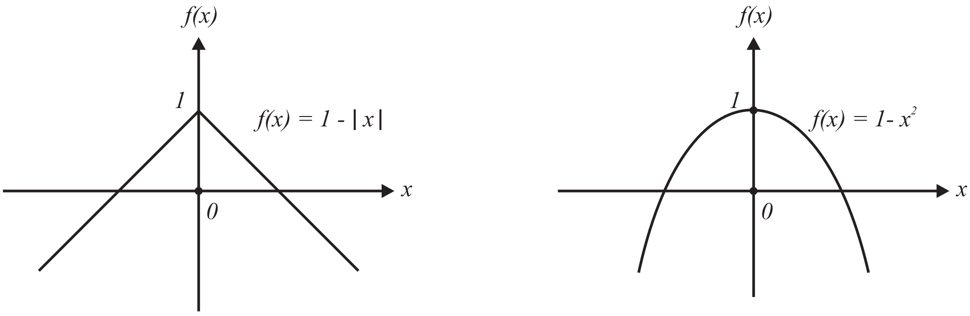

Fig - 7

What is the striking difference between the two graphs at the origin (apart from one being linear and the other, non-linear)? The first has a sharp, sudden turn at the y-axis while the second passes 'smoothly' through the y-axis.

Lets make this idea more concrete. Imagine a person called Theta walking on the graph of $f(x) = 1 - |x|$, towards the y-axis, once from the left and once from the right:

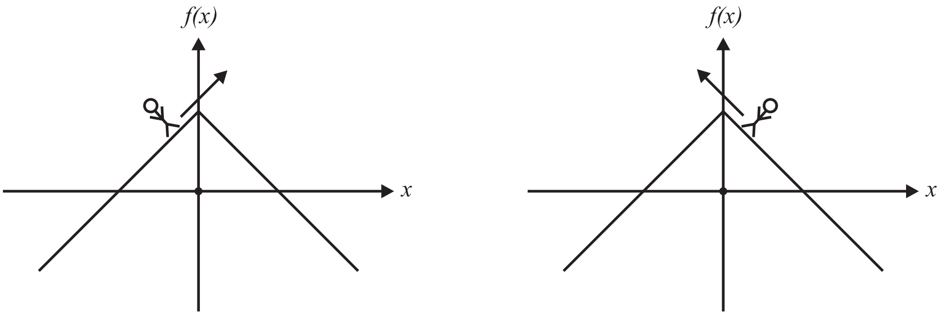

Fig - 8

While walking from the left, Theta will be moving in a north-east direction as he approaches the y-axis; while walking from the right, he will be walking in a north-west direction.

Now consider Theta walking on $f(x) = 1 - x^2$, once from the left and once from the right, towards $x = 0$.

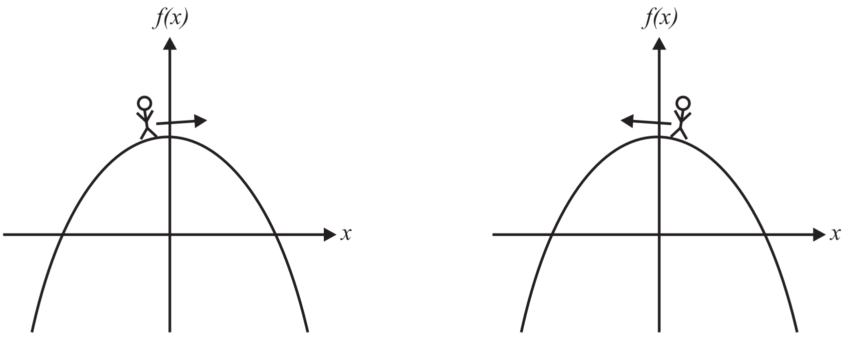

Fig - 9

As Theta approaches the y-axis, we see that he moves 'almost' horizontally near the y-axis, in both the cases. The line of travel becomes almost the same from either side near $x = 0$. At $x = 0$, the line of travel becomes precisely horizontal (for an instant), whether Theta is walking from the left or the right. (This unique line of travel is obviously the tangent drawn at $x = 0$)

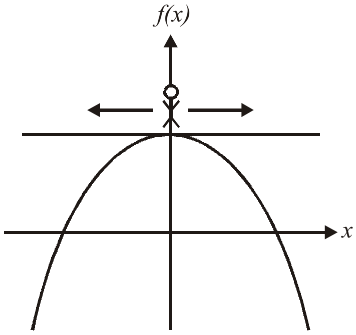

Fig - 10

In mathematical language, since the lines of travel from both sides tend to become the same as the y-axis is approached or more precisely, the tangent drawn to the immediate left of $x = 0$ and the one to the immediate right, become precisely the same at $x = 0$ (a unique tangent), we say that the function $f(x) = 1 - x^2$ is differentiable at $x = 0$. This means that the graph is "smoothly varying" around $x = 0$ or there is no sharp turn at $x = 0$.

In the case of $f(x) = 1 - |x|$, the lines of travel from the left hand and the right hand sides are different. The line of travel (or tangent) to the immediate left of $x = 0$ is inclined at 45º to the x-axis while the one to the immediate right is inclined at 135º. Precisely at $x = 0$, there is no unique tangent that can be drawn to $f(x)$. We therefore say that $f(x) = 1 - |x|$ is non-differentiable at $x = 0$. This means that the graph has a sharp point (or turn) at $x = 0$, as is evident from Fig. - 7.

Section - 3 | LEFT AND RIGHT DERIVATIVES

Let us now put the discussions we have done above in concrete mathematical form.

Consider the curve $f(x) = 1 + |x| - x^2$.

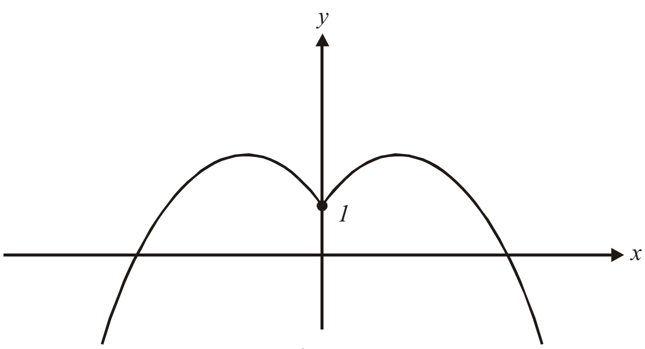

Fig - 11

Theta is walking on this curve towards the y-axis from the right. When he is infinitesmally close to the y-axis, his direction of travel will be along the tangent drawn to the right segment of the graph, at an x-coordinate in the immediate right neighbourhood of the origin ; or equivalently, at a point on the right segment of the graph which is infinitesmally close to the point (0, 1).

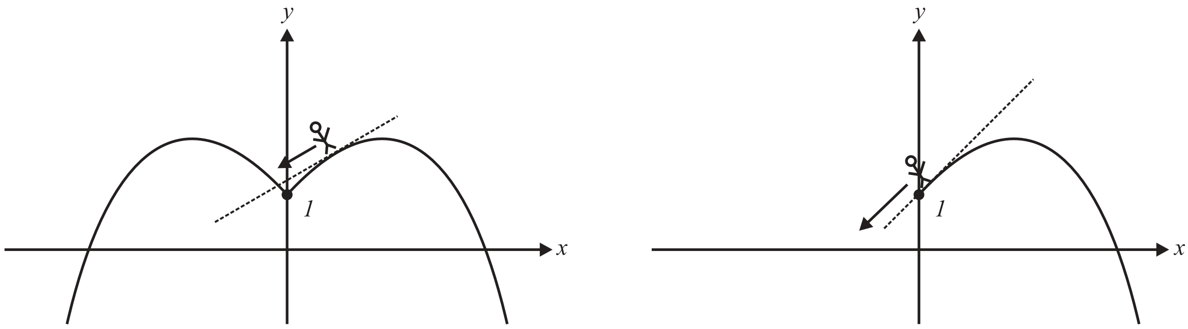

Fig - 12

How do we find out this direction of travel near the point (0, 1)? In other wards, how do we find out the slope of a tangent drawn to the right part of the graph, at a point extremely (infinitesmally) near to (0, 1)?

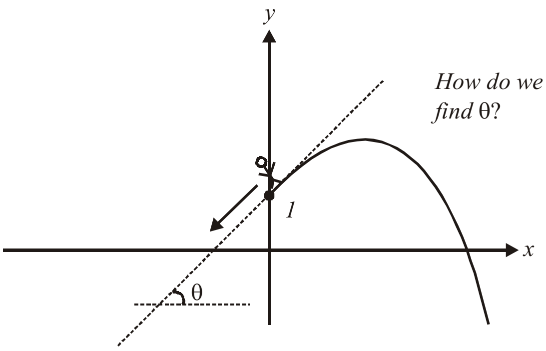

Fig - 13

To evaluate this slope, we first drawn a secant on this graph, passing through (0, 1), as shown in the figure below:

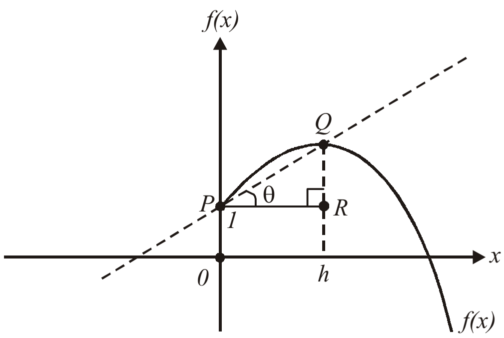

Fig - 14

Let the x-coordinate of the point Q be h. The slope of the secant PQ is

$$\tan\theta = \frac{QR}{PR}$$

Notice that QR is $f(h) - f(0)$ and PR is h. Therefore,

$$\tan\theta = \frac{f(h) - f(0)}{h} \qquad \text{...(i)}$$

Now we make this secant closer to a tangent by reducing h: look at the figure below; as h is reduced or as $h \to 0$, the secant PQ 'tends' to become a tangent drawn 'at' P (or more accurately, a tangent at a point infinitesmally close to the point P):

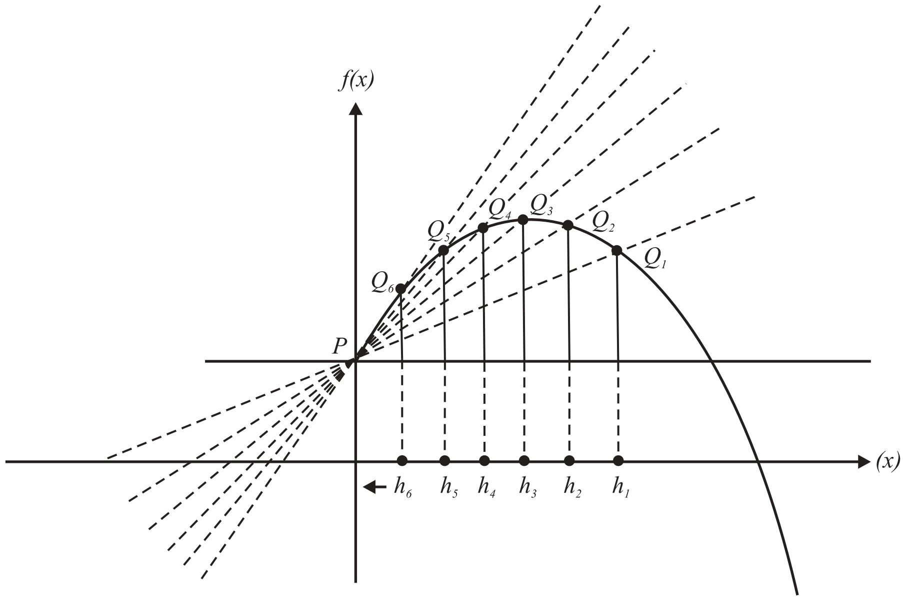

Fig - 15

We see that as the point $Q \to P$ or as $h \to 0,$ the secant PQ tends to become a tangent to the curve; to find the slope of this tangent, we find $\lim_{h\to 0}(\tan\theta)$ where $\tan\theta$ is given by (i)

$$\text{Slope of tangent} = \lim_{h\to 0}\frac{f(h)-f(a)}{h}$$

This limit gives us the slope of the tangent 'at' the point P. (By 'at' we mean 'just near'). Lets evaluate this limit for this particular function:

$$f(x) = 1 + |x| - x^2$$

so $\quad f(h) = 1 + h - h^2$

and $\; f(0) = 1$

$$\text{Slope of tangent} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h-h^2}{h}$$

$$= 1$$

This means that the tangent drawn at P (on the right part of the graph) is inclined at 45º to the x-axis.

In mathematical jargon, the limit we have just evaluated is called the Right Hand Derivative (RHD) of $f(x)$ at $x = 0$. This quantity, as we have seen, gives us the behaviour of the curve (its slope) in the immediate right side vicinity of $x = 0$.

Obviously, there will exist a Left Hand Derivative (LHD) also that will give us the behaviour of the curve in the immediate left side vicinity of $x = 0$. In other words, the LHD will give us the direction of travel of Theta as he is 'just about' to reach the point (0, 1) travelling from the left towards the y-axis.

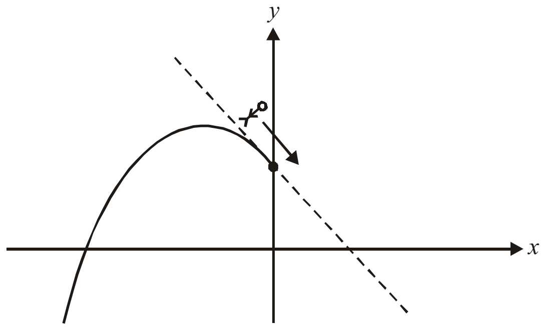

Fig - 16

To evaluate the LHD, we follow a procedure similar to the one we used to evaluate the RHD ; only this time we will drawn the secant PQ on the left side of the graph.

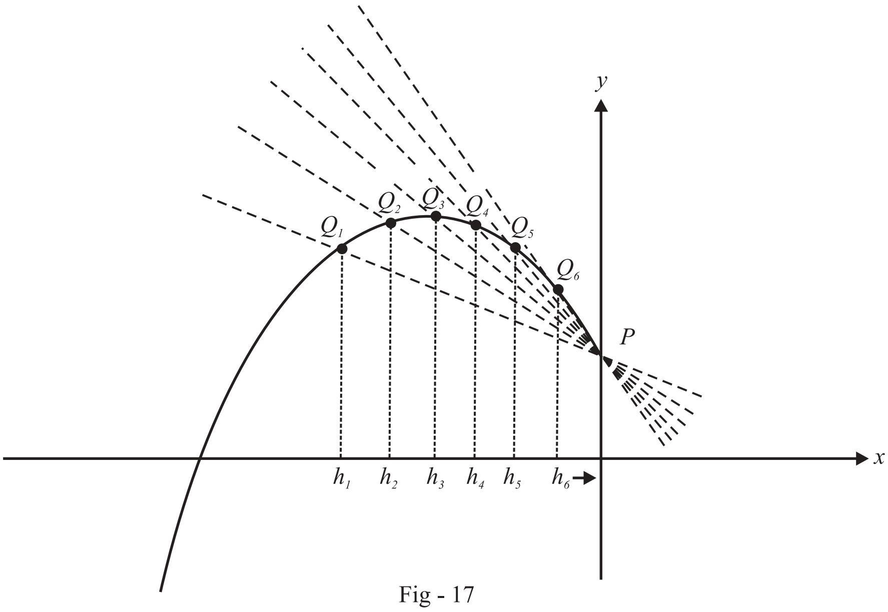

Fig - 17

As the point $Q \to P$ or as $h \to 0$, the secant PQ again tends to become a tangent. As for the previous case, the slope of this tangent will be given by;

$$\underset{(\text{LHD})}{\text{Slope of tangent}} = -\lim_{h \to 0}\left(\frac{f(h) - f(0)}{h}\right) \qquad \{\text{why did we put a negative sign?}\}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{-h}$$

For this particular case:

$$\text{LHD} = \lim_{h \to 0} \frac{(1 + h - h^2) - 1}{-h}$$

$$= -1$$

The tangent drawn to the left part of the graph 'just near' P will be inclined at 135^o to the x-axis.

Notice that because there is a sharp point at $x = 0$, or in other words Theta's direction of travel will change when crossing the y-axis, the LHD and RHD have different values. We would say that this function is non-differentiable at $x = 0$. No tangent can be drawn to $f(x)$ precisely at $x = 0$.

On the other hand, for a 'smooth' function, the LHD and RHD at that point will be equal and such a function would be differentiable at that point. This means that a unique tangent can be drawn at that point.

Before summarizing, let us write down the general expressions for the LHD and RHD

$$\boxed{\begin{array}{l}\text{LHD at } x = a \\ \text{for } y = f(x)\end{array}}$$

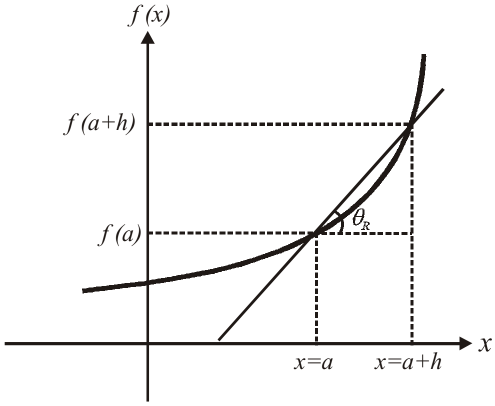

Fig - 18

$$\tan\theta = \frac{f(a) - f(a-h)}{h} = \frac{f(a-h) - f(a)}{-h}$$

$$\Rightarrow \boxed{\text{LHD}(\text{at } x = a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}}$$

$$\boxed{\begin{array}{l}\text{RHD at } x = a \\ \text{for } y = f(x)\end{array}}$$

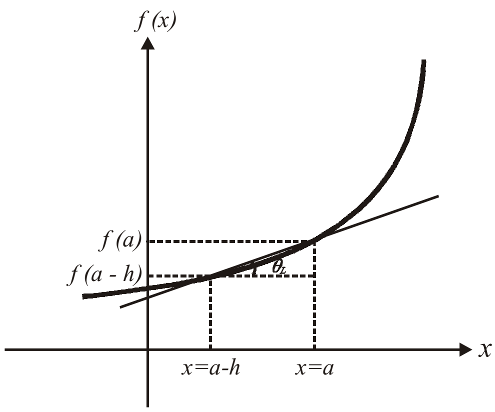

Fig - 19

$$\tan\theta = \frac{f(a+h)-f(a)}{h}$$

$$\Rightarrow \boxed{\text{RHD}(\text{at } x=a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

To summarize:

Consider a continuous function $y = f(x)$

- * If at a particular point $x = a$, the LHD and RHD have equal numerical values, we say that $f(x)$ is differentiable at $x = a$. In graphical terms, this means that the graph crosses $x = a$ 'smoothly', without any sharp turn. This also means that a unique tangent can be drawn to the curve $y = f(x)$ at $x = a$.
  The *derivative at* $x = a$ implies the slope of the tangent at $x = a$; i.e.
  Derivative (at $x = a$) = LHD = RHD
  {Obviously, the derivative exists only if $f(x)$ is differentiable at $x = a$}
  The derivative at $x = a$ is denoted by $f'(a)$
- * If at a particular point $x = a$, the LHD and RHD have non-equal values or one (or both) of them does not exist, we say that $f(x)$ is non differentiable at $x = a$. Graphically, this means that the graph does not pass through $x = a$ 'smoothly', there is a sharp turn at $x = a$.

For a discontinuous function $f(x)$ at $x = a$ we can define LHD and RHD separately. (We 'll have to slightly modify our technique to evaluate LHD and RHD; can you see what modification is required?).

But for any discontinuous function at $x = a$, $f(x)$ would always be non differentiable at $x = a$ since no unique tangent could be drawn to $f(x)$ at $x = a$.

Therefore, for differentiability at $x = a$ the necessary and sufficient conditions that $f(x)$ has to satisfy are:

(i) $f(x)$ must be continuous at $x = a$.

(ii) LHD = RHD at $x = a$.

We will consider some examples to make this discussion more clear.

(i) $\boxed{f(x) = x^2}$

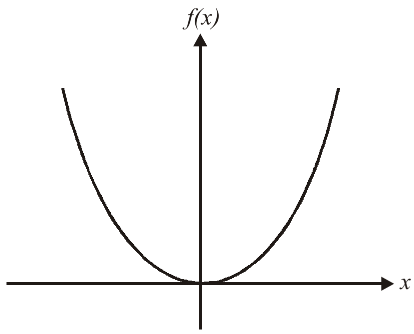

Fig - 20

The graph of $f(x)$ is 'smooth'. Lets verify this by evaluating the LHD and RHD at any general x-coordinate.

$$\text{LHD} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

$$= \lim_{h\to 0}\frac{(x-h)^2 - x^2}{-h}$$

$$= \lim_{h\to 0}\frac{x^2 + h^2 - 2xh - x^2}{-h}$$

$$= \lim_{h\to 0}\frac{h^2 - 2xh}{-h}$$

$$= 2x$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh + h^2}{h}$$

$$= 2x$$

We see that LHD = RHD for every x value. Hence $f(x)$ is everywhere differentiable (smooth). Also the slope at any coordinate x has a numerical value $2x$ (equal to the LHD and RHD).
We say that the derivate of $f(x) = x^2$ is $2x$.

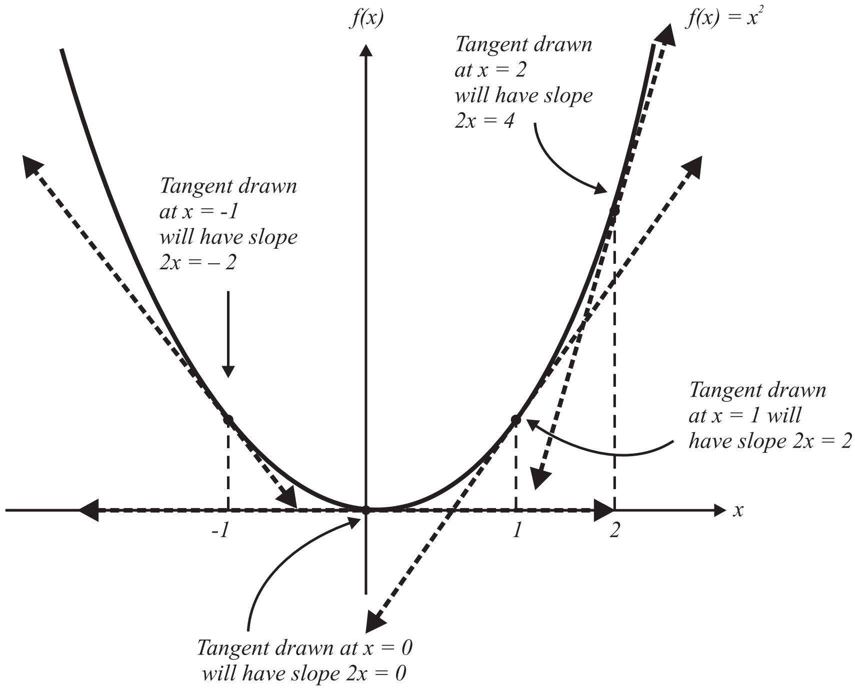

Fig - 21

(ii) $f(x) = |x|$

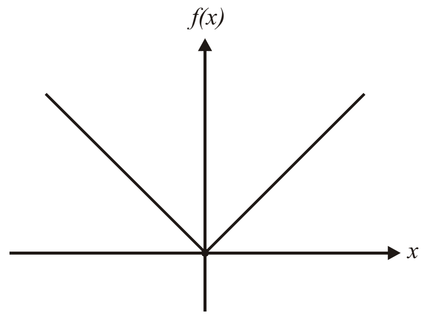

Fig - 22

The graph has a sharp turn (corner) at $x = 0$. This means that at $x = 0$, $f(x) = |x|$ should be non-differentiable. Lets verify this again :

$$\text{LHD} = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} \quad \Big\} \text{ we want to evaluate LHD at } x = 0$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= -1$$

Similarly,

$$\text{RHD} = 1$$

This result is visually obvious from Fig - 22; the left segment is $y = -x$ which has a slope -1, while the right segment is $y = x$, which has slope 1.

(iii) $f(x) = [x]$ and $f(x) = \{x\}$

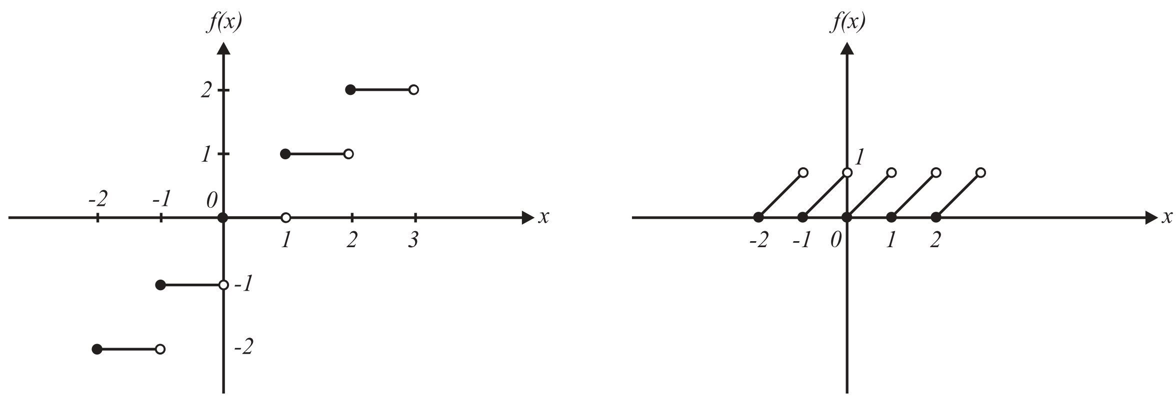

Fig - 23

The analysis of these two functions is straightforward graphically.

For $f(x) = [x]$, the LHD and RHD at any integer are both 0, but $f(x)$ is discontinuous too. Hence, although LHD = RHD, $f(x)$ is non-differentiable at integral points. At all other values of x, it is differentiable with the derivative's value being 0.

Similarly, for $f(x) = \{x\}$, the LHD and RHD at any integer are both 1, but due to discontinuity at all integers, $f(x)$ is non-differentiable at all integers. At all other values of x, $f(x)$ is differentiable with the derivative's value being 1.

(iv) $\boxed{f(x) = \sin x}$

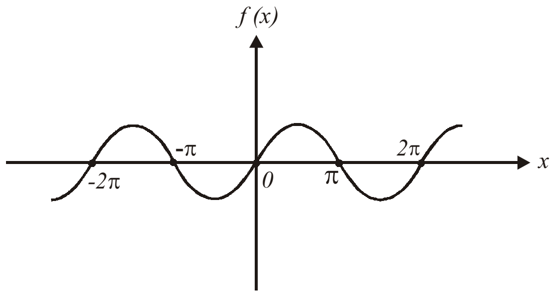

Fig - 24

This function seems 'smooth' everywhere. We will now verify this:

$$\text{LHD} = \lim_{h\to 0} \frac{\sin(x-h) - \sin x}{-h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(x - \frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{-h}$$

$$= \cos x$$

$$\text{RHD} = \lim_{h\to 0} \frac{sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \cos x$$

Hence, LHD = RHD for all values of x; $\sin x$ is everywhere differentiable. The derivative at x (slope of tangent at x) has the numerical value $\cos x$.

$\Rightarrow$ The derivative of sin x is cos x.

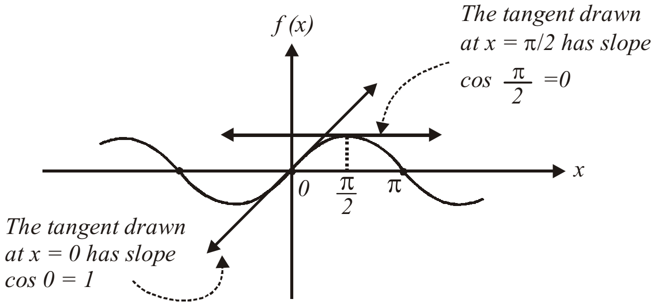

Fig - 25

By now you should be pretty clear about the meaning and graphical significance of differentiability and evaluating the derivative. Evaluating the derivative using the process we have described above is called differentiation using first principles.

This discussion is extremely important to whatever we'll study afterwards. You must ensure that you've fully understand this.

Example – 11

If $f(x) = \begin{cases} 3 - x^2 & , -1 \le x < 2 \\ 2x - 4 & , 2 \le x \le 4 \end{cases}$, discuss its continuity and differentiability.

Solution: In questions involving evaluation of continuity and differentiability, we can of course proceed analytically by writing down the expressions for LHD and RHD (at those points where we feel the function might not be continuous or differentiable); but we will always try to follow a graphical approach also, wherever possible, since it gives much useful information about the behaviour of the function.

Note that in piecewise defined functions such as the one above we have to use different definitions of $f(x)$ in different intervals. So for example, to evaluate $f(a-h)$ we will have to use the function definition for $x < a$, whereas for $f(a+h)$ we have to use the definition for $x > a$.

In this question, there is only one point, namely $x = 2$, where this function could be possibly discontinuous and /or non-differentiable. The two functions $3 - x^2$ and $2x - 4$ are otherwise continuous and differentiable in their separate intervals. Hence, we need to analyse the continuity and differentiability only at $x = 2$ {such points are sometimes referred to as critical points}

Analytical approach:

$$\text{LHL}(\text{at } x=2) = \lim_{x\to 2^-} f(x)$$

$$= \lim_{x\to 2}(3 - x^2)$$

$$= -1$$

$$\text{RHL}(\text{at } x=2) = \lim_{x\to 2^+} f(x)$$

$$= \lim_{x\to 2} 2x-4$$

$$= 0$$

$$f(2)=2(2)-4 \qquad \left\{\begin{array}{l}\text{Note that we used the lower}\\ \text{function to evaluate } f(2) \text{ because of}\\ \text{the placement of equality signwith 2}\\ \text{in } \quad 2x-4, \qquad 2\le x\le 4\end{array}\right\}$$

$$= 0$$

This function is discontinuous at $x = 2$ and therefore also non-differentiable.

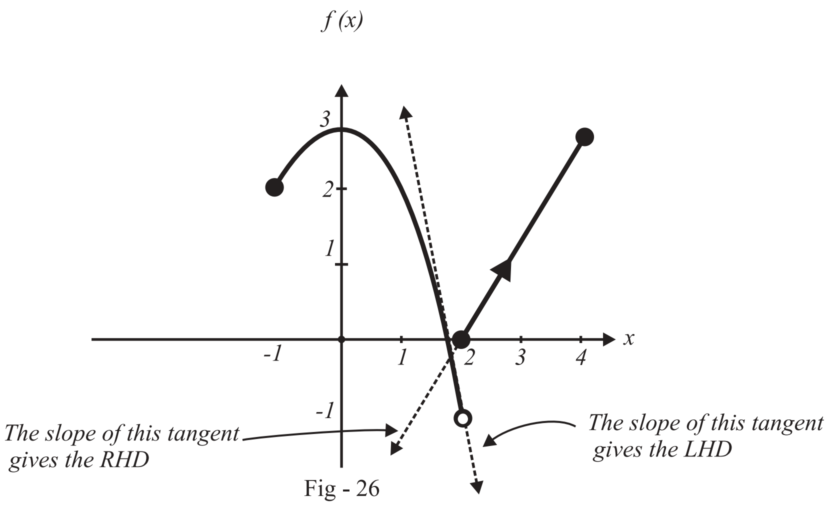

Fig - 26

$$\text{LHD}(\text{at } x=2)=\lim_{h\to 0}\frac{f(2-h)-f(2)}{h}$$

This should be the expression to evaluate LHD at $x = 2$. But notice that for $f(2)$, we cannot substitute '0' since $x = 2$ is part of the other segment i.e. $x = 2$ does not lie in the domain of the left hand function. So what should we use for $f(2)$? Think about this carefully and you will realise that we would have to use $\lim\limits_{x\to 2^-} f(x)$ or LHL at $x = 2$ in place of $f(2)$. (This is the modification we mentioned a little while back). This LHL value is –1.

Therefore,

$$\text{LHD}(\text{at } x=2)=\lim_{h\to 0}\frac{f(2-h)-(-1)}{h}$$

$$=\lim_{h\to 0}\frac{\left\{3-(2-h)^2\right\}+1}{-h}$$

$$= \lim_{h \to 0} \frac{4h - h^2}{-h} = -4$$

$$\text{RHD}(\text{at } x = 2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= 2$$

◀

Example – 12

A function $f(x)$ is defined as:

$$f(x) = \begin{cases} -x^2, & x \le 0 \\ 5x - 4, & 0 < x \le 1 \\ 4x^2 - 3x, & 1 < x \le 2 \\ 3x + 5, & x > 2 \end{cases}$$

. Discuss the continuity and differentiability of $f(x)$.

Solution: The critical points for this function are $x = 0, 1, 2$.

Lets analyse $f(x)$ for each of these critical points separately.

(i) $\boxed{x = 0 \quad \{f(0) = 0\}}$

$$\text{LHL} = \lim_{x \to 0^-} (-x^2) = 0$$

$$\text{LHD} = \lim_{h \to 0} \frac{-(-h)^2 - 0}{-h} = 0$$

$$\text{RHL} = \lim_{h \to 0^+} (5x - 4) = -4$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(h) - (\text{RHL at } x = 0)}{h}$$

$$= \lim_{h \to 0} \frac{5h - 4 + 4}{h} = 5$$

Therefore, this function is non-continuous (and non-differentiable) at $x = 0$.

(ii) $\boxed{x = 1 \quad \{f(1) = 1\}}$

$$\text{LHL} = \lim_{x \to 1^-} (5x - 4) = 1$$

$$\text{LHD} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{5(1-h)-4-1}{-h}$$

$= 5$

$$\text{RHL} = \lim_{h\to 1^+}\left(4x^2-3x\right)=1$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(1+h)-(\text{RHL at } x=1)}{h}$$

$$= \lim_{h\to 0}\frac{4(1+h)^2-3(1+h)-1}{h}$$

$= 5$

Since LHL = RHL and LHD = RHD, $f(x)$ is continuous and differentiable at $x = 1$

(iii) $\boxed{x = 2\{f(2)=10\}}$

$$\text{LHL} = \lim_{x\to 2^-}\left(4x^2-3x\right)=10$$

$$\text{LHD} = \lim_{h\to 0}\frac{f(2-h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{\left\{4(2-h)^2-3(2-h)\right\}-10}{-h}$$

$= 13$

$$\text{RHL} = \lim_{x\to 2^+}(3x+5)=11$$

$$\text{RHD} = \lim_{h\to 0}\frac{f(2+h)-(\text{RHL at } x=2)}{h} \quad \begin{Bmatrix}\text{Observecarefullyhowwe}\\ \text{wrotetheexpressionforRHD}\end{Bmatrix}$$

$$= \lim_{h\to 0}\frac{\{3(2+h)+5\}-11}{h}$$

$= 3$

We see that $f(x)$ is non-continuous (and non-differentiable) at $x = 0$ and $x = 2$.

The graph is plotted below:

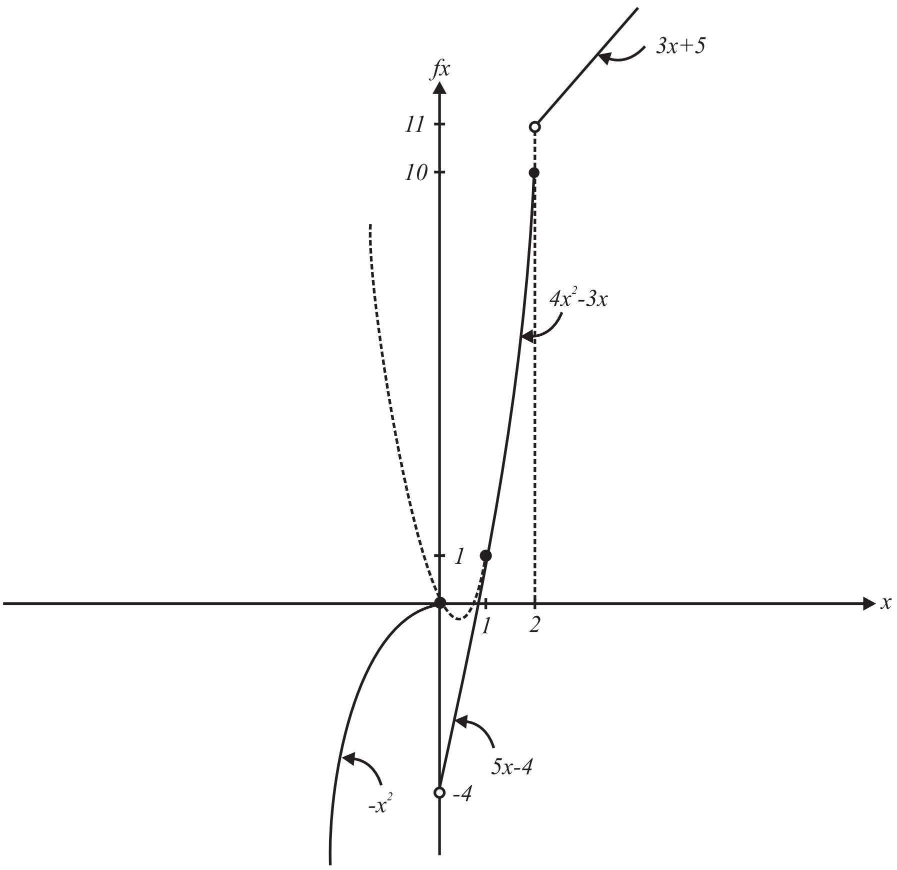

Fig - 27

The elaborate analysis done for this question will not be done for subsequent questions where we will try focus more on the graphical approach.

Example – 13

Let $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x-1, & 0 < x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$.

Evaluate the continuity and differentiability of $g(x)$ in the interval [–2, 2] by drawing the graph.

Solution: From the graph of $f(x)$, we can easily derive the graphs of $f(|x|)$ and $|f(x)|$, and add them point by point to get the graph of $g(x)$.

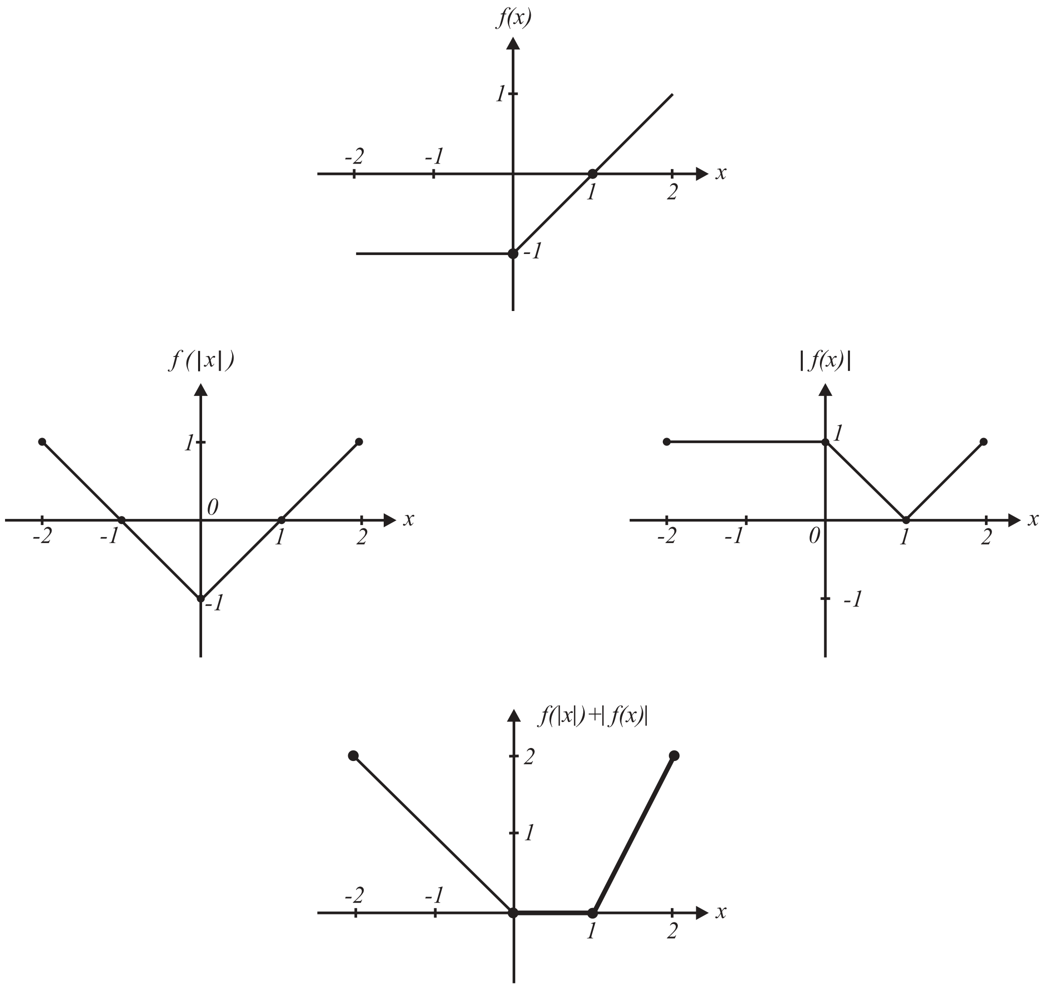

Fig - 28

Verify for yourself the result of the addition of the two graphs.

It is obvious from the resultant graph that $g(x)$ is continuous every where but non-differentiable at $x = 0$ and $x = 1$

Example – 14

If $f(x) = \begin{cases} 2x^2 + 12x + 16, & -4 \le x \le -2 \\ 2 - |x| \quad , & -2 < x \le 1 \\ 4x - x^2 - 2 \quad , & 1 < x \le 3 \end{cases}$, evaluate the continuity and differentiability of $f(x)$

Solution: This graph will have 3 different segments which can easily be plotted as shown in below (you are urged to carry out the plotting on your own):

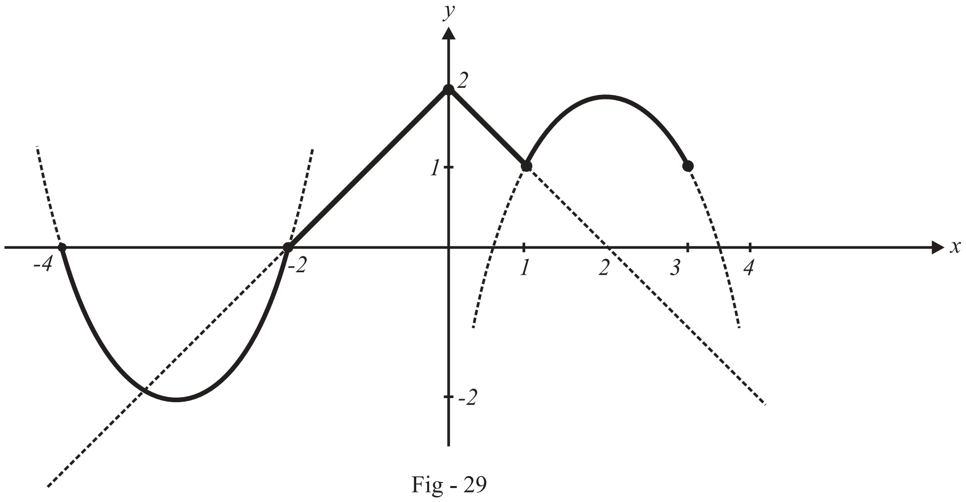

Fig - 29

We first draw all the three functions' graphs on the axis using dotted curves and then darken the relevant portions as specified in the piecewise domain.

For example, we selected only that segment of $2x^2+12x+16$ for which $x \in [-4, -2]$ and so on.

It is clear from the graph that $f(x)$ is continuous in its entire domain [–4, 3] but is non-differentiable at 3 points $x = -2, 0, 1$.

The evaluation of left and right derivatives on each of these three points is left to you as an exercise. The results are:

$\boxed{x=-2}$	LHD = 4	RHD = –1
$\boxed{x=0}$	LHD = 1	RHD = –1
$\boxed{x=1}$	LHD = –1	RHD = 2

◀

Example – 15

A function $f:\mathbb{R}\to\mathbb{R}$ satisfies $\left|f(x)-f(y)\right| \le |x-y|^3$ for all $x, y \in \mathbb{R}$. Prove that $f(x)$ is a constant function.

Solution: For a constant function $f(x) = k$, the tangent drawn at any point would be the same line $y = k$ itself. Hence, the slope of tangent at any point (the derivative at any point) has the value 0.

Therefore, our aim in this question should be to somehow show that $f(x)$ has derivative 0 at all points.

Let us evaluate the RHD at any x:

$$\text{RHD at } x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Therefore,

$$|\text{RHD at } x| = \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right|$$

$$= \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

$$\leq \lim_{h \to 0} \frac{|(x+h) - (x)|^3}{|h|} \qquad \left\{ \text{Using the given relation} \right\}$$

$$= \lim_{h \to 0} |h|^2 = 0$$

$$\Rightarrow |RHD| \leq 0$$

Since $|RHD|$ cannot be negative, it has to be 0.

$$\Rightarrow RHD = 0 \text{ for all } x$$

Similarly, LHD = 0 for all x

$$\Rightarrow f(x) \text{ is a constant function.}$$

◀

Example – 16

Let $f(x) = \begin{cases} xe^{-\left\{\frac{1}{|x|} + \frac{1}{x}\right\}}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$, Evaluate the continuity and differentiability of $f(x)$

Solution: We should first of all write $f(x)$ separately for $x > 0$ and $x < 0$; using $|x| = \begin{cases} x \ , & x > 0 \\ -x, & x < 0 \end{cases}$

$$f(x) = \begin{cases} xe^{-\frac{2}{x}} & x > 0 \\ 0 & x = 0 \\ x & x < 0 \end{cases}$$

At this juncture, we do not have sufficient knowledge to accurately plot the graph for $x > 0$; we will hence follow the analytical approach.

The critical point is only $x = 0$

$$\text{LHL}\left(\text{at } x = 0\right) = \lim_{x \to 0^-} \left(x\right) = 0$$

$$\text{RHL}\left(\text{at } x = 0\right) = \lim_{x \to 0^+} \left(xe^{-2/x}\right) = 0 \quad \left(\text{Verify}\right)$$

$$\text{LHD}\left(\text{at } x = 0\right) = \lim_{h \to 0} \frac{f\left(-h\right) - f\left(0\right)}{-h}$$

$$= 1$$

$$\text{RHD}\left(\text{at } x = 0\right) = \lim_{h \to 0} \frac{f\left(h\right) - f\left(0\right)}{h}$$

$$= \lim_{h \to 0} \frac{he^{-2/h}}{h}$$

$$= 0$$

Therefore, this function is continuous at $x = 0$ (and everywhere else)

but not differentiable at $x = 0$ ◀

TRY YOURSELF - II

Q. 1 If $f(x) = \min\{|x|, |x-2|, 2-|x-1|\}$, draw the graph of $f(x)$ and discuss its continuity and differentiability.

Q. 2 If $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ discuss the continuity and differentiability of $f(x)$ at $x = 0$.

Q. 3 Let $f(x)$ be defined in the interval [–2, 2] such that

$$f(x) = \begin{cases} [x] & , -2 \le x \le -1/2 \\ 2x^2 - 1 & , \dfrac{-1}{2} < x \le 2 \end{cases} \text{ and } g(x) = f(|x|) + |f(x)|.$$

Discuss the differentiability of $g(x)$ in [–2, 2].

Q. 4 If $f(x) = \begin{cases} |x-1|([x]-x), & x \neq 1 \\ 0 & x-1 \end{cases}$, test its differentiability at $x = 1$.

Q. 5 Given $|f(x) - f(y)| < |x-y|^2$, and $f(0) = 1$, find $f(x)$.

Q. 6 If $f(x) = \begin{cases} k\left\{1 - \dfrac{|x|}{l}\right\} & |x| \le l \\ 0 & |x| > l \end{cases}$ discuss the continuity and differentiability of

$g(x) = f(x+1) + f(x-1)$

Q. 7 If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x) + 2f(y)}{3}$ for all $x, y \in \mathbb{R}$, and $f'(0) = 1$, prove that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Q. 8 Find the values of *a*, *b* and *c* if the function $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ is differentiable at $x = 0$.

❖ ❖ ❖ ❖ ❖ ❖ ❖

SOLVED EXAMPLES

Example – 1

If $f(x)=\begin{cases}\dfrac{\sin 3x+a\sin 2x+b\sin x}{x^5}, & x\neq 0\\ c, & x=0\end{cases}$ is continuous at $x = 0$, find a, b and c.

Solution: For continuity, we require:

$$\lim_{x\to 0} f(x)=c \qquad \left[\begin{array}{l}\text{LHL and RHL will be}\\ \text{the same so we do not}\\ \text{need to evaluate them}\\ \text{separately}\end{array}\right]$$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left\{\frac{\sin 3x+a\sin 2x+b\sin x}{x^5}\right\}$$

$$=\lim_{x\to 0}\frac{\left\{3x-\dfrac{(3x)^3}{3!}+\dfrac{(3x)^5}{5!}-\ldots\right\}+a\left\{2x-\dfrac{(2x)^3}{3!}+\dfrac{(2x)^5}{5!}-\ldots\right\}+b\left\{x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots\right\}}{x^5}$$

$$=\lim_{x\to 0}\frac{(3+2a+b)\,x-\left(\dfrac{27}{6}+\dfrac{8a}{6}+\dfrac{b}{6}\right)x^3+\left(\dfrac{243}{120}+\dfrac{32a}{120}+\dfrac{b}{120}\right)x^5-\ldots}{x^5}$$

These terms must be 0.

As in Section -1, example-3, note that

$3 + 2a + b = 0$ ... (i)

$27 + 8a + b = 0$ ... (ii)

$$\left[\begin{array}{l}\text{If this does not hold,}\\ \qquad \displaystyle\lim_{x\to 0}\frac{(3+2a+b)x}{x^5}\ \text{and}\ \lim_{x\to 0}\frac{(27+8a+b)x^3}{x^5}\\ \text{will become infinite}\end{array}\right]$$

Solving (i) and (ii), we get

$a = -4 \qquad b = 5$

The limit now reduces to

$$\lim_{x\to 0}\frac{\left(\frac{243}{120}+\frac{32a}{120}+\frac{b}{120}\right)x^5+\ldots.(\text{higher order terms})}{x^5}$$

$$=\frac{243+32a+b}{120}$$

$= 1$

Hence,

c = 1 ◀

Example – 2

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x+y)=f(x)f(y)$ for all $x, y\in\mathbb{R}$. If $f(x)$ is continuous at $x=0$, show that $f(x)$ is continuous for all $x\in\mathbb{R}$.

Solution: Since $f(x)$ is continuous at $x=0$,

$$\lim_{h\to 0}f(-h)=\lim_{h\to 0}f(h)=f(0) \qquad \ldots \text{(i)}$$

To evaluate $f(0)$, we substitute $x=y=0$ in the given functional relation to get.

$$f(0)=f(0)^2$$

$$\Rightarrow f(0)=0 \text{ or } f(0)=1$$

(i) If $f(0)=0$, then $f(x)=f(x+0)=f(x)\cdot f(0)=0$ i.e

$f(x)=0$ for all values of x so that $f(x)$ is continuous everywhere.

(ii) We now assume $f(0)=1$.

Now,

LHL at any $x=\lim_{h\to 0}f(x-h)$

$=\lim_{h\to 0}f(x)\cdot f(-h)$

$=f(x)\lim_{h\to 0}f(-h)$

$=f(x)\cdot f(0)$ (From (i))

$=f(x)$

Similarly,

$$\text{RHL at any } x = \lim_{h\to 0} f(x+h)$$

$$= \lim_{h\to 0} f(x)\, f(h)$$

$$= f(x)$$

Hence, $f(x)$ is continuous for all $x \in \mathbb{R}$ ◀

Example – 3

Let $f(x) = \begin{cases} x\sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$.

Evaluate the continuity and differentiability of $f(x)$ and $g(x)$.

Solution: We will first try to graphically understand the behaviour of these two functions and then verify our results analytically. Notice that no matter what the argument of the sin function is, its magnitude will always remain between –1 and 1.

Therefore,

$$\left| x\sin\frac{1}{x} \right| \leq |x|$$

and

$$\left| x^2 \sin\frac{1}{x} \right| \leq |x|^2$$

This means that the graph of $x\sin\frac{1}{x}$ will always lie between the lines $y = \pm x$ and the graph of $x^2\sin\frac{1}{x}$ will always lie between the two curves $y = \pm x^2$.

Also, notice that as $|x|$ increases, $\frac{1}{x}$ decreases in a progressively slower manner while when $|x|$ is close to 0, the increase in $\frac{1}{x}$ is very fast (as $|x|$ decreases visualise the graph of $y = \frac{1}{x}$). This means that near the origin, the variation in the graph of $\sin\frac{1}{x}$ will be extremely rapid because the successive zeroes of the graph will become closer and closer. As we keep on increasing x, the variation will become slower and slower and the graph will 'spread out'. For example, for $x > \frac{1}{\pi}$ there will be no finite root of the function. Only when $x \to \infty$ will $\sin\frac{1}{x}$ again approach 0.

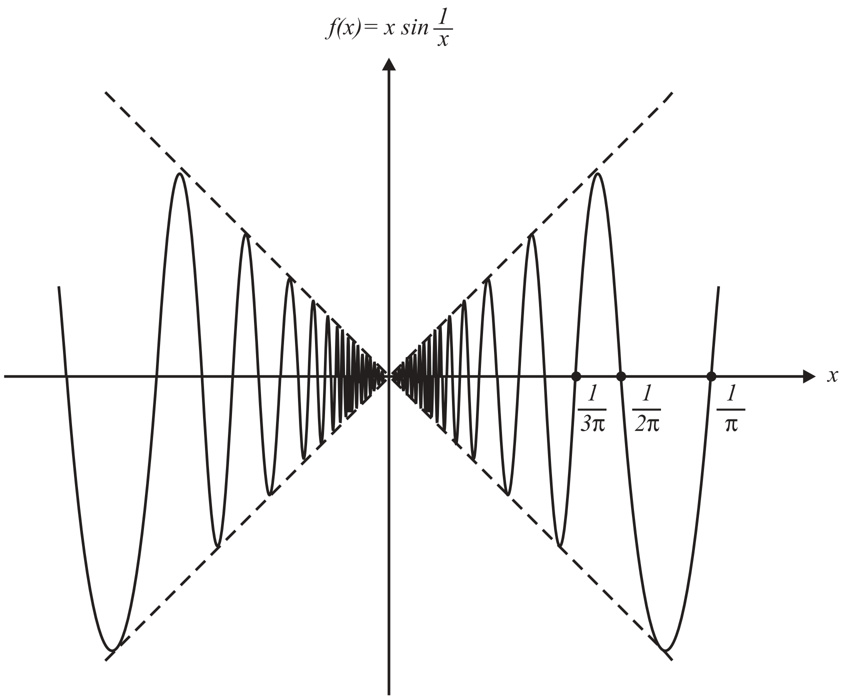

Fig. - 30

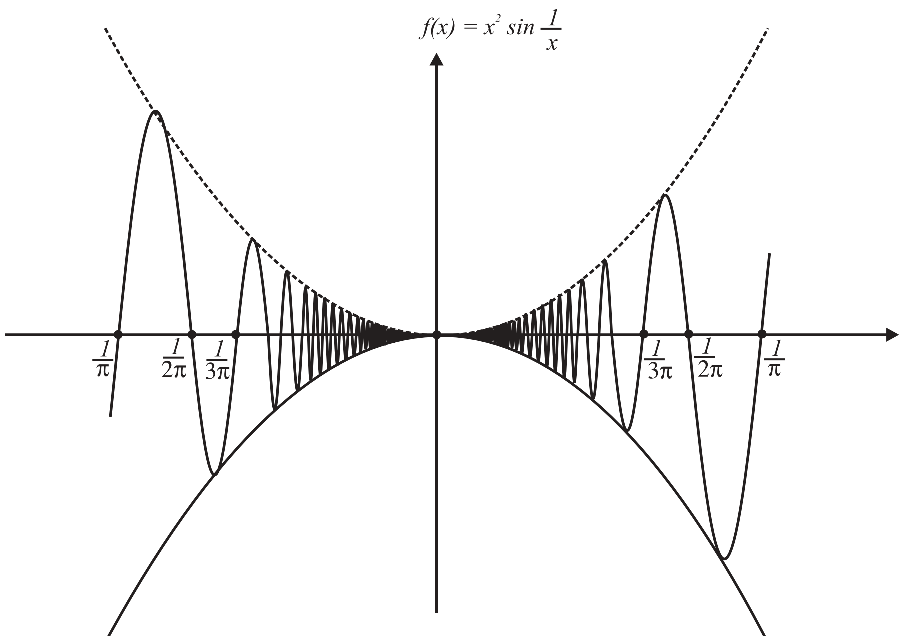

Fig - 31

Figures 30 and 31 show the approximation variation we should expect for these two functions.

Notice how the lines $y = \pm x$ 'envelope' the graph of the function in the first case and the curves $y = \pm x^2$ 'envelope' the graph of the function in the second case.

The envelopes shrink to zero vertical width at the origin in both cases. Therefore, we must have:

$$\lim_{x \to 0} x \sin\frac{1}{x} = 0 \text{ and } \lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$

(This is also analytically obvious; $\sin\frac{1}{x}$ is a finite number between –1 and 1; when it gets multiplied by x(where $x \to 0$), the whole product gets infinitesmally small).

Now lets try to get a 'feel' on what will happen to the derivatives of these two functions at the origin.

For $f(x) = x\sin\frac{1}{x}$, the slope of the envelope is constant (±1). Thus, the sinusoidal function inside the envelope will keep on oscillating as we approach the origin, while shrinking in width due to the shrinking envelope. The slope of the curve also keeps on changing and does not approach a fixed value.

However, for $g(x) = x^2 \sin\frac{1}{x}$, the slope of the envelope is itself decreasing as we approach the origin, apart from shrinking in width. This envelope will 'compress' or 'hammer out' or 'flatten' the sin oscillations near the origin. What should therefore happen to the derivative? It should become 0 at the origin!

Let us 'zoom in' on the graphs of both the functions around the origin, to see what is happening:

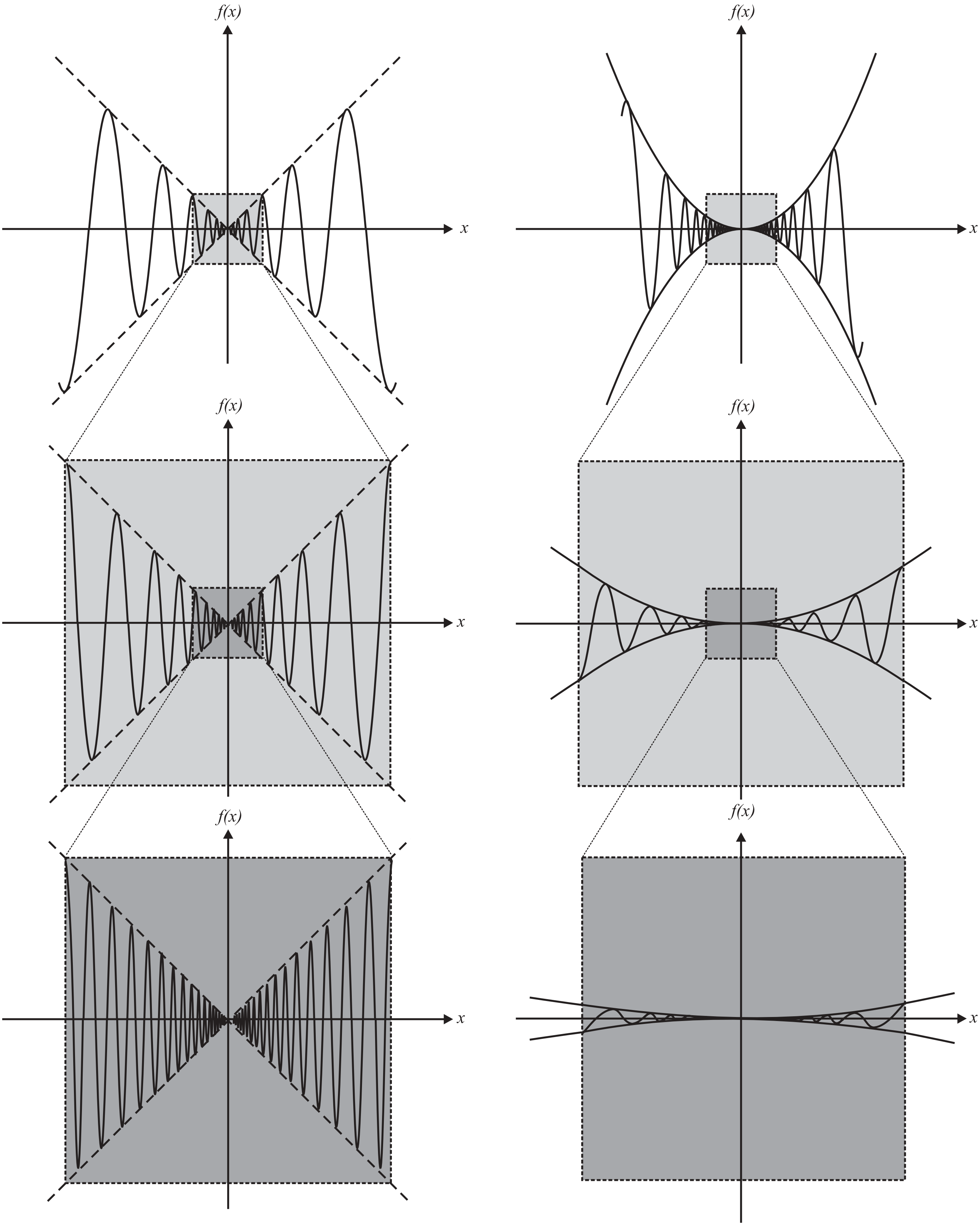

Fig - 32

These graphs are not very accurate and are only of an approximate nature; but they do give us some feeling on the behaviour of these two functions near the origin.

The $x\sin\frac{1}{x}$ graph keeps 'continuing' in the same manner no matter how much we zoom in; however, in the $x^2\sin\frac{1}{x}$ graph, the decreasing slope of the envelope itself tends to flatten out the curve and make its slope tend to 0. Hence, the derivative of $x\sin\frac{1}{x}$ at the origin will not have any definite value, while the derivative of $x^2\sin\frac{1}{x}$ will be 0 at the origin .

Lets verify this analytically:

(i) $\boxed{f(x) = x\sin\frac{1}{x} \text{ at } x = 0}$

$$\text{LHD} = \text{RHD} = \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\frac{1}{h}}{h}$$

$$= \lim_{h\to 0}\left(\sin\frac{1}{h}\right)$$

This limit, as we know, does not exist; hence, the derivative for $f(x)$ does not exist at $x = 0$

(ii) $\boxed{f(x) = x^2\sin\frac{1}{x} \text{ at } x = 0}$

$$\text{LHD=RHD} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h\sin\frac{1}{h}$$

$$= 0$$

This verifies our earlier assertion ◀

Example – 4

If $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and $f(x) = 1 + g(x)G(x)$, where $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} G(x)$ exists, prove that $f(x)$ is continuous for all $x \in \mathbb{R}$.

Solution: We have $\lim_{h\to 0} g(h) = \lim_{h\to 0} g(-h) = 0$

and $\lim_{h\to 0} G(h) = \lim_{h\to 0} G(-h) = k\{\text{some finite number}\}$

Now, LHL of $f(x)$ (at any x)

$$= \lim_{h\to 0} f(x-h) = \lim_{h\to 0} f(x)f(-h)$$

$$= f(x)\lim_{h\to 0}(1 + g(-h)G(-h))$$

$$= f(x)$$

Similarly, RHL (at any x) of $f(x) = f(x)$

$$\Rightarrow f(x) \text{ is continuous for all } x$$

◀

Example – 5

If $f(x) = \begin{Bmatrix} |x^2 - 1| - 1 & , & x \le 1 \\ |2x-3| - |x-2| & , & x > 1 \end{Bmatrix}$, discuss the continuity and differentiability of $f(x)$.

Solution: Note that $f(x)$ can equivalently be rewritten as:

$$f(x) = \begin{Bmatrix} x^2 - 2 & x \le -1 \\ -x^2 & -1 < x \le 1 \\ 1 - x & 1 < x \le 3/2 \\ 3x - 5 & 3/2 < x < 2 \\ x - 1 & 2 \le x \end{Bmatrix}$$

We will now simply draw the graph:

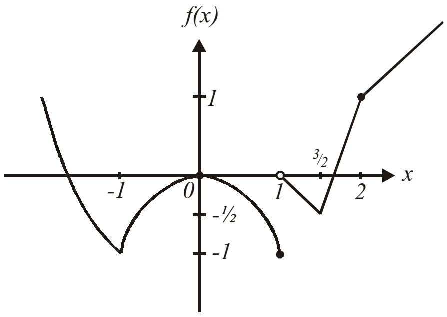

Fig - 33

We see that $f(x)$ is non-continuous at $x = 1$ and non-differentiable at 4 points, $x = -1, 1, \frac{3}{2}, 2$ ◀

Example – 6

Let a function $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x+y)=f(x)f(y)$ for all $x, y\in\mathbb{R}$ and $f(x)\neq 0$ for any $x\in\mathbb{R}$. If the function $f(x)$ is differentiable at $x = 0$, show that $f'(x)=f'(0)f(x)$ for all $x\in\mathbb{R}$.

Solution: Substituting $x = y = 0$ in the given relation, we get

$$f(0)=f(0)^2 \Rightarrow f(0)=1 \qquad \{\text{since } f(x)\neq 0 \text{ for any } x\}$$

It is given that $f(x)$ is differentiable at $x = 0$, i.e., $f'(0)$ exists.

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)-1}{h} \qquad \text{...(i)}$$

Now we write down the expression for $f'(x)$:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h}$$

$$=f(x).\lim_{h\to 0}\frac{f(h)-1}{h}$$

$$=f(x)\cdot f'(0) \qquad \{\text{from (i)}\}$$

$$\Rightarrow \quad f'(x)=f'(0)\cdot f(x) \text{ for all } x$$ ◀

Example – 7

If $f(xy) = f(x) + f(y)$ for all $x, y > 0$ and $f(x)$ is differentiable at $x = 1$, then prove that $f(x)$ is differentiable for all $x > 0$

Solution: Substituting $x = y = 1$, we get $f(1) = 0$

Since $f(x)$ is differentiable at $x = 1$, $f'(1)$ exists

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h)}{h} \qquad \text{...(i)}$$

Now, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

To evaluate $f'(x)$, we have to somehow manipulate its expression so that we are able to use the expression for $f'(1)$ we evaluated in (i)

We do this as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f\left\{x\left(1 + \frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left\{f(x) + f\left(1 + \frac{h}{x}\right)\right\} - f(x)}{h} \qquad \left\{\begin{matrix}\text{Using the relation} \\ \text{given in the question}\end{matrix}\right\}$$

$$= \lim_{h \to 0} \frac{f(1 + h/x)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{x.\frac{h}{x}} \qquad \left\{\begin{matrix}\text{Introduction of } x \text{ in} \\ \text{the denominator}\end{matrix}\right\}$$

$$= \lim_{\theta \to 0} \frac{f(1+\theta)}{x \cdot \theta} \qquad \left\{\theta = \frac{h}{x}\right\}$$

$$= \frac{f'(1)}{x} \qquad \{\text{Using (i)}\}$$

Therefore, $f'(x)$ has a finite value for all $x > 0$

$\Rightarrow$ $f(x)$ is differentiable everywhere. ◀

Example – 8

If $f(x) = x^2 - 2|x|$ and $g(x) = \begin{cases} Min\{f(t) : -2 \le t \le x, & -2 \le x < 0\} \\ Max\{f(t) : \ 0 \le t \le x, & 0 \le x \le 3\} \end{cases}$ draw the graph of $g(x)$ and discuss its continuity and differentiability.

Solution: The graph of *f*(*x*) is sketched below (in the relevant domain):

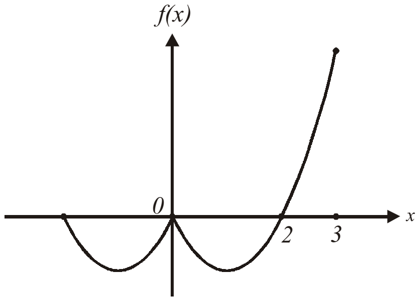

Fig - 34

Now we must understand what the definition of *g*(*x*) means.

Consider the upper definition of *g*(*x*):

$$g(x) = Min\{f(t) : -2 \le t \le x, \quad -2 \le x < 0\}$$

To evaluate $g(x)$ at any *x*, we scan the entire interval from –2 to *x*, (this is what the variable *t* is for), select that value of *f*(*t*) which is minimum in this interval; this minimum value of *f*(*t*) becomes the value of the function at *x*.

The figure below illustrates this graphically for four different values of *x* (we are considering the interval $-2 \le x < 0$, the upper definition of *g*(*x*)):

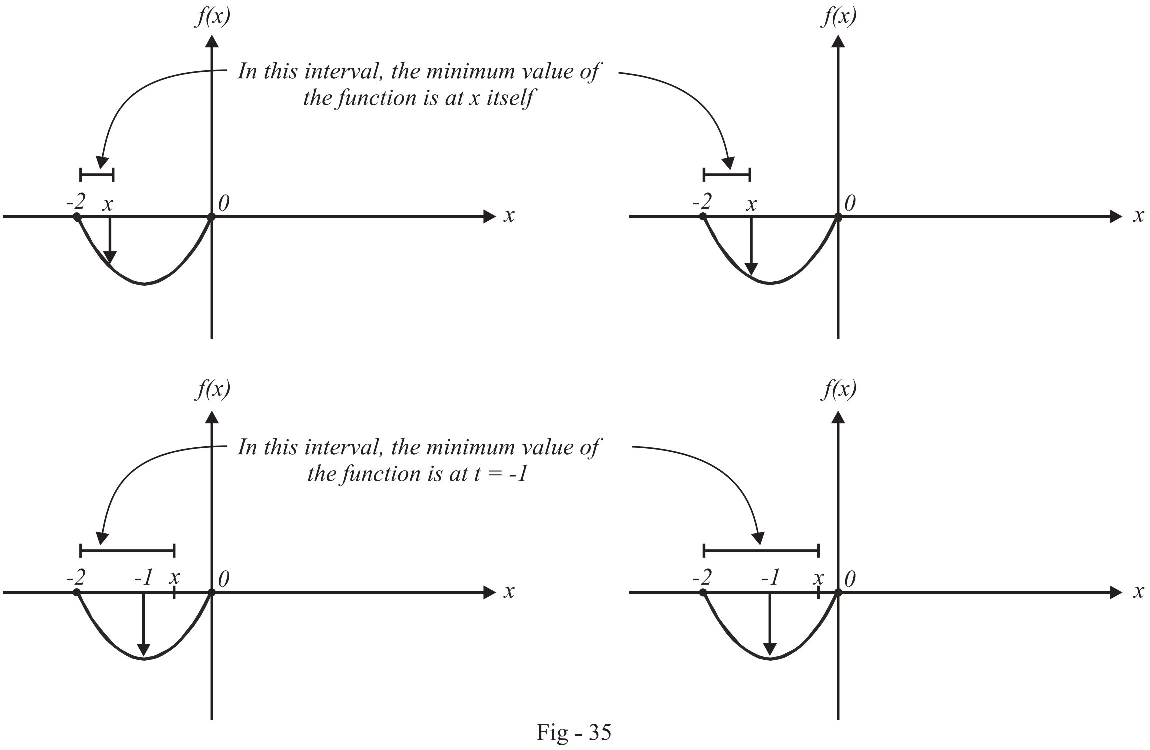

Fig - 35

Notice that where x crosses –1 (when $-1 < x < 0$), the minimum value of the function in the interval $[-2, x]$ becomes fixed at $t = -1$. When $-2 < x < -1$, the minimum value of the function in the interval $[-2, x]$ is at $t = x$.

So, how do we draw the graph of $g(x)$? In $[-2, -1]$, the graph of $g(x)$ will the same as $f(x)$ (because the minimum value of $f(x)$ is at $t = x$ itself, as described above). In $[-1, 0]$, the minimum value becomes fixed at $t = -1$, equal to –1, so that in this interval the graph of $g(x)$ is constant; $g(x) = -1$ for $x \in [-1, 0]$. The graph of $g(x)$ for $x \in [-2, 0]$ is sketched below;

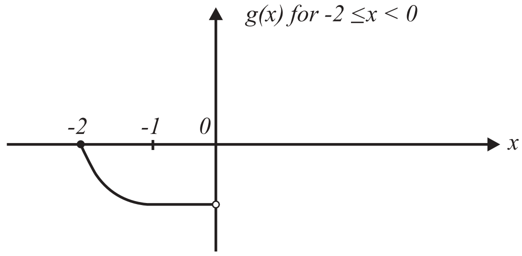

Fig - 36

For $0 \le x \le 3$, the definition is

$g(x) = Max\{f(t); 0 \le t \le x, 0 \le x \le 3\}$. The figure below illustrates how to obtain $g(x)$ in this case for four different values of x:

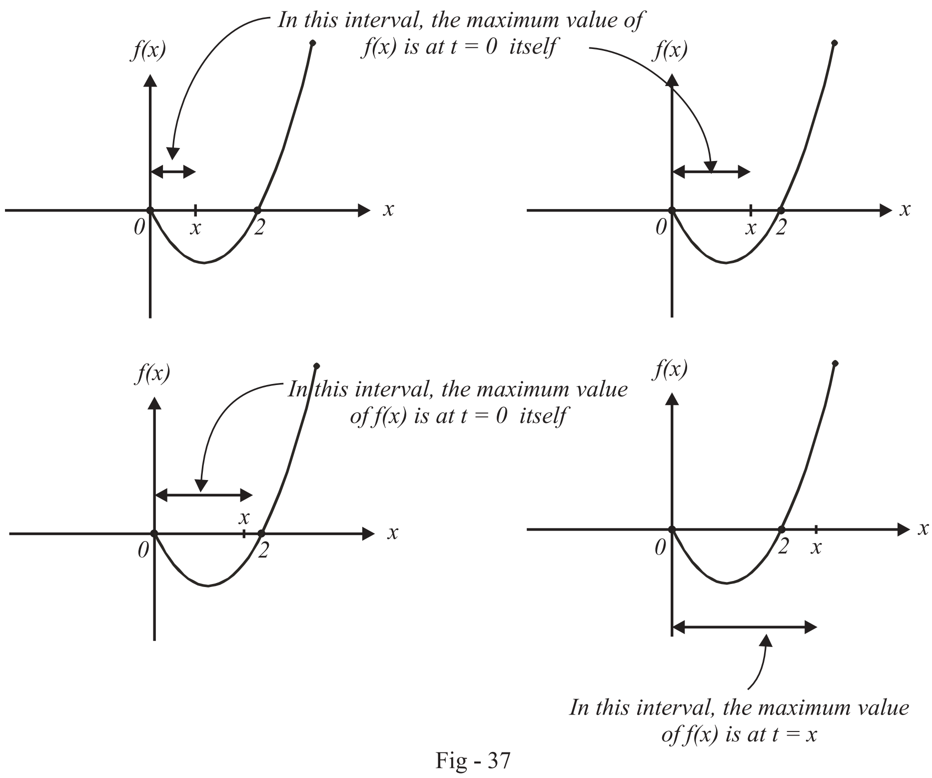

Fig - 37

Notice then untill x lies in the interval [0, 2], the maximum value of $f(x)$ in the interval $[0, x]$ is at $t = 0$, equal to 0.
As soon as x becomes greater than 2, the maximum value of $f(x)$ in the interval $[0, x]$ is now at $t = x$.
The graph of $g(x)$ for the interval [0, 3] is sketched below:

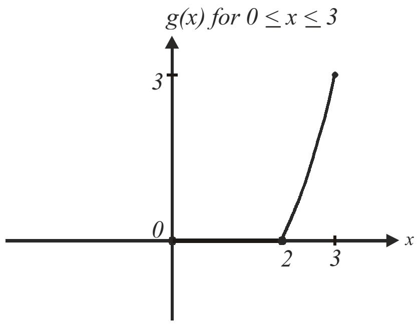

Fig - 38

The overall graph for $g(x)$ is therefore:

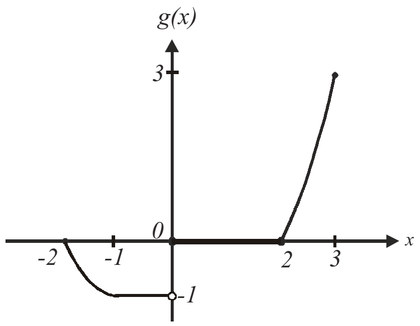

Fig - 39

We see that $g(x)$ is discontinuous at $x = 0$ and non-differentiable at $x = 0, 2$. ◀

Try to draw the following graphs on your own:

(i) $f(x) = \max\{2x - x^2; 0 \le t \le x, 0 \le x \le 2\}$

(ii) $f(x) = \min\{|x^2 - 1|; -2 \le t \le x, -2 \le x \le 2\}$

Example – 9

Let $f(x) = \begin{Bmatrix} x + a & , x < 0 \\ |x - 1| & , x \ge 0 \end{Bmatrix}$ and $g(x) = \begin{Bmatrix} x + 1 & , x < 0 \\ (x-1)^2 + b & , x \ge 0 \end{Bmatrix}$

where a and b are non-negative real numbers. Determine the composite function $g(f(x))$. If $g(f(x))$ is continuous for all real x, determine the values of a and b. For these values of a and b, will $g(f(x))$ be differentiable at $x = 0$?

Solution: Evaluation of the composition of piecewise defined functions can be tricky; hence follow the solution to this problem carefully.

$$g(f(x)) = \begin{cases} f(x)+1 & f(x)<0 \\ (f(x)-1)^2+b & f(x)\geq 0 \end{cases}$$

The conditions '$f(x) < 0$' and '$f(x) \geq 0$' have to be written in terms of x.

Notice from the definition of $f(x)$ that $f(x) < 0$ only when $x + a < 0$ i.e.

$$f(x) < 0 \Rightarrow x < -a.$$

so, $f(x) \geq 0$ when $x \geq -a$. But also notice that the definition of $f(x)$ changes at $x = 0$.

Hence, $g(f(x))$ can be rewritten as.

$$g(f(x)) = \begin{cases} f(x)+1 & x<-a \\ (f(x)-1)^2+b & x\geq -a \end{cases}$$

$$= \begin{cases} x+a+1 & x<-a \\ (x+a-1)^2+b & -a\leq x<0 \\ (|x-1|-1)^2+b & x\geq 0 \end{cases}$$

$$= \begin{cases} x+a+1 & x<-a \\ (x+a-1)^2+b & -a\leq x<0 \\ x^2+b & 0\leq x<1 \\ (x-2)^2+b & 1\leq x \end{cases}$$

This is the 'simplified' definition of $g(f(x))$. Reread the whole discussion above carefully till you fully understand it.

Now our task is easy. We just need to equate LHL and RHL at each of the critical points $x = -a$, 0, 1 to find out a and b.

The remaining part of this question is left as an exercise to you. The answers are:

$a = 1$ $\qquad$ $b = 0$

For these values, $f(x)$ is non differentiable at $x = \pm 1$ ◀

Example – 10

Let $f(x)$ be the function (defined in Example -14, Section-3)

$$f(x)=\begin{cases} 2x^2+12x+16 & , \quad -4\le x\le -2 \\ 2-|x| & , \quad -2<x\le 1 \\ 4x-x^2-2 & , \quad 1<x\le 3 \end{cases}$$. Plot $[f(x)]$ and discuss its continuity.

Solution: We drew the graph of this function in Fig. 29. We are replicating it in more detail here:

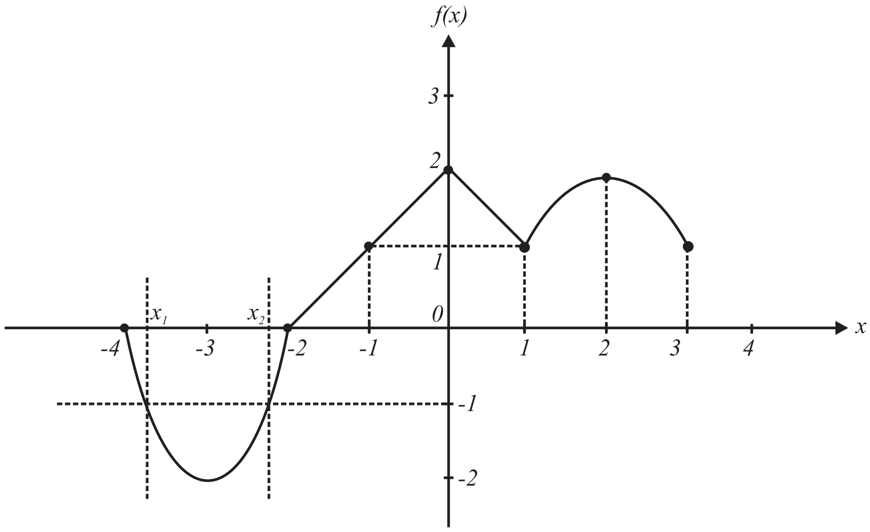

Fig - 40

To draw the required graph, notice that the value of $[f(x)]$ will change every time (the value of) $f(x)$ crosses an integer.

For example, notice from the graph the following facts:

When $x\in(-4,x_1],0<f(x)\le -1\Rightarrow[f(x)]=-1$

When $x\in(x_1,x_2),-2\le f(x)<-1\Rightarrow[f(x)]=-2$

When $x=0$ or $x=2,\ f(x)=2\Rightarrow[f(x)]=2$

and so on.

x_1 and x_2 can be evaluated by solving

$$f(x)=2x^2+12x+16=-1$$

$$\Rightarrow 2x^2+12x+17=0$$

$$\Rightarrow x=-3\pm\frac{\sqrt{10}}{4}$$

$$\Rightarrow x_1=-3-\frac{\sqrt{10}}{4},\ x_2=-3+\frac{\sqrt{10}}{4}$$

For the sake of completeness, the complete definition of $[f(x)]$ is given below :

$$[f(x)] = \begin{cases} 0 & x = -4 \\ -1 & -4 < x \le x_1 \\ -2 & x_1 < x < x_2 \\ -1 & x_2 \le x < -2 \\ 0 & -2 \le x < -1 \\ 1 & -1 \le x < 0 \\ 2 & x = 0 \\ 1 & 0 < x < 2 \\ 2 & x = 2 \\ 1 & 2 < x \le 3 \end{cases}$$

The graph of $[f(x)]$ is:

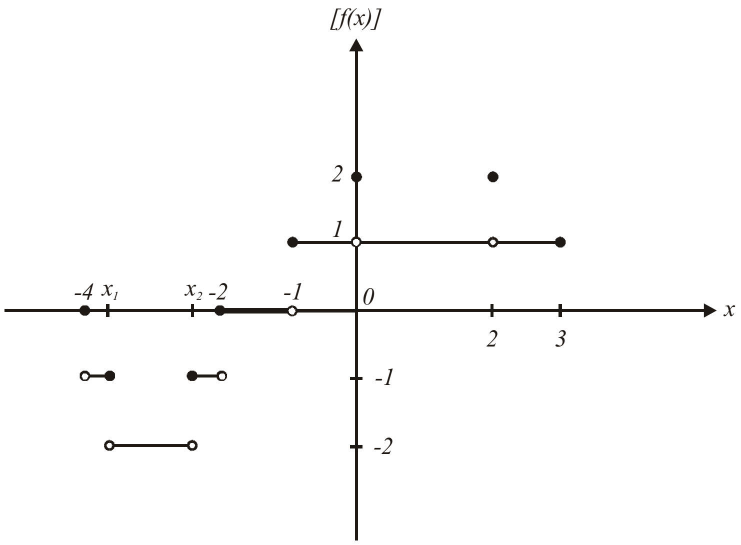

Fig - 41

This function is discontinuous at the following points:

$x = -4,\ x_1,\ x_2,\ -2,\ -1,\ 0,\ 2$

Example – 11

Let $\alpha \in \mathbb{R}$. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at α if and only if there exists a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in \mathbb{R}$.

Solution: The proof that we seek is two-way that is, when we say that

(*P* is true) if and only if (*Q* is true)

we mean that

(*P* is true) implies (*Q* is true)

and (*Q* is true)implies (*P* is true).

For this question, we first assume the existence of a function $g(x)$ where $g(x)$ satisfies

$$f(x) - f(\alpha) = g(x)(x-\alpha) \text{ and } g(x) \text{ is continuous at } x = \alpha.$$

Due to this continuity,

$$\lim_{x\to\alpha} g(x) \text{ exists}$$

But,

$$\lim_{x\to\alpha} g(x) = \lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha}$$

$$= \lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h}$$

$$= f'(\alpha)$$

Hence, $f'(\alpha)$ exists or $f(x)$ is differentiable at $x = \alpha$.

The other way proof is left as an exercise to the reader ◀

Example – 12

Evaluate the differentiability of $f(x) = [x] + \sqrt{\{x\}}$

Solution: We drew the graph of this function in the unit on Functions. That graph is replicated here:

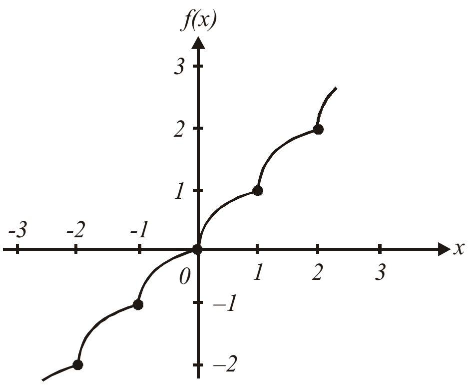

Fig - 42

$f(x)$ is obviously continuous every where but non-differentiable at all integer points. Let us evaluate the LHD and RHD at each integer. For that ,we analyse the segment of the curve between any two adjacent integers. Lets pick up the segment between 0 and 1; this segment is part of the segment

$f(x) = \sqrt{x}$:

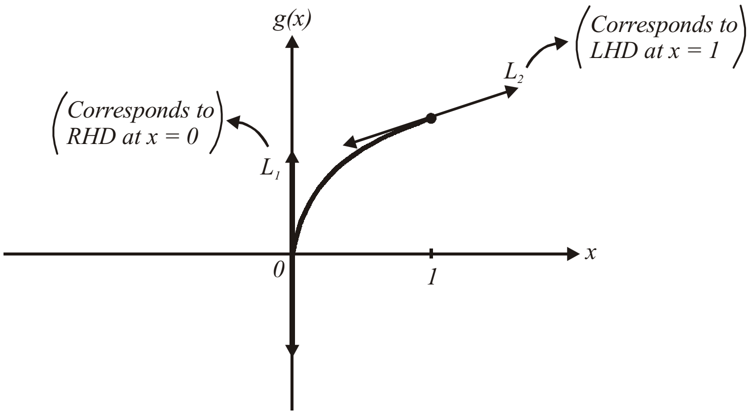

Fig - 43

The lines L_1 and L_2 correspond to the RHD at $x = 0$ and LHD at $x = 1$ respectively.

$$RHD(at\ x=0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h}$$

$$= \lim_{h\to 0}\left\{\frac{\sqrt{h}-0}{h}\right\}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{h}}$$

$$= \infty \qquad \text{(This tangent is vertical).}$$

$$LHD\ (at\ x=1) = \lim_{h\to 0}\frac{g(1-h)-g(1)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{1-h}-1}{-h}$$

$$= \lim_{h\to 0}\frac{-h}{-h\left(\sqrt{1-h}+1\right)} \quad \text{(By rationalization)}$$

$$= \frac{1}{2}$$

Therefore LHD at each integer is $\frac{1}{2}$ and RHD at each integer is ∞ ◀

EXERCISE

Q. 1 Find the value of a and b if f is continuous at $x=\pi/2$, where

$$f(x)=\begin{cases}\left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & 0<x<\pi/2\\ a+2 & x=\pi/2\\ \left(1+|\cot x|\right)^{\frac{b}{a}|\tan x|} & \dfrac{\pi}{2}<x<\pi\end{cases}$$

Q. 2 Evaluate the continuity of the function $f(x)=[x^2]-[x]^2$.

Q. 3 Find the value of $f(0)$ so that $f(x)=\dfrac{2^{|x|}e^{|x|}-|x|-|x|\ln 2-1}{x^2}, x\neq 0$ is continuous at $x=0$.

Q. 4 A function $f(x)$ is defined as $f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1} & x^2\neq 1\\ 0 & x^2=1\end{cases}$. Discuss the continuity of $f(x)$ at $x=1$.

Q. 5 Discuss the continuity of the function $f(x)=\lim\limits_{n\to\infty}\dfrac{\ln(2+x)-x^{2n}\sin x}{1+x^{2n}}$ at $x=1$.

Q. 6 Discuss the continuity of $f(x)=\lim\limits_{n\to\infty}\left\{\dfrac{\sin \pi x}{2}\right\}^{2n}$ in the interval [0, 2].

Q. 7 Let $f(x)=\begin{cases}x^m\sin\dfrac{1}{x} & ,x\neq 0\\ 0 & ,x=0\end{cases}$. Find the set of values of m for which

(i) $f(x)$ is continuous at $x=0$

(ii) $f(x)$ is differentiable at $x=0$

(iii) $f(x)$ is continuous but not differentiable at $x=0$.

Q. 8 Let $f(x)=\begin{cases}\left|1-4x^2\right| & ,0\le x<1\\ \left[x^2-2x\right] & ,1\le x<2\end{cases}$. Discuss the continuity and differentiability of $g(x)$ in $(0, 2)$.

Q. 9 Let $f(x)=x^3-x^2+x-1$ and $g(x)=\begin{cases}\max\{f(t):0\le t\le x;0\le x\le 1\}\\ 3-x, & 1<x\le 2\end{cases}$.

Discuss the continuity and differentiability of $g(x)$ in the interval $(0, 2)$

Q. 10 Let $f(x)=|x+1|\left(|x|+|x-1|\right)$. Draw the graph of $f(x)$ and analyze its continuity and differentiability.

Q. 11 If $f(x)=x^3-3x$ and $g(x)=\begin{cases}\min\{f(t):0\le t\le x\}, & 0\le x\le 2\\ (2x-5) & , \quad 2<x\le 3\\ (x-2)^2 & , \quad x>3\end{cases}$,

draw the graph of $g(x)$ and discuss its continuity and differentiability.

Q. 12 Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined as:

$$f(x)=\begin{cases}g(x) & ,x\le 0\\ \left(\dfrac{x+1}{x+2}\right)^{1/x} & ,x>0\end{cases}.$$

Find $g(x)$ such that $f(x)$ is continuous and $f'(1)=f(-1)$.

Q. 13 Examine the continuous function $f(x)=\begin{cases}x^2+1 & ,|x|\le 1\\ x^3+ax^2+bx & ,|x|>1\end{cases}$ for differentiability.

Q. 14 Consider the function

$$f(x)=\begin{cases}\min\left\{|x|,\sqrt{1-x^2}\right\}, & -1\le x\le 1\\ \left[|x|\right], & 1<|x|<2\end{cases}$$

Plot the graph for $f(x)$ and discuss its continuity and differentiability.

Q. 15 If a function $f:[-2a,2a]\to\mathbb{R}$ is an odd function such that $f(x)=f(2a-x)$ for $x\in[a,2a]$ and the left hand derivative at $x=a$ is 0, then find the left hand derivative at $x=-a$.