MECHANICS

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1 Measurements

A measurement should always be regarded as an estimate. The precision of the final result of an experiment cannot be better than the precision of the measurements made during the experiment, so the aim of the experimenter should be to make the estimates as good as possible.

There are many factors which contribute to the accuracy of a measurement. Perhaps the most obvious of these is the level of attention paid by the person making the measurements: a careless experimenter gets bad results! However, if the experiment is well designed, one careless measurement will usually be obvious and can therefore be ignored in the final analysis. In the following discussion of errors and level of precision we assume that the experiment is being performed by a careful person who is making the best use of the apparatus available.

1.1 Systematic Errors

If a voltmeter is not connected to anything else it should, of course, read zero. If it does not, the "zero error" is said to be a systematic error: all the readings of this meter are too high or too low. The same problem can occur with stop-watches, thermometers etc. Even if the instrument can not easily be reset to zero, we can usually take the zero error into account by simply adding it to or subtracting it from all the readings. (It should be noted however that other types of systematic error might be less easy to deal with.) Similarly, if 10 ammeters are connected in series with each other they should all give exactly the same reading. In practice they probably will not. Each ammeter could have a small constant error. Again this will give results having systematic errors.

For this reason, note that a precise reading is not necessarily an accurate reading. A precise reading taken from an instrument with a systematic error will give an inaccurate result.

1.2 Random Errors

Try asking 10 people to read the level of liquid in the same measuring cylinder. There will almost certainly be small differences in their estimates of the level. Connect a voltmeter into a circuit, take a reading, disconnect the meter, reconnect it and measure the same voltage again. There might be a slight difference between the readings. These are random (unpredictable) errors. Random errors can never be eliminated completely but we can usually be sure that the correct reading lies within certain limits.

To indicate this to the reader of the experiment report, the results of measurements should be written as

$$\text{Result} \pm \text{Uncertainty}$$

• Any measurement that we made without a knowledge of it's uncertainty is meaningless.

For example, suppose we measure a length, l to be 25 cm with an uncertainty of 0.1cm. We write the result as

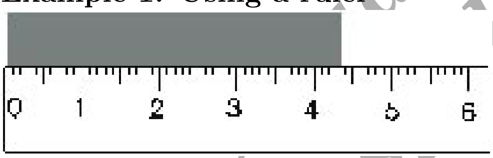
$l = 25\text{cm} \pm 0 \cdot 1\text{cm}$

By this, we mean that all we are sure about is that l is somewhere in the range 24.9cm to 25.1cm.

1.3 Quantifying the Uncertainty

The number we write as the uncertainty tells the reader about the instrument used to make the measurement. (As stated above, we assume that the instrument has been used correctly.) Consider the following examples.

Example 1: Using a ruler



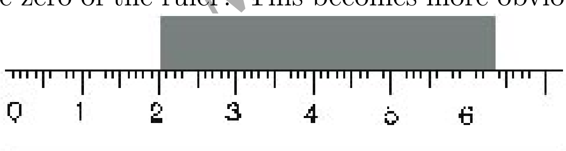
The length of the object being measured is obviously somewhere near 4·3cm (but it is certainly not exactly 4·3cm).

The result could therefore be stated as

4·3cm ± half the smallest division on the ruler

In choosing an uncertainty equal to half the smallest division on the ruler, we are accepting a range of possible results equal to the size of the smallest division on the ruler.

However, do you notice something which has not been taken into account? A measurement of length is, in fact, a measure of two positions and then a subtraction. Was the end of the object exactly opposite the zero of the ruler? This becomes more obvious if we consider the measurement again, as shown below.



Notice that the left-hand end of the object is not exactly opposite the 2cm mark of the ruler. It is nearer to 2cm than to 2·1cm, but this measurement is subject to the same level of uncertainty.

Therefore the length of the object is

$$(6 \cdot 3 \pm 0 \cdot 05)\text{cm} - (2 \cdot 0 \pm 0 \cdot 05)\text{cm}$$

so, the length can be between

$$(6 \cdot 3 + 0 \cdot 05) - (2 \cdot 0 - 0 \cdot 05) \text{ and } (6 \cdot 3 - 0 \cdot 05) - (2 \cdot 0 + 0 \cdot 05)$$

that is, between

4·4 cm and 4·2 cm

We now see that the range of possible results is 0·2cm, so we write

$$\text{length} = 4 \cdot 3\text{cm} \pm 0 \cdot 1\text{cm}$$

In general, we state a result as

reading ± the smallest division on the measuring instrument

Example 2: Using a Stop-Watch

Consider using a stop-watch which measures to 1/100 of a second to find the time for a pendulum to oscillate once. Suppose that this time is about 1s. Then, the smallest division on the watch is only about 1% of the time being measured. We could write the result as

$$T = 1s \pm 0 \cdot 01s$$

which is equivalent to saying that the time T is between 0·99s and 1·01s.

This sounds quite good until you remember that the reaction-time of the person using the watch might be about 0·1s. Let us be pessimistic and say that the person's reaction-time is 0·15s. Now considering the measurement again, with a possible 0·15s at the starting and stopping time of the watch, we should now state the result as

$$T = 1s \pm (0 \cdot 01 + 0 \cdot 3)s$$

In other words, T is between about **0·7s** and **1·3s**.

We could probably have guessed the answer to this degree of precision even without a stop-watch!

Conclusions from the preceding discussion

If we accept that an uncertainty (sometimes called an indeterminacy) of about 1% of the measurement being made is reasonable, then

a) a ruler, marked in mm, is useful for making measurements of distances of about 10cm or greater.

b) a manually operated stop-watch is useful for measuring times of about 30s or more (for precise measurements of shorter times, an electronically operated watch must be used)

1.4 How many Decimal Places?

Suppose you have a timer which measures to a precision of 0·01s and it gives a reading of 4·58s. The actual time being measured could have been 4·576s or 4·585s etc. However, we have no way of knowing this, so we write

$$t = 4 \cdot 58s \pm 0 \cdot 01s$$

If we now repeat the experiment using a better timer which measures to a precision of 0·0001s. The timer might still give us a time of 4·58s but now we would indicate the greater precision of the instrument being used by stating the result as

$$t = 4 \cdot 5800s \pm 0 \cdot 0001s$$

So, as a general rule, look at the precision of the instrument being used and state the result to that number of decimal places.

Another point to remember is that very often we will be using our results to plot a graph. On most graph paper you can represent a result to a precision of 3 significant figures (3 sig. fig.). So (assuming that your measurements allow for this level of precision) convert your table of results to lists of numbers in standard form and give them to two decimal places. (By "standard form" we mean a number between 0·00 and 9·99 multiplied by the appropriate power of 10.)

1.5 How does an uncertainty in a measurement affect the FINAL result?

The measurements we make during an experiment are usually not the final result; they are used to calculate the final result.

When considering how an uncertainty in a measurement will affect the final result, it will be helpful to express uncertainties in a slightly different way. Remember, what really matters is that the uncertainty in a given measurement should be much smaller than the measurement itself. For example, if you write, "I measured the time to a precision of 0·01s", it sounds good: unless you then inform your reader that the time measured was 0·02s ! The uncertainty is 50% of the measured time so, in reality, the measurement is useless. We will define the quantity Relative Uncertainty as follows

$$\text{Relative Uncertainty} = \frac{\text{Absolute Uncertainty}}{\text{Measured Value}}$$

(to emphasise the difference, we use the term "absolute uncertainty" where previously we simply said "uncertainty").

We will now see how to answer the question in the title.

It is always possible, in simple situations, to find the effect on the final result by straightforward calculations but the following rules can help to reduce the number of calculations needed in more complicated situations.

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1.6 Percentage Error

If there is an error ΔL in measurement of any physical quantity L, then $\Delta L/L$ is called fraction error and $\Delta L/L \times 100$ is called *percentage error*.

1.7 Combination of Errors

If we do an experiment involving several measurements, we must know how the errors in all the measurements combine.

(a) Error of a sum or a difference

Suppose two physical quantities A and B have measured values A ± ΔA, B ± ΔB respectively where ΔA and ΔB are their absolute errors. We wish to find the error ΔZ in the sum

Z = A + B.

We have by addition, Z ± ΔZ = (A ± ΔA) + (B ± ΔB).

The maximum possible error in Z

ΔZ = ΔA + ΔB

For the difference Z = A – B, we have

Z ± Δ Z = (A ± ΔA) – (B ± ΔB) = (A – B) ± ΔA ± ΔB

or, ± ΔZ = ± ΔA ± ΔB

The maximum value of the error ΔZ is again ΔA + ΔB.

Hence the rule : When two quantities are added or subtracted, the absolute error in the final result is the sum of the absolute errors in the individual quantities.

➤ Example 1. The temperatures of two bodies measured by a thermometer are t1 = 20 0C ± 0.5 0C and t2 = 50 0C ± 0.5 0C. Calculate the temperature difference and the error theirin.
Sol. $t' = t^2 - t^1 = (500C \pm 0.50C) - (200C \pm 0.50C)t' = 300C \pm 10C$

(b) Error of a product or a quotient

Suppose Z = AB and the measured values of A and B are A ± ΔA and B ± ΔB.Then

Z ± ΔZ = (A ± ΔA) (B ± ΔB) = AB ± B ΔA ± A ΔB ± ΔA ΔB.

Dividing LHS by Z and RHS by AB we have,

1±(ΔZ/Z) = 1 ± (ΔA/A) ± (ΔB/B) ± (ΔA/A)(ΔB/B).

Since ΔA and ΔB are small, we shall ignore their product.

Hence the maximum relative error

ΔZ/ Z = (ΔA/A) + (ΔB/B).

Hence the rule : When two quantities are multiplied or divided, the relative error in the result is the sum of the relative errors in the multipliers.

➤ **Example2. The resistance R = V/I where V = (100 ± 5)V and I = (10 ± 0.2)A. Find the percentage error in R.**
Answer . The percentage error in V = $\frac{\Delta V}{V} \times 100 = \frac{5}{100} \times 100 = 5\%$
and percentage error in I = $\frac{\Delta I}{I} \times 100 = \frac{0.2}{10} \times 100 = 2\%$.
The total error in R would therefore be 5% + 2% = 7%.

➤ **Example 3. Two resistors of resistances $R_1 = 100 \pm 3$ ohm and $R_2 = 200 \pm 4$ ohm are connected (a) in series, (b) in parallel. Find the equivalent resistance of the (a) series combination, (b) parallel combination. Use for (a) the relation $R = R_1 + R_2$, and for (b) $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$ and $\frac{\Delta R'}{R'^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}$.**
Answer. (a) The equivalent resistance of series combination
$R = R_1 + R_2 = (100 \pm 3)\text{ohm} + (200 \pm 4)\text{ohm} = 300 \pm 7$ ohm.
(b) The equivalent resistance of parallel combination
$R' = \frac{R_1 R_2}{R_1 + R_2} = \frac{200}{3} = 66.7\text{ohm}$
Then, from $\frac{1}{R'} = \frac{1}{R_1} + \frac{1}{R_2}$ we get,
$\frac{\Delta R'}{R'^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2}$
$\Delta R' = (R'^2)\frac{\Delta R_1}{R_1^2} + (R'^2)\frac{\Delta R_2}{R_2^2} = \left(\frac{66.7}{100}\right)^2 3 + \left(\frac{66.7}{200}\right)^2 4 = 1.8$
Then, $R' = 66.7 \pm 1.8\,\text{ohm}$
(Here, ΔR is expresed as 1.8 instead of 2 to keep in confirmity with the rules of significant figures.)

(c) Error in case of a measured quantity raised to a power

Suppose $Z = A^2$,

Then, $\Delta Z/Z = (\Delta A/A) + (\Delta A/A) = 2(\Delta A/A)$.

Hence, the relative error in A^2 is two times the error in A.

In general, if $Z = A^p B^q / C^r$ Then, $\Delta Z/Z = p(\Delta A/A) + q(\Delta B/B) + r(\Delta C/C)$.

Hence the rule : The relative error in a physical quantity raised to the power k is the k times the relative error in the individual quantity.

➤ **Example 4. Find the relative error in Z, if $Z = A^4 B^{1/3}/CD^{3/2}$.**
Answer. The relative error in Z is $\Delta Z/Z = 4(\Delta A/A) + (1/3)(\Delta B/B) + (\Delta C/C) + (3/2)(\Delta D/D)$.

➤ **Example5. The period of oscillation of a simple pendulum is $T = 2\pi\sqrt{\frac{L}{g}}$.** Measured value of L is 20.0 cm known to 1 **mm accuracy and time for** 100 **oscillations of the pendulum is found to be** $90s$ **using a wrist watch of** $1s$ **resolution. What is the accuracy in the determination of** g **?**

Answer Given that $T = 2\pi\sqrt{\frac{L}{g}}$, therefore

$g = 4\pi^2 L/T^2$ Here, $T = \frac{t}{n}$ and $\Delta T = \frac{\Delta t}{n}$.

Therefore, $\frac{\Delta T}{T} = \frac{\Delta t}{t}$.

The errors in both L and t are the least count errors. Therefore,

$(\Delta g/g) = (\Delta L/L) + 2(\Delta T/T)$

$= \frac{0.1}{20.0} + 2\left(\frac{1}{90}\right) = 0.32$

Thus, the percentage error in g is

$= (\Delta g/g) \times 100 = (\Delta L/L) \times 100 + 2(\Delta T/T) \times 100 = 3\%$

2 Dimensional Analysis

2.1 The International System Of Units

Unit: It is the smallest part of physical quantity (a quantity which can be measured) in any system of units. Any standard unit should have the following two properties:

(a) Invariability: The standard unit must be invariable. Thus assuming your height as standard as standard unit of length is not invariable because your height changes with your age.

(b) Availability: The standard unit should be easily made available for comparing with other quantities. Also you will not be available every where for comparision.

Types of units

(i) F.P.S. system of units: It is the british engineering system of units which uses foot as the unit of length, pound as the unit of mass and second as the unit of time.

(ii) C.G.S. system of units: It is the Gaussian system which uses centimeter as the unit of length, gram as the unit of mass and second as the unit of time.

(iii) M.K.S. system of units: In this system, meter is the unit of length, kilogram is the unit of mass and second is the unit of time.

(iv) International system of units (SI units): This system of units was introduced in 1960, by the general conference of weights and measures and was internationally accepted. Our calcullations mainly will be in this system of units.

Units are divided in two groups as **fundamental units** and **Derived units.**

(A) Fundamental unit: If the unit of a physical quantity is independent of of the other unit, the physical quantity is said to be fundamental quantity and its unit as fundamental unit.

The SI unit is based on the following seven fundamental units and two supplementary units:

S. No.	**Fundamental Quantity**	**Fundamental Unit**	**Unit Symbol Used**
1.	Mass	kilogram	Kg
2.	Length	meter	m
3.	Time	second	s
4.	Temperature	kelvin	K
5.	Electric Current	ampere	A
6.	Luminous Intensity	candela	cd
7.	Amount of Matter	mole	mol

S. No.	**Supplementary Physical Quantity**	**Supplementary Unit**	**Unit Symbol Used**
1.	Plane Angle	Radian	rad
2.	Solid Angle	Steradian	sr

Note:

(*i*) Unit cannot be plurel e.g. writing 5kgs is wrong concept, the correct is 5kg.

(*ii*) If the name of a unit is the name of a scientist and you are writing the complete name, then start from a small letter, e.g. 5 ampere and if you are writing the single latter use big letter, e.g., 5A.

(B) Derived Unit: If the unit of a physical quantity depends on the units of the fundamental quantities then the quantity is said to be dependent physical quantity (or derived quantity) and its unit is dependent unit (or derived unit). e.g. unit of velocity is m/s, which depends on the unit of length and time and hence the velocity is said to be dependent physical quantity and its unit is derived unit.

2.2 Dimensions

The word dimension has a special meaning in physics. It denotes the physical nature of a quantity. Whether a distance is measured in units of feet or meters or fathoms, it is still a distance. We say its dimension is *length*.

All the physical quantities represented by derived units can be expressed in terms of some combination of seven fundamental or base quantities. We shall call these base quantities as the seven dimensions of the physical world, which are denoted with square brackets [ ]. Thus, length has the dimension [L], mass [M], time [T], electric current [A], thermodynamic temperature [K], luminous intensity [cd], and amount of substance [mol]. The dimensions of a physical quantity are the powers (or exponents) to which the base quantities are raised to represent that quantity. Note that using the square brackets [ ] round a quantity means that we are dealing with 'the dimensions of' the quantity.

In mechanics, all the physical quantities can be written in terms of the dimensions [L], [M] and [T]. For example, the volume occupied by an object is expressed as the product of length, breadth and height, or three lengths. Hence the dimensions of volume are $[L] \times [L] \times [L] = [L]^3 = [L^3]$. As the volume is independent of mass and time, it is said to possess zero dimension in mass $[M^0]$, zero dimension in time $[T^0]$ and three dimensions in length.

Similarly, force, as the product of mass and acceleration, can be expressed as

Force = mass × acceleration = mass× (length)/(time)^2

The dimensions of force are $[M][L]/[T]^2 = [MLT^{-2}]$. Thus, the force has one dimension in mass, one dimension in length, and -2 dimensions in time. The dimensions in all other base quantities are zero.

2.3 Dimensional Formulae And Dimensional Equations

The expression which shows how and which of the base quantities represent the dimensions of a physical quantity is called the dimensional formula of the given physical quantity.

For example, the dimensional formula of the volume = $[M^0L^3T^0]$,
Dimensional formula of speed or velocity is = $[M^0LT^{-1}]$
Dimensional formula of acceleration = $[M^0LT^{-2}]$
Dimensional formula of mass density = $[ML^{-3}T^0]$

An equation obtained by equating a physical quantity with its dimensional formula is called the dimensional equation of the physical quantity. Thus, the dimensional equations are the equations, which represent the dimensions of a physical quantity in terms of the base quantities. For example, the dimensional equations of volume $[V]$, speed $[v]$, force $[F]$ and mass density $[\rho]$ may be expressed as

$[V] = [M^0L^3T^0]$
$[v] = [M^0LT^{-1}]$
$[F] = [MLT^2]$
$[\rho] = [ML^{-3}T^0]$

Note:

1. Numerical values are dimensionless. e.g., dimensional formula of two (2) is
   $[2] = [M^0L^0T^0]$

2. All trigonometrical ratios are dimensionless. e.g., If
   $y = \sin\theta$, then $[y] = [\sin\theta] = [M^0L^0T^0]$

3. Exponential functions are dimensionless. e.g., If
   $y = e^x$, then $[x] = [M^0L^0T^0]$, $[y] = [e^x] = [M^0L^0T^0]$

4. Logarithmic functions are dimensionless. e.g., if
   $y = \log_a x$, then $[x] = [M^0L^0T^0]$, $[a] = [M^0L^0T^0]$ $[y] = [\log_a x] = [M^0L^0T^0]$

5. All ratios are dimensionless, e.g.,
   (a) mole$=\frac{wt.\,of\,compound}{molecular\,wt.\,of\,compound}$, then [mole]$=[M^0L^0T^0]$
   (b) Specific gravity $= \frac{\text{weight of certain volume of a substance}}{\text{weight of same volume of water at } 4^\circ C}$, hence [sp. gr.] $= [M^0L^0T^0]$
   (c) Relative density $= \frac{\text{density of a substance}}{\text{density of water at } 4^\circ C}$, [Relative density]$=[M^0L^0T^0]$.

2.4 Uses of Dimensions

2.4.1 Homogeneity of dimensions in an equation

It states that in a correct physical equation the dimensions of each term added or subtracted must be same.

- With the help of above statement we can check the correctness of a physical equation.
- A dimensionally correct equation may be incorrect also.
- But a dimensionally incorrect equation will be always incorrect equation.

Let us check the equation $v = u + at$. Here u is the initial velocity, v is the final velocity, a is constant acceleration and t is the time considered for motion between a segment of a path.
$[u] = \left[\frac{m}{sec}\right] = \left[LT^{-1}\right]$
$[v] = \left[\frac{m}{sec}\right] = \left[LT^{-1}\right]$
$[at] = [acc.][time] = \left[LT^{-2}\right][T] = \left[LT^{-1}\right]$
Thus, the equation is correct as the dimensions of each term are the same. So, we can say that the equation is dimensionally correct.

2.4.2 Conversion of units

The numerical value of a physical quantity in a system of units can be changed to another system of units using the equation $nu=$ const, i.e. $n_1u_1 = n_2u_2$ or $n_2 = n_1\left[\frac{u_1}{u_2}\right]$ or $n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c$ where n_1 is the numerical value in one system having unit $[u_1]$ and n_2 is the numerical value in other system having unit $[u_2]$.

Ex. The unit of work in SI system is joule. Now, how many CGS energy is equal to 1 joule ?
Dimensional formula of energy is $\left[ML^2T^{-2}\right]$, therefore
$n_2 = n_1\left[\frac{M_1}{M_2}\right]^a\left[\frac{L_1}{L_2}\right]^b\left[\frac{T_1}{T_2}\right]^c \Longrightarrow n_2 = n_1\left[\frac{\text{kg}}{\text{gram}}\right]^1\left[\frac{\text{meter}}{\text{cm}}\right]^2\left[\frac{\text{sec}}{\text{sec}}\right]^{-2} = 1\left[\frac{1000\text{gram}}{\text{gram}}\right]^1\left[\frac{100\text{ cm}}{\text{cm}}\right]^2\left[\frac{\text{sec}}{\text{sec}}\right]^{-2} = 10^7$
$\Longrightarrow$ 1 joule$=10^7 CGS$ units
Thus, knowing the conversion factors for the base quantities, one can find the numerical value of a physical quantity from one system of units to other system of units.

2.4.3 To find relations among the physical quantities

Suppose we want to find relation between force, mass and acceleration. Let force depends on mass and acceleration as follows

$F = Km^{\alpha}a^{\beta}$ where K is dimensionless constant and α and β are the powers of mass and acceleration.

According to principle of homogeneity

$[F] = [K][m]^{\alpha}[a]^{\beta}$

$\implies \left[MLT^{-2}\right] = \left[M^0L^0T^0\right][M]^{\alpha}[LT^{-2}]^{\beta}$

$\implies \left[MLT^{-2}\right] = \left[M^{\alpha}L^{\beta}T^{-2\beta}\right]$

Equating the dimensions on both sides we get $\alpha = 1$, $\beta = 1$, $-2\beta = -2$

$\implies \ \alpha = 1, \beta = 1$

Now putting the values of α and β in required equation we will get a mathematical equation $F = Kma$. The value of K can be found experimentally.

2.5 Limitations of the Dimensional Method

1. It cannot provide the exact expression for a physical quantity.

2. A dimensionally correct equation may be incorrect also.e.g.
   $v = 2u + 3at$
   have $[v] = [LT^{-1}]$, $[2u] = [2][u] = \left[M^0L^0T^0\right][LT^{-1}] = [LT^{-1}]$
   and $[6at] = [6][a][t] = \left[M^0L^0T^0\right][LT^{-2}][T] = [LT^{-1}]$
   Hence, the equation is dimensionally correct.
   But, experimentally found result prove that the above equation is numericaly inconsistent.

Assignments

Q1. The dimension of permeability is

(a) $\left[ML^2T^{-2}A^{-1}\right]$ (b) $ML^2T^{-2}A^{-2}$ (c) $MLT^{-2}A^{-1}$ (d) $MLT^{-3}A^{-1}$

Q2. In which of the following pairs, the two physical quantities do not have the same dimensional formula

(a) pressure and stress (b) impulse and momentum

(c) work and energy (d) moment of inertia and angular momentum

Q3. Dimensions of young's modulus and shear modulus are

(a) different (b) same, and are that of work (c) same, and are that of force (d) same, and are that of pressure

Q4. The dimension of co-efficient of viscosity is

(a) $\left[ML^{-1}T^{-1}\right]$ (b) $\left[M^0L^{-1}T^{-2}\right]$ (c) $\left[ML^2T^{-1}\right]$ (d) $\left[MLT^2\right]$

Q5. Given the dimension of potential difference $\left[ML^2T^{-3}A^{-1}\right]$. The dimension of resistance is

(a) $[MLT^{-3}A^{-2}]$ (b)$[ML^2T^{-3}A^{-2}]$ (c) $[ML^2T^{-3}A^{-1}]$ (d) $[ML^2T^{-3}A^{-3}]$

Q6. The dimension of Planck's constant is

(a) it is dimensionless, since it is a constant (b) same as that of frequency (c) $[ML^2T^{-1}]$ (d) $[ML^2T^{-2}]$

Q7. The period of oscillation, T of a floating cylinder with length h immersed in a liquid of density ρ is, $T = 2\pi\sqrt{\frac{h\rho}{c}}$, where c is a constant. The dimension of C is

(a) it is dimensionless (b) MLT^{-2} (c) $ML^{-2}T^{-2}$ (d) L^2T^{-2}

Q8. Which one of the following is not a base SI unit

(a) coulomb (b) kilogram (c) metre (d) candela

Q9. The speed of a particle varies in time according to $v = AT - Bt^3$. The dimensions of A and B respectively are

(a) L, LT^{-2} (b) L, LT^{-3} (c) LT^{-2}, LT^{-4} (d) L^2, LT^{-4}

Q10. In the equation $\left(P+\frac{a}{V^2}\right)(V-b) = RT$, where V is the volume, P is the pressure, T is the absolute temperature and R is the universal gas constant, the constants a and b have the dimensions

(a) $[M^2L^4T^{-2}\theta]$, $[L^4]$ (b) $[ML^5T^{-2}\theta]$, $[L^3]$ (c) $[ML^5T^{-2}\theta]$, $[L^3]$ (d) $[ML^5T^{-2}]$, $[L^3]$

Q11. The speed v of a particle of mass m as a function of time t is given by $v = \alpha A \sin\left[\left(\frac{k}{m}\right)^{1/2} t\right]$, where dimension of A is L. The dimensions of α and k respectively are

(a) LT^{-1}, MLT^{-2} (b) T^{-1}, MT^{-2} (c) LT^{-2}, MLT^{-2} (d) LT^{-2}, $ML^{-1}T^{-2}$

Q12. If A and B have different dimensions, which one of the following operations is not possible.

(a) AB (b) $1 - A/B$ (c) $A - B$ (d) $AB^{1/2}$

Q13. The planck's constant has the same dimension as

(a) angular velocity (b) energy (c) angular momentum (d) momentum

Q14. The quantity $eh/(4pm)$ has the same dimension as

(a) electric dipole moment (b) magnetic dipole moment (c) potential difference (d) current density

Q15. Which one of these is dimensionless

(a) refractive index (b) resistivity (c) universal constant of gravitation (d) solar constant

Q16. The product of resistance and capacitance has the same dimension as

(a) permittivity (b) potential difference (c) magnetic flux (d) time

Q17. The equation of motion of a rocket is $v = V_0 \log\left(1 - \frac{\alpha t}{M_0}\right) + v_0$ where v is the velocity of the rocket at time t, v_0 is its initial velocity, M_0 is its initial mass V_0 is the velocity of the fuel ejecting out of the vehicle with respect to the vehicle. The constant α has the dimension

(a) it is dimensionless (b) M (c) T (d) MT^{-1}

Answers

1. (c) using the formula of force between two current carrying conductors Force/unit length $= \frac{\mu_0}{2\pi}\frac{I_1 I_2}{r}$

2. (d) 3. (d) 4. (a)

5. (b): obtain it from $V = iR$

6. (c) 7. (c) dimension of two sides of equation must be equal

8. (a), 9. (c) same as 7

10. (d): a/v^2 has dimension of P and b has dimension of V

11. (b): the argument of a trigonometric function must be dimensionless

12. (c), 13. (c) 14.(b) 15. (a) 16. (d) 17. (d)

3 Vectors

3.1 Scalar Quantities

Physical quantities which can completely be specified by a number (magnitude) having an appropriate unit are known as "SCALAR QUANTITIES".

- Scalar quantities are comparable only when they have the same physical dimensions.
- Two or more than two scalar quantities measured in the same system of units are equal if they have the same magnitude and sign.
- Scalar quantities are denoted by letters in ordinary type.
- Scalar quantities are added, subtracted, multiplied or divided by the simple rules of algebra.

EXAMPLES

Work, energy, electric flux, volume, refractive index, time, speed, electric potential, potential difference, viscosity, density, power, mass, distance, temperature, electric charge etc.

3.2 Vectors Quantities

Physical quantities having both magnitude and direction with appropriate unit are known as "VECTOR QUANTITIES".

- We can't specify a vector quantity without mention of direction.
- vector quantities are expressed by using bold letters or with arrow sign such as: **V, F, A,** $\boldsymbol{\tau}$ or $\vec{V}$, $\vec{F}$, $\vec{A}$, $\vec{\tau}$.
- vector quantities can not be added, subtracted, multiplied or divided by the simple rules of algebra.
- vector quantities added, subtracted, multiplied or divided by the rules of trigonometry and geometry.

EXAMPLES

Velocity, electric field intensity, acceleration, force, momentum, torque, displacement, electric current, weight, angular momentum etc.

REPRESENTATION OF VECTORS

On paper vector quantities are represented by a straight line with arrow head pointing the direction of vector or terminal point of vector. A vector quantity is first transformed into a suitable scale and then a line is drawn with the help of the scale chosen in the given direction.

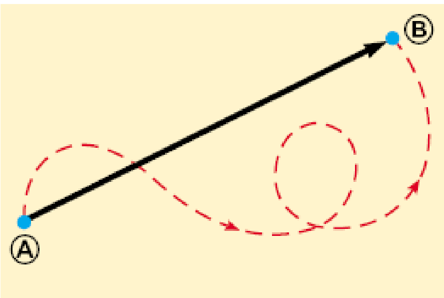


Figure . As a particle moves from Ⓐ to Ⓑ along an arbitrary path represented by the broken line, its displacement is a vector quantity shown by the arrow drawn from Ⓐ to Ⓑ.

3.3 Some Properties of Vectors

Equality of Two Vectors

For many purposes, two vectors A and B may be defined to be equal if they have the same magnitude and point in the same direction. That is, $\vec{A} = \vec{B}$ only if $A = B$ and if $\vec{A}$ and $\vec{B}$ point in the same direction along parallel lines. For example, all the vectors in Figure 1. are equal even though they have different starting points. This property allows us to move a vector to a position parallel to itself in a diagram without affecting the vector.

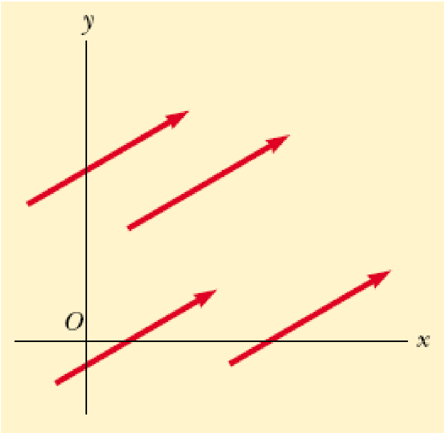


Figure 1 These four vectors are equal because they have equal lengths and point in the same direction.

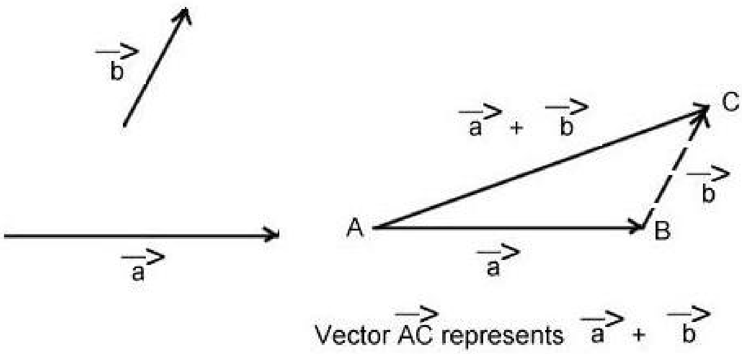
Addition of Vectors

1. Triangle Law:

Let $\vec{a}$ and $\vec{b}$ are two vectors in different directions.

When you shift the vector $\vec{b}$ without changing its magnitude and direction to coincide its terminal point to the same of vector $\vec{a}$, you find the two vectors are acting at single point but acting in different directions.

Complete the triangle by joining AC. The vector $\vec{AC}$ is the addition of vector $\vec{AB}$ and vector $\vec{BC}$.

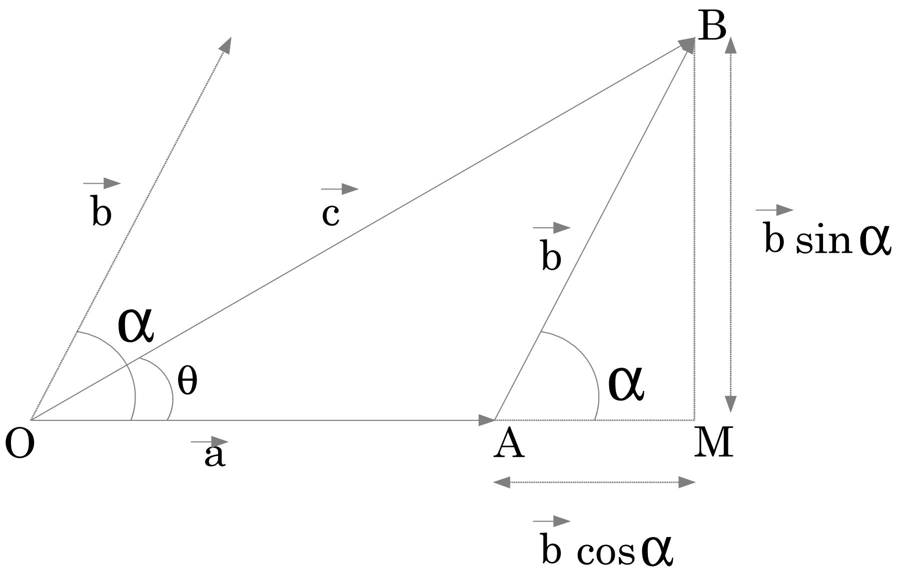


In following figure, $\vec{a}$ and $\vec{b}$ are any two vectors and angle between them is α. Let $\vec{c}$ is their resultant vector and it makes an angle θ with vector $\vec{a}$, then in ΔOMB

$OB^2 = OM^2 + MB^2$

$\implies c^2 = (a + b\cos\alpha)^2 + (b\sin\alpha)^2 = a^2 + b^2 + 2ab\cos\alpha$

$\implies c = \sqrt{a^2 + b^2 + 2ab\cos\alpha}$



Again in ΔOBM
$\tan\theta = \frac{BM}{OM} = \frac{b\sin\alpha}{a+b\cos\alpha}$
$\implies \theta = \tan^{-1}\left(\frac{b\sin\alpha}{a+b\cos\alpha}\right)$

A geometric construction can also be used to add more than two vectors. This is shown in following figure for the case of four vectors. The resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$ is the vector that completes the polygon. In other words, *$\vec{R}$ is the vector drawn from the tail of the first vector to the tip of the last vector.*

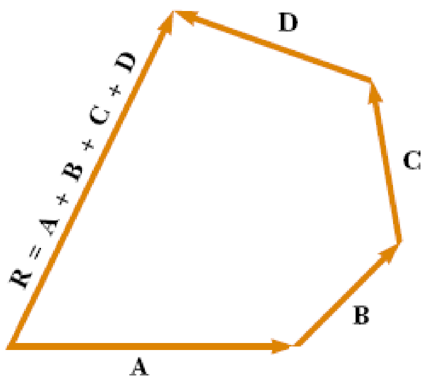


Figure . Geometric construction for summing four vectors. The resultant vector **R** is by definition the one that completes the polygon.

2. Parallelogram Law:

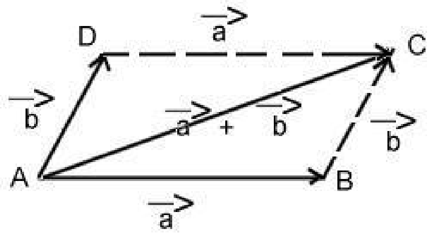
Consider two vectors $\vec{a}$ and $\vec{b}$ acting at a point A. Without changing the magnitude and direction, shift the vector $\vec{a}$ to point D and the vector $\vec{b}$ to point B.

Complete the parallelogram ABCD and join the diagonal AC. The vector $\vec{AC}$ is the sum of the vectors $\vec{AB}$ and $\vec{AC}$.

This result can also be arrived by applying triangular law of addition of vectors by considering the triangle ABC.

Therefore, the parallelogram law of addition of vectors is equivalent to the triangular law of addition of vectors.

Now look at the triangles ABC and ADC in the parallelogram.



The triangle ABC gives the result as the vector $\vec{AC}$ is the sum of vectors $\vec{a}$ and $\vec{b}$.
The triangle ADC gives the result as the vector $\vec{AC}$ is the sum of vectors $\vec{b}$ and $\vec{a}$.
Commutative Law Of Addition: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

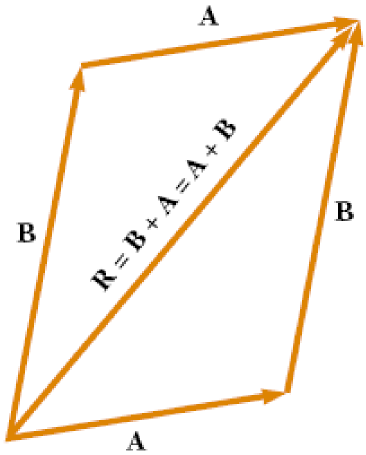


Figure 1. This construction shows that **A** + **B** = **B** + **A**—in other words, that vector addition is commutative.

Associative Law of Addition:
Vectors addition is associative. i.e., $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

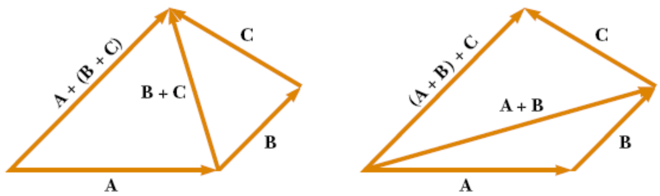
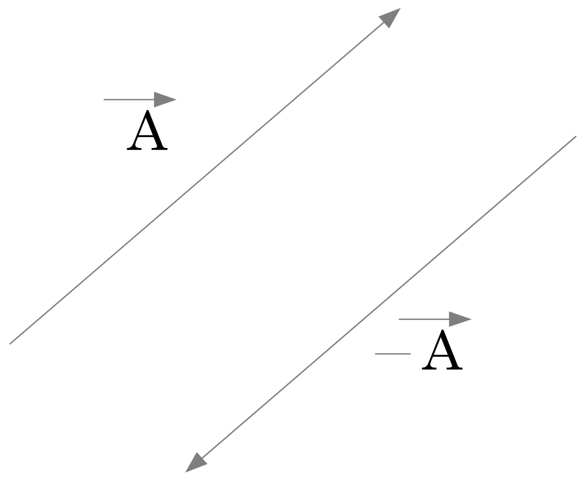
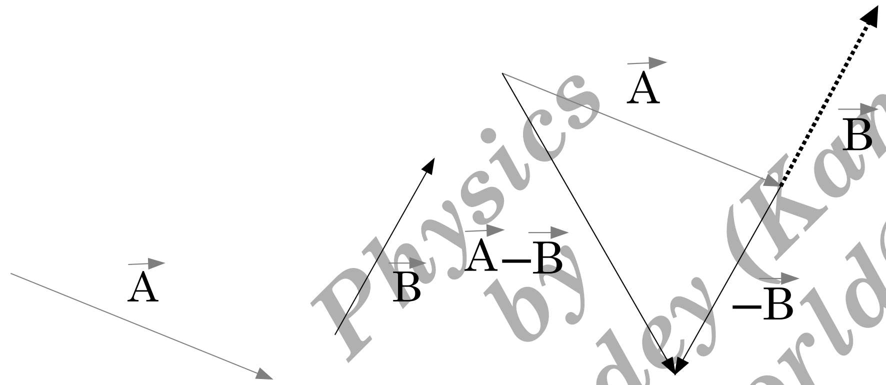


Figure .1 Geometric constructions for verifying the associative law of addition.

Vector Subtraction
You probably know that subtraction is the same thing as adding a negative: $8 - 5$ is the same thing as $8 + (-5)$. The easiest way to think about vector subtraction is in terms of adding a negative vector. What's a negative vector? It's the same vector as its positive counterpart, only pointing in the opposite direction.



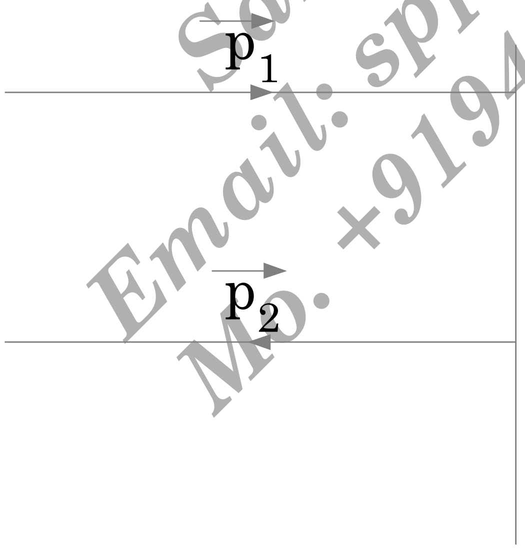
$\vec{A} - \vec{B}$, then, is the same thing as $\vec{A} + (-\vec{B})$. For instance, let's take the two vectors $\vec{A}$ and $\vec{B}$:



To subtract $\vec{B}$ from $\vec{A}$, take a vector of the same magnitude as $\vec{B}$, but pointing in the opposite direction, and add that vector to $\vec{A}$, using either the tip-to-tail method or the parallelogram method.
The operation of vector subtraction makes use of the definition of the negative of a vector. We define the operation $\vec{A} - \vec{B}$ as vector $-\vec{B}$ added to vector $\vec{A}$:
$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Examples of Subtractions

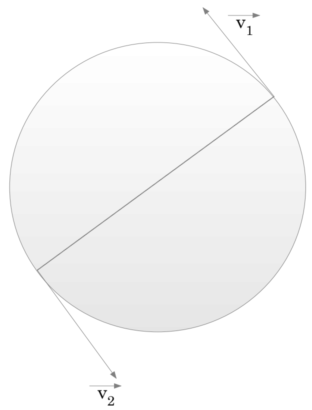
(i) If a particle makes an elastic collision with a wall in a direction normal to its surface and gets reflected just in opposite direction with same momentum, then



$|\vec{p_1}| = |\vec{p_2}| = p$ and $\vec{p_1} = +\vec{p}$, $\vec{p_2} = -\vec{p}$

Hence change in momentum is $\Delta\vec{p} = \vec{p_2} - \vec{p_2} = (-\vec{p}) - \vec{p} = -2\vec{p}$

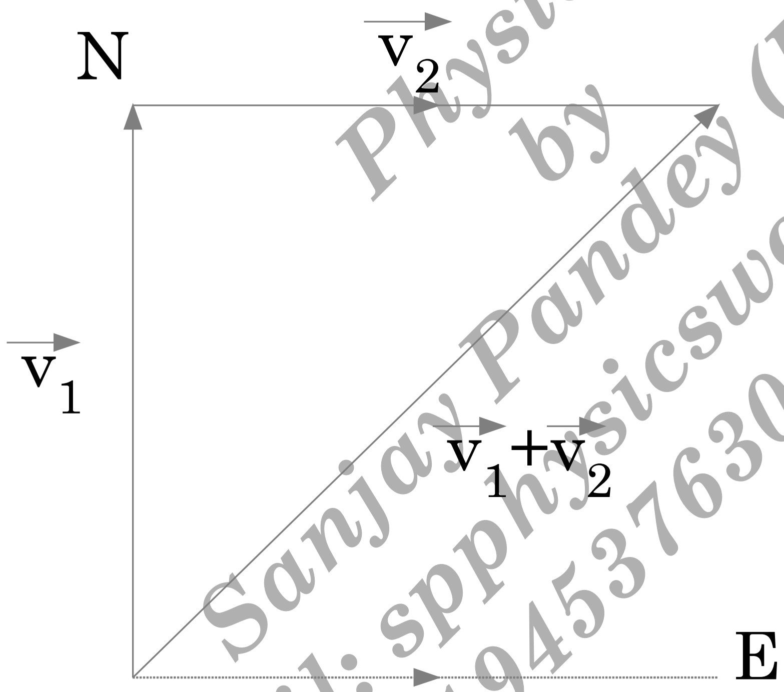
(ii) A particle is moving on a circular path of constant radius. What will be the change in its velocity when it completes half the revolution?



Here, if $\vec{v_1} = +\vec{v}$, then $\vec{v_2} = -\vec{v}$

Thus change in velocity $\Delta\vec{v} = \vec{v_2} - \vec{v_1} = \vec{v} - (-\vec{v}) = -2\vec{v}$

(iii) A particle is moving with a velocity $\vec{v_1}$ due north in a tube. If the tube is also moving with same velocity $\vec{v_2}$ due east, then what will be its resultant velocity?



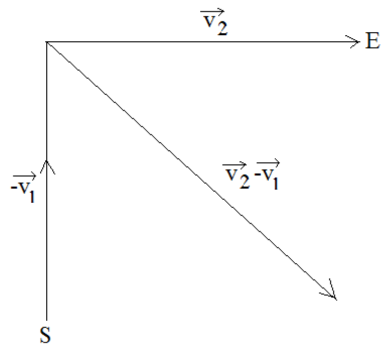
$R = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos 90^\circ}$

If $|\vec{v_1}| = |\vec{v_2}| = v$, then

$R = \sqrt{v^2 + v^2} = v\sqrt{2}$

Resultant velocity will be along North-east direction.

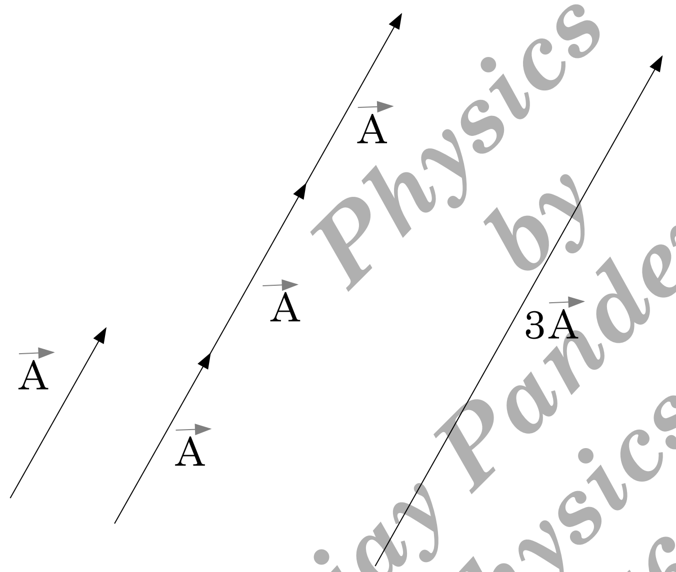
(iv) If a particle moving towards north changes its direction and moves towards east with the same speed, then what will be the change in its velocity?



$\Delta\vec{v} = \vec{v_2} - \vec{v_1}$
and $|\Delta\vec{v}| = \sqrt{v_1^2 + v_2^2 + 2v_1v_2\cos 90^\circ} = \sqrt{v^2 + v^2} = v\sqrt{2}$
Now the, direction $\Delta\vec{v}$ will be along south-east direction.

Multiplication by a Scalar:

Multiplication is like repeated addition. Multiplying 4 by 3 means adding four three times: . The multiplication of a vector times a scalar works in the same way. Multiplying the vector $\vec{A}$ by the positive scalar c is equivalent to adding together c copies of the vector A. Thus $3\vec{A} = \vec{A} + \vec{A} + \vec{A}$. Multiplying a vector by a scalar will get you a vector with the same direction, but different magnitude, as the original.

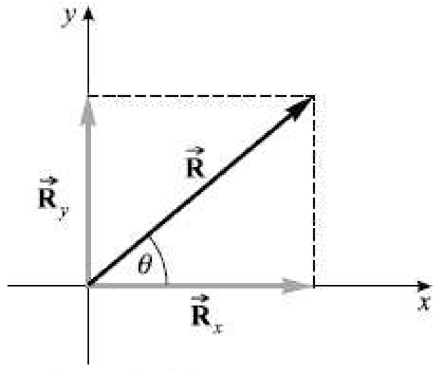


The result of multiplying $\vec{A}$ by c is a vector in the same direction as $\vec{A}$, with a magnitude of $c\vec{A}$. If c is negative, then the direction of $\vec{A}$ is reversed by scalar multiplication.

Example 1. Two equal forces act on a particle. Find the angle between them when the square of their resultant is equal to three times their product.
Solution. Let the angle between the two equal forces, P, P be α and let R be their resultant.
Hence, $R^2 = P^2 + P^2 + 2P \times P\cos\alpha$
$\implies 3P \times P = 2P^2(1 + \cos\alpha)$
$\implies \frac{3}{2} = 1 + \cos\alpha = 2\cos^2\frac{\alpha}{2}$
$\implies \cos\frac{\alpha}{2} = \frac{\sqrt{3}}{2}$
$\implies \frac{\alpha}{2} = 30^\circ \implies \alpha = 60^\circ$
Hence the required angle is 60°.

- **Component of a Vector**

A COMPONENT OF A VECTOR is its effective value in a given direction. For example, the x-component of a displacement is the displacement parallel to the x-axis caused by the given displacement. A vector in three dimensions may be considered as the resultant of its component vectors resolved along any three *mutually perpendicular* directions. Similarly, a vector in two dimensions



may be resolved into two component vectors acting along any two mutually perpendicular directions. Figure shows the vector $\vec{\mathbf{R}}$ and its x and y vector components, $\vec{\mathbf{R}}_x$ and $\vec{\mathbf{R}}_y$, which have magnitudes

$|\vec{\mathbf{R}}_x| = |\vec{\mathbf{R}}| \cos\theta$ and $|\vec{\mathbf{R}}_y| = |\vec{\mathbf{R}}| \sin\theta$

or equivalently

$$R_x = R\cos\theta \quad \text{and} \quad R_y = R\sin\theta$$

COMPONENT METHOD FOR ADDING VECTORS: Each vector is resolved into its x-, y-, and z-components, with negatively directed components taken as negative. The scalar x-component R_x of the resultant $\vec{\mathbf{R}}$ is the algebraic sum of all the scalar x-components. The scalar y- and z-components of the resultant are found in a similar way. With the components known, the magnitude of the resultant is given by

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

In two dimensions, the angle of the resultant with the x-axis can be found from the relation

$$\tan\theta = \frac{R_y}{R_x}$$

• Unit vectors

A vector of **unit magnitude** is called a **unit vector**. If any vector $\vec{a}$ is divided by its own magnitude, the result is a unit vector having the same direction as $\vec{a}$. This new vector is denoted by $\hat{a}$ so that

$\hat{a} = \frac{\vec{a}}{a}$

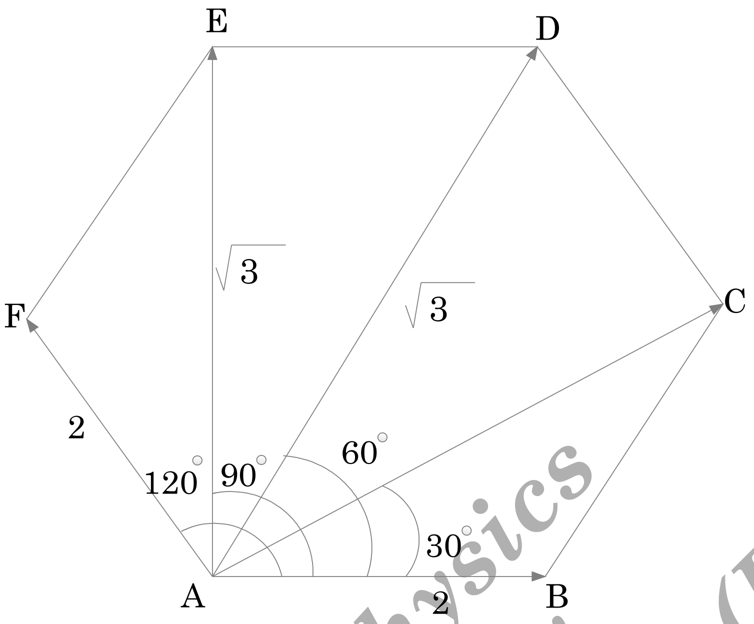
The unit vector along axes X, Y and Z are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively.

Thus any vector $\vec{A}$ can be written as $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

and $\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Example 2. Forces of magnitudes 2N, $\sqrt{3}N$, 5N, $\sqrt{3}$N and 2N act respectively at one of the angular points of a regular hexagon towards the five other in order. Find the magnitude and direction of the resultant force.

Solution. Let the forces of magnitudes 2N, $\sqrt{3}N$, 5N, $\sqrt{3}$N and 2N act at A of the regular hexagon ABCDEF along AB, AC, AD, AE and AF respectively



Clearly, $\angle BAC = \angle CAD = \angle DAE = \angle EAF = 30°$

Hence, AC, AD, AE and AF make angles 30°, 60°, 90° and 120° with AB. We take the perpendicular lines AB and AE as the X and Y-axes respectively and resolve all the forces along these two lines. If X and Y be the sums of the resolved parts of all the forces along X- and Y-axes respectively, then

$$\begin{aligned} X &= 2\cos 0° + \sqrt{3}\cos 30° + 5\cos 60 + \sqrt{3}\cos 90° + 2\cos 120° \\ &= 2 \times 1 + \sqrt{3} \times \frac{\sqrt{3}}{2} + 5 \times \frac{1}{2} + 0 - 2 \times \frac{1}{2} = 5 \end{aligned}$$

$$\begin{aligned} Y &= 2\sin 0° + \sqrt{3}\sin 30° + 5\sin 60 + \sqrt{3}\sin 90° + 2\sin 120° \\ &= 2 \times 0 + \sqrt{3} \times \frac{1}{2} + 5 \times \frac{\sqrt{3}}{2} + \sqrt{3} \times 1 + 2 \times \frac{\sqrt{3}}{2} = 5\sqrt{3} \end{aligned}$$

Hence, the resultant R of all the forces is given by

$$R = \sqrt{X^2 + Y^2} = \sqrt{5^2 + \left(5\sqrt{3}\right)^2} = 10$$

If θ be the angle made by the resultant with X-axis, then

$\tan\theta = \frac{Y}{X} = \frac{5\sqrt{3}}{5} = \sqrt{3} = \tan 60°$

$\therefore \theta = 60°$

Hence, the magnitude of the resultant is 10 N and it will act in the direction AD.

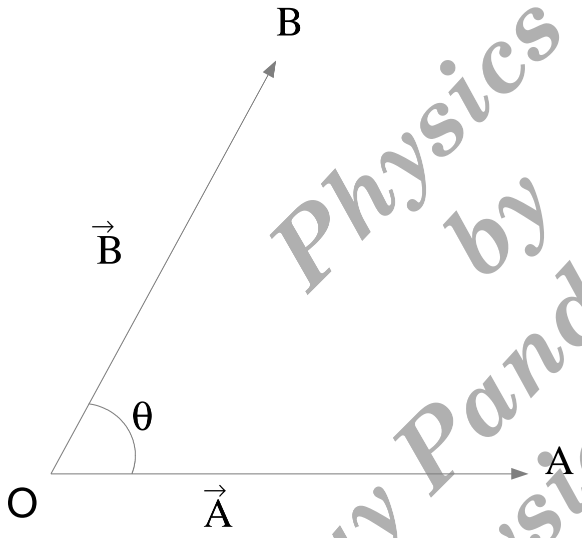
3.4 Multiplication of Vectors

Multiplication

- By a Scalar
  - $m\vec{V} = mV\hat{n} = R$, the resultant is a vector whose magnitude is m times the magnitude of $\vec{V}$ while direction is the same as that of $\vec{V}$
- By a Vector
  - SCALAR PRODUCT (dot product) $\vec{A}.\vec{B}$
  - VECTOR PRODUCT (cross product) $\vec{A}\times\vec{B}$

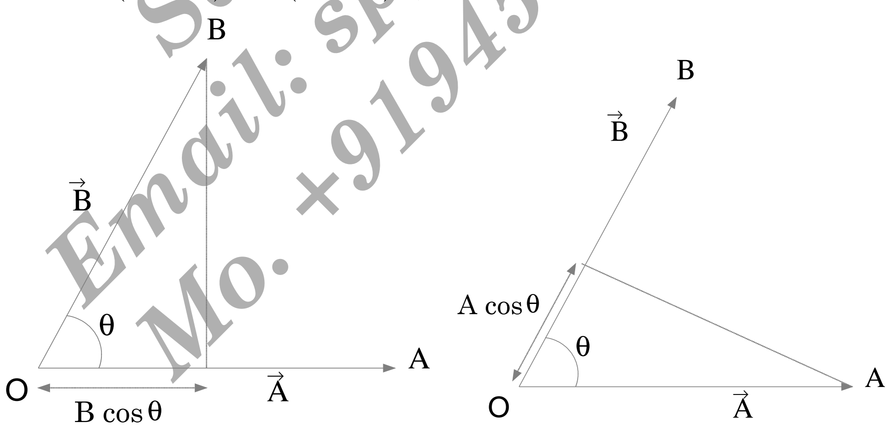
- **The Scalar Product $\vec{A}.\vec{B}$**

1. Suppose the vectors $\vec{A}$ and $\vec{B}$ have representations $\vec{OA}$ and $\vec{OB}$. Then the scalar product $\vec{A}.\vec{B}$ of $\vec{A}$ and $\vec{B}$ is defined by
   $\vec{A}.\vec{B} = AB\cos\theta$



   where θ is the angle between $\vec{OA}$ and $\vec{OB}$. [Note that $\vec{A}.\vec{B}$ is a scalar quantity.]

2. Above equation can also be expressed as below:
   $\vec{A}.\vec{B} = A(B\cos\theta) = B(A\cos\theta)$



   where $B\cos\theta$ is the magnitude of component of $\vec{B}$ along the direction of vector $\vec{A}$ and $A\cos\theta$ is the magnitude of component of vector $\vec{A}$ along the direction of $\vec{B}$[See Figure]. Therefore, **the dot product of two vectors can also be interpreted as the product of magnitude of one**

vector and the magnitude of the component of other vector along the direction of first vector.

3. Dot product of two vectors can be positive, zero or negative depending upon whether θ is less than $90°$, equal to $90°$ or $90° < \theta < 180°$.

4. **Dot product of two vectors is always commutative i.e.,**
$\vec{A}.\vec{B} = \vec{B}.\vec{A}$, $\vec{A}.\vec{B} = AB\cos\theta$ (θ is the angle between $\vec{A}$ and $\vec{B}$ measured in anticlockwise(ACW) direction)
and $\vec{B}.\vec{A} = BA\cos(-\theta) = AB\cos\theta$
Thus $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

5. The dot product of a vector with itself gives square of its magnitude i.e.
$\vec{A}.\vec{A} = AA\cos 0° = A^2$

6. The dot product of two mutually perpendicular vectors is zero i.e., if two vectors $\vec{A}$ and $\vec{B}$ are perpendicular then $\vec{A}.\vec{B} = AB\cos 90° = 0$

7. The dot product obeys the distributive law i.e., $\vec{A}.\left(\vec{B}+\vec{C}\right) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

8. Two vectors are collinear, if their dot product is numerically equal to product of their magnitudes i.e.,
when $\theta = 0°$ or $180°$, $\left|\vec{A}.\vec{B}\right| = AB$

9. In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$, we have the following two properties:
(a) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ (as $\theta = 0°$)
(b) $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$ (as $\theta = 90°$)

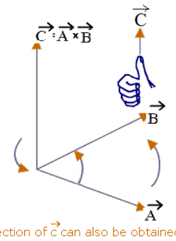
10. **Dot product of two vectors in terms of their rectangular components in three dimensions:**
$\vec{A}.\vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right).\left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right) = A_xB_x + A_yB_y + A_zB_z$

11. **Examples of some physical quantities which can be expressed as scalar product of two vectors:**
(a) **Work (W)** is defined as the scalar product of force $\left(\vec{F}\right)$ and the displacement $(\vec{s})$ i.e., $W = \vec{F}.\vec{s}$
(b) **Power (P)** is defined as the scalar product of force $\left(\vec{F}\right)$ and the velocity $(\vec{v})$ i.e., $P = \vec{F}.\vec{v}$

➤ **Vector Product of Two Vectors**

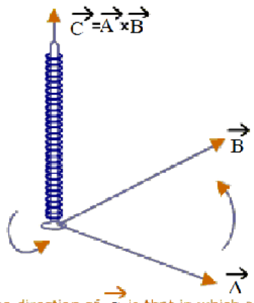
1. The vector product of two vectors is is defined as a vector having magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them and direction $\perp$ to the plane containing the two vectors in accordance with right handed screw rule or right hand thumb rule.

2. Mathematically, if θ is the angle between vectors $\vec{A}$ and $\vec{B}$, then
$\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$
The direction of vector $\vec{A}\times\vec{B}$ is the same as that of unit vector $\hat{n}$. It is decided by the following two rules:
(a) Right Hand Rule: The direction of the vector product can be visualized with the right-hand rule. If you curl the fingers of your right hand so that they follow a rotation from vector $\vec{A}$ to vector $\vec{B}$, then the thumb will point in the direction of the vector product $\vec{A}\times\vec{B}$.



The direction of $\vec{c}$ can also be obtained from the right hand rule

(b) Right hand screw rule: Rotate a right handed screw from vector $\vec{A}$ to $\vec{B}$ through the smaller angle between them; then the direction of motion of screw gives the direction of vector $\vec{A} \times \vec{B}$



The direction of $\vec{C}$ is that in which a right handed screw advances when turned from $\vec{A}$ to $\vec{B}$

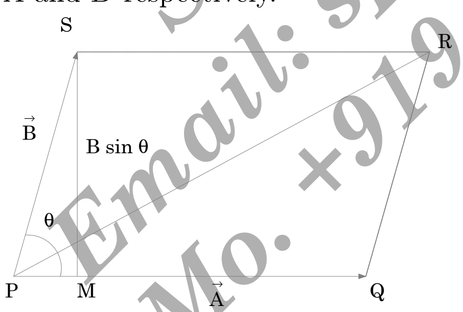
In case of vector $\vec{B} \times \vec{A}$, the direction of advancement of screw, on rotating it from vector $\vec{B}$ to $\vec{A}$, will be just opposite to that observed in the case of vector $\vec{A} \times \vec{B}$. Hence vector $\vec{B} \times \vec{A}$ is directed just in opposite direction to that of vector $\vec{A} \times \vec{B}$ i.e., $\vec{B} \times \vec{A} = -\left(\vec{A} \times \vec{B}\right)$.

3. The cross product of two vectors does not obey commutative law.
   As discussed above
   $\vec{A} \times \vec{B} = -\left(\vec{B} \times \vec{A}\right)$ *i.e.,* $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

4. The cross product follows the distributive law i.e.,
   $\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

5. The cross product of a vector with itself is a NULL vector i.e.,
   $\vec{A} \times \vec{A} = AA \sin 0° \hat{n} = 0$

6. The cross product of two vectors represents the area of the parallelogram formed by them,
   Figure, shows a parallelogram $PQRS$ whose adjacent sides $\vec{PQ}$ and $\vec{PS}$ are represented by vectors $\vec{A}$ and $\vec{B}$ respectively.



   Now, area of parallelogram $= QP \times SM = QP.AB \sin\theta$
   Because, the magnitude of vectors $\vec{A} \times \vec{B}$ is $AB \sin\theta$, hence cross product of two vectors represents the area of parallelogram formed by it. It is worth noting that the area vector $\vec{A} \times \vec{B}$acts along the perpendicular to the plane of two vectors $\vec{A}$ and $\vec{B}$.

7. In case of unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$, we obtain following two important properties:

(a) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = (1)(1)\sin 0^\circ\,(\hat{n}) = 0$
(b) $\hat{i} \times \hat{j} = (1)(1)\sin 90^\circ \left(\hat{k}\right) = \hat{k}$
where, $\hat{k}$ is a unit vector $\perp$ to the plane of $\hat{i}$ and $\hat{j}$ in a direction in which a right hand screw will advance, when rotated from $\hat{i}$ to $\hat{j}$
Also, $-\hat{j} \times \hat{i} = -(1)(1)\sin 90^\circ \left(-\hat{k}\right) = \hat{k}$
Similarly, $\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$ and $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

8. **Cross product of two vectors in terms of their rectangular components:**

$$\begin{aligned}
\vec{A} \times \vec{B} &= (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\
&= \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \\
&= \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)
\end{aligned}$$

9. **Examples of some physical quantities which can be expressed as cross product of two vectors:**
(a) The instantaneous velocity ($\vec{v}$) of a particle is equal to the cross product of its angular velocity ($\vec{\omega}$)and the position vector ($\vec{r}$) i.e.,
$\vec{v} = \vec{\omega} \times \vec{r}$
(b) The tangential acceleration ($\vec{a_t}$) of a particle is equal to the cross product of its angular acceleration ($\vec{\alpha}$) and the position vector ($\vec{r}$) i.e.,
$\vec{a_t} = \vec{\alpha} \times \vec{r}$
(c) The centripetal acceleration ($\vec{a_c}$) of a particle is equal to the cross product of its angular velocity and the linear velocity ($\vec{v}$) i.e.,
$\vec{a_c} = \vec{\omega} \times \vec{v}$
(d) The torque of a force ($\vec{F}$) is equal to cross product of the position vector ($\vec{r}$) and the applied force ($\vec{F}$) i.e.,
$\vec{\tau} = \vec{r} \times \vec{F}$
(e) The angular momentum ($\vec{L}$) is equal to the cross product of the position vector ($\vec{r}$) and linear momentum ($\vec{p}$) of the particle i.e.,
$\vec{L} = \vec{r} \times \vec{p}$

Note: If a physical quantity itself has no direction but has different values in different directions, then it is said to be TENSOR. We consider here following two examples:
(a) Moment of Inertia: It is not a vector as its direction is not to be specified but has different values in different directions. Hence it is neither scalar nor a vector but a tensor.
(b) Stress: It is defined as force per unit area (acting at a point). It has no direction so it is not a vector but at a point it has different values for different areas so it is not a scalar also. It is actually a tensor.
(c) Normally, density, dielectric constant are scalars but in case these quantities have different values in different directions, they will be called tensors

Solved Examples

Ex.1. If $\vec{v} = 6\hat{i} - 3\hat{j} + 15\hat{k}$ and $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$, find the component of $\vec{v}$ in the direction of $\vec{a}$.

Sol. $|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = 3$

$$\hat{a} = \frac{\vec{a}}{a} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

The required magnitude of component of $\vec{v}$ is therefore

$$\vec{v}.\hat{a} = (6\hat{i} - 3\hat{j} + 15\hat{k}).\frac{2\hat{i} - \hat{j} - 2\hat{k}}{3} = \frac{12 + 3 - 30}{3} = -5$$

Therefore, component of vector $\vec{v}$ along $\vec{a} = (\vec{v}.\hat{a})\hat{a} = (-5)\frac{2\hat{i}-\hat{j}-2\hat{k}}{3} = -\frac{5}{3}(2\hat{i} - \hat{j} - 2\hat{k})$

Ex.2. If $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 3\hat{k}$, find a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Sol. The vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Now

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ -1 & 0 & -3 \end{vmatrix} \\
&= (3-0)\hat{i} - ((-6) - (-2))\hat{j} + (0-1)\hat{k} \\
&= 3\hat{i} + 4\hat{j} - \hat{k}
\end{aligned}$$

The magnitude of this vector is $\left|\vec{a} \times \vec{b}\right| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$

Hence the required unit vector can be either $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \pm\frac{3\hat{i}+4\hat{j}-\hat{k}}{\sqrt{26}}$

Ex.3. If $\left|\vec{A} + \vec{B}\right| = \left|\vec{A} - \vec{B}\right|$, then find the angle between $\vec{A}$ and $\vec{B}$.

Sol. Here,

$$\begin{aligned}
\left|\vec{A} + \vec{B}\right| &= \left|\vec{A} - \vec{B}\right| \\
\Longrightarrow \left|\vec{A} + \vec{B}\right|^2 &= \left|\vec{A} - \vec{B}\right|^2 \\
\Longrightarrow \left(\vec{A} + \vec{B}\right).\left(\vec{A} + \vec{B}\right) &= \left(\vec{A} - \vec{B}\right).\left(\vec{A} - \vec{B}\right) \\
\Longrightarrow \vec{A}.\vec{A} + \vec{A}.\vec{B} + \vec{B}.\vec{A} + \vec{B}.\vec{B} &= \vec{A}.\vec{A} - \vec{A}.\vec{B} - \vec{B}.\vec{A} + \vec{B}.\vec{B} \\
\Longrightarrow A^2 + 2AB\cos\theta + B^2 &= A^2 - 2AB\cos\theta + B^2, \quad \text{here } \theta \text{be the angle between } \vec{A} \text{ and } \vec{B}. \\
\Longrightarrow 4AB\cos\theta &= 0 \\
\Longrightarrow \cos\theta &= 0 \ [\because 4AB \neq 0] \\
\Longrightarrow \theta &= 90^\circ
\end{aligned}$$

3.5 Assignment

Questions

1. Two vectors have unequal magnitudes. Can their sum be zero? Explain.

2. Can the magnitude of a particle's displacement be greater than the distance traveled? Explain.

3. The magnitudes of two vectors A and B are $A = 5$ units and $B = 2$ units. Find the largest and smallest values possible for the magnitude of the resultant vector $R = A + B$.

4. Which of the following are vectors and which are not: force, temperature, the volume of water in a can, the ratings of a TV show, the height of a building, the velocity of a sports car, the age of the Universe?

5. A vector A lies in the xy plane. For what orientations of A will both of its components be negative? For what orientations will its components have opposite signs?

6. A book is moved once around the perimeter of a tabletop with the dimensions $1.0m \times 2.0m$. If the book ends up at its initial position, what is its displacement? What is the distance traveled?

7. While traveling along a straight interstate highway you notice that the mile marker reads 260. You travel until you reach mile marker 150 and then retrace your path to the mile marker 175. What is the magnitude of your resultant displacement from mile marker 260?

8. If the component of vector A along the direction of vector B is zero, what can you conclude about the two vectors?

9. Can the magnitude of a vector have a negative value? Explain.

10. Under what circumstances would a nonzero vector lying in the xy plane have components that are equal in magnitude?

11. If $A = B$, what can you conclude about the components of A and B?

12. Is it possible to add a vector quantity to a scalar quantity? Explain.

13. The resolution of vectors into components is equivalent to replacing the original vector with the sum of two vectors, whose sum is the same as the original vector. There are an infinite number of pairs of vectors that will satisfy this condition; we choose that pair with one vector parallel to the x axis and the second parallel to the y axis. What difficulties would be introduced by defining components relative to axes that are not perpendicular—for example, the x axis and a y axis oriented at 45° to the x axis?

14. In what circumstance is the x component of a vector given by the magnitude of the vector times the sine of its direction angle?

4 Kinematics

Kinematics is the study of the motion of material bodies without regard to the forces that cause their motion.

Everything moves-even things that appear to be at rest. They move relative to the Sun and stars. As you're reading this, you're moving at about 107,000 kilometers per hour relative to the Sun, and you're moving even faster relative to the center of our galaxy. When we discuss the motion of something, we describe the motion relative to something else. If you walk down the aisle of a moving bus, your speed relative to the floor of the bus is likely quite different from your speed relative to the road. When we say a racing car reaches a speed of 300 kilometers per hour, we mean relative to the track. Unless stated otherwise, when we discuss the speeds of things in our environment, we mean relative to the surface of Earth. Motion is relative.

- **Description of motion**

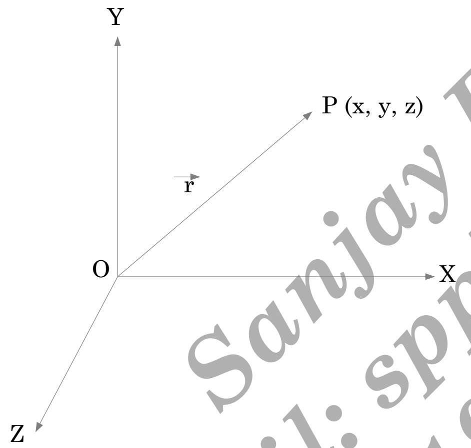
Motion, the change of the position of a body during a time interval. To describe the motion, numerical values (coordinates) are assigned to the position of the body in a **coordinate system**. The time variation of the coordinates characterizes the motion.

Uniform motion exists if the body moves equal distances in equal time intervals. *Opposite*: **non-uniform motion**.

- **Basic Definitions**

Position Vector (also called Radius vector)

The vector drawn from the reference point (origin of the co-ordinate system) to the particle to specify its position.



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

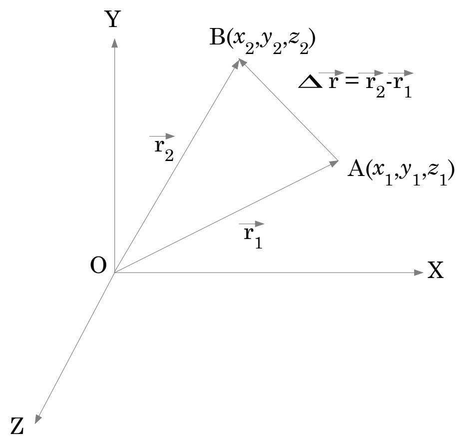
where $\vec{r}$ is the position vector of point P with respect to origin O.

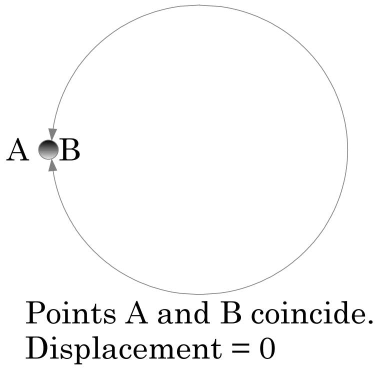
Displacement:

Change in position vector of a moving particle in a time interval is defined as displacement. The following figures illustrate the displacement of the particle in different situations.

$$\begin{aligned} \text{Displacement} &= \Delta\vec{r} = \vec{AB} = \vec{OB} - \vec{OA} = \vec{r_2} - \vec{r_1} \\ &= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \end{aligned}$$

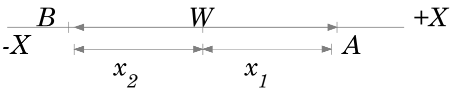
$$\text{Magnitude of displacement from A to B } = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$





Displacement is not merely "the shortest distance between the initial and final positions of the particle". This phrase stands for the separation between the initial and final positions.

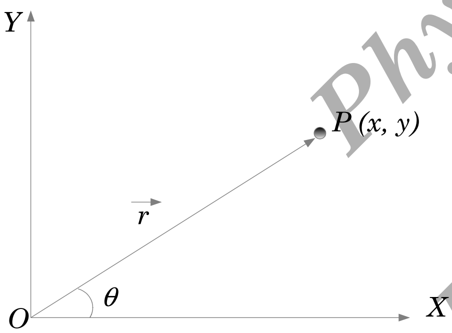
For the one dimensional motion shown in figure:



Position vector of A is, $\vec{OA} = +x_1\hat{i}$

Position vector of B is, $\vec{OB} = -x_2\hat{i}$

For the two dimensional motion:



To locate the particle P in the coordinate system shown we draw vector $\vec{OP}$. Let it be represented by $\vec{r}$.

Polar coordinates = $(r,\, \theta)$, Cartesian coordinates = $(x,\, y)$

You can change $(x,\, y)$ into $(r,\, \theta)$ or $(r,\, \theta)$ into $(x,\, y)$ as follows:

If $(x,\, y)$ are known, then

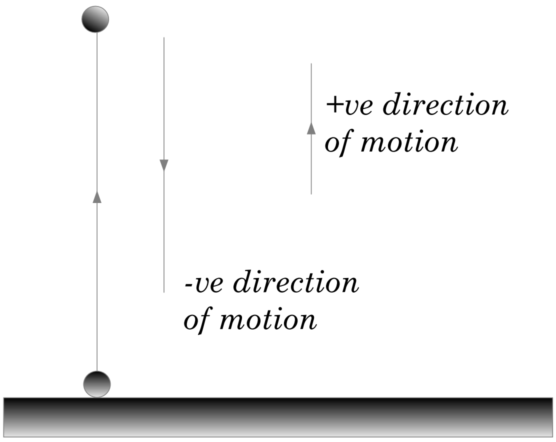
$r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

And if $(r,\, \theta)$ are known, then $x = r\cos\theta$, $y = r\sin\theta$.

Vector $\vec{r}$ can also be written as $\vec{r} = x\hat{i} + y\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along the X and Y axes.

☹ **Common Misconceptions**

1. Distance ? Is it "**path covered by a particle**" ? No. It's length of the path the particle travels.
2. Displacement ? Is it **"shortest distance between initial and final positions"** ? No. It is change in position, represented by the vector drawn from the initial position to the final position.
3. There is no difference between deceleration and retardation! No. **The do differ.** In a decelerated motion, which basically means that acceleration in a given coordinate system is $-ve$, speed may decrease, may increase depending on the direction of motion and when the speed of the particle decreases, it physically slows down, it is called retardation.



Motion is continuously decelerated (both up and down). Only upward motion is retardation

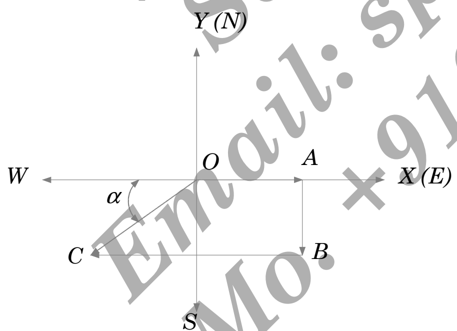
4. For motion in one-dimension, distance $d = |\vec{s}|$, $\vec{s}$ is the displacement? Not necessarily. $d = |\vec{s}|$ if motion takes place in one dimension. If the particle changes the direction of motion during a time interval, $d \neq |\vec{s}|$. In such cases the distance is calculated by tracing the path followed and calculating its length.

Distance:

In strict sense of definition it is length of the (actual path) followed by a body (precisely speaking a particle). Many a time distance is used rather loosely for 'separation' between two points.

Example.1. An Indian policeman chasing a terrorist at the time of attack on Indian parliament reported as follows. The terrorist moved 40 m towards east, made a perpendicular right turn, ran for 50 m, made a perpendicular right turn, ran for 160 m and finally he was gunned down. Find
(i) the component of displacement in east direction,
(ii) the component of displacements of displacement in north direction,
(iii) the total displacement
(iv) the total distance traversed by the terrorist, from the point when police man started chasing.

Sol. Let the path followed by terrorist is $40m$ from O to A, $50m$ from A to B and $160m$ from B to C. The total displacement is $\vec{OC} = \vec{OA} + \vec{AB} + \vec{BC} = 40\hat{i} + 50(-\hat{j}) + 160(-\hat{i}) = -120\hat{i} - 50\hat{j}$



(i) the component of displacement in east $= -120m$ i.e. $120m$ in west direction from starting point.
(ii) the displacement in in north direction $= -50m$ i.e., $50m$ in south direction from starting point.
(iii) the net displacement is $130m$ at $22°37'$south of west
(iv) total distance=path length traversed $= OA + AB + BC = 40 + 50 + 160 = 250m$

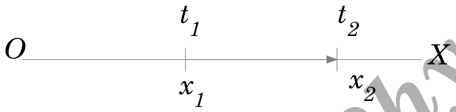
Instantaneous Speed

FIGURE
A speedometer gives readings in both miles per hour and kilometers per hour.

Things in motion often have variations in speed. A car, for example, may travel along a street at 50 km/h, slow to 0 km/h at a red light, and speed up to only 30 km/h because of traffic. You can tell the speed of the car at any instant by looking at its speedometer. The speed at any instant is the instantaneous speed. A car traveling at 50 km/h usually goes at that speed for less than one hour. If it did go at that speed for a full hour, it would cover 50 km. If it continued at that speed for half an hour, it would cover half that distance: 25 km. If it continued for only one minute, it would cover less than 1 km.

Average Velocity and Average Speed

Average velocity $\bar{v}$ is defined as the ratio of displacement divided by the corresponding time interval.

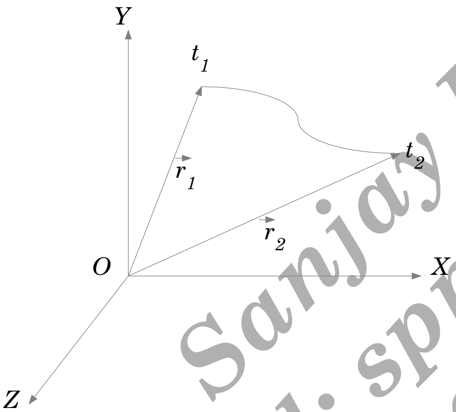


$\bar{v} = \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{x_2}-\vec{x_1}}{t_2-t_1}$

whereas *average speed* is just the total distance divided by the time interval,

average speed $= \frac{\text{total distance}}{\Delta t}$

For the two and three dimensional motion



average velocity $= \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r_2}-\vec{r_1}}{t_2-t_1}$

The direction of average velocity is that of change in position vector.

The **instantaneous velocity** is given as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\vec{\Delta r}}{\Delta t} = \frac{\vec{dr}}{dt}$$

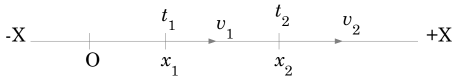
When the instantaneous velocity of a particle is constant over entire time interval of motion, it is said to be moving with **uniform velocity**. Or "when the average velocity of the particle calculated over different time intervals chosen randomly turns out to be the same, the velocity of the particle is uniform".

Acceleration:

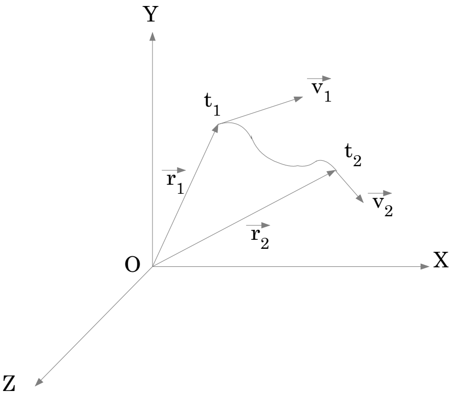
Rate of change of velocity is called acceleration.

Average acceleration $= \frac{\text{net change in velocity}}{\text{total time elapsed}}$

For the one-dimensional motion shown in the figure,



average acceleration
$< a >= \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}$
Acceleration of the particle at any given instant of time given by



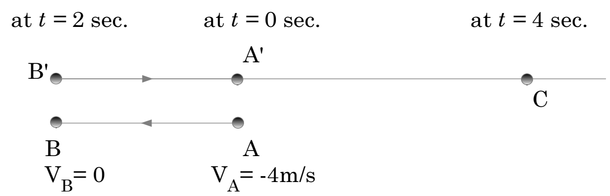
$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$,
is called the **instantaneous acceleration**.

★ When the instantaneous acceleration is same over entire time interval of the motion, the motion is said to be uniformly accelerated.

★ When the average acceleration calculated over different time intervals chosen randomly turns out to be the same the motion of the particle is uniformly accelerated.

Example.2: The position of a particle moving on a straight line is defined by the relation $x = t^3 - 2t^2 - 4t$, where x is expressed in meter and t in second. Determine
(a) the velocity at $t = 1$ second and $t = 4$ second,
(b) the distance traveled in first 4 seconds,
(c) the average velocity during the interval of 1 second to 4 second,
(d) The acceleration at $t = 4$ second and
(e) the average acceleration during 1 second to 4 second

Solution. Equation of motion of the particle is
$x = t^3 - 2t^2 - 4t$ ... (1)
Instantaneous velocity
$v = \frac{dx}{dt} = 3t^2 - 4t - 4$ ... (2)
Instantaneous acceleration
$a = \frac{dv}{dt} = 6t - 4$ ... (3)
(a) $v_1(t = 1) = 3 \times 1^2 - 4 \times 1 - 4 = -5 m/s$ (using equation (2))
$v_4(t = 4) = 3 \times 4^2 - 4 \times 4 - 4 = 28 m/s$
(b) Let us check, whether particle continues to move in same direction or it changes its direction during the considered section of motion. Using equation (2) if we put $v = 0$ we will find
$3t^2 - 4t - 4 = 0$
$\Longrightarrow 3t^2 - 6t + 2t - 4 = 0$
$\Longrightarrow 3t(t - 2) + 2(t - 2) = 0$
$\Longrightarrow (t - 2)(3t + 2) = 0$
$\Longrightarrow (t = 2)$ [because $t \neq -\frac{2}{3}$]
i.e., at $t = 2$ second velocity becomes zero and the particle reverses the direction of motion. Let the displacement in right direction be taken as positive and left direction as negative. The figure given below is the path followed by the particle in first four seconds from A to C.



$x_2(t=2) = 2^3 - 2 \times 2^2 - 4 \times 2 = -8\text{m}$
(-ve sign indicates that particle is in left direction relative to starting point A)
i.e., $AB = +8m$ and $x_4(t=4) = 4^3 - 2 \times 4^2 - 4 \times 4 = +16m$ i.e., $A'C = 16m$.

Note: In the above figure the path of the particle during returning is shown along $B'A'C$ to understand the problem easily, actually the path of particle during returning will be same as its initial path (line AB).

From above figure the distance traversed by the particle is given by
$AB + B'C = AB + B'A' + A'C = 8 + 8 + 16 = 32m$
(c) $x_1(t=1) = 1^3 - 2(1^2) - 4(1) = -5m$
$x_2(t=4) = 4^3 - 2(4^2) - 4(4) = 16m$
Therefore the average velocity during 1 second to 4 second is given by:
$v_{av} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{16-(-5)}{4-1} = \frac{21}{3} = 7m/s$
(d) $a(t=4) = (6 \times 4) - 4 = 20m/s^2$
(e) $v_1(t=1) = 3(1^2) - 4(1) - 4 = -5m/s$
$v_2(t=4) = 3(4^2) - 4(4) - 4 = 28m/s$
$a_{av} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{28-(-5)}{4-1} = \frac{33}{3} = 11m/s$

Equations of motion of a particle moving with uniform acceleration along a straight line:

Equation Number	Equation	Missing Quantity
1.	$v = u + at$	s
2.	$s = ut + \frac{1}{2}at^2$	v
3.	$s = vt - \frac{1}{2}at^2$	u
4.	$s = \frac{1}{2}(u+v)t$	a
5.	$v^2 = u^2 + 2as$	t
6.	$< v >= \frac{u+v}{2}$	
7.	$s_n = u + \frac{a}{2}(2n-1)$	

Note: **(i)** (u, v, a and s are $+ve$ if they are directed along the $+ve$ direction of motion and $-ve$ if they are oppositely directed). You can take opposite sign also, but final equation of motion will be same. However to avoid the miscalculation we generally take parameters in the direction of motion as positive and in reverse direction as negative.
(ii) These equations are applicable only when acceleration a remains constant in magnitude and direction both.

Example 3:
A particle, moving with a uniform acceleration along a straight line ABC, crosses point A at $t = 0$ with a velocity $12m/s$. B is $40m$ away from A and C is $64m$ from A. The particle passes B at $t = 4sec$.
(i) After what time will the particle be at C ?
(ii) What is its velocity at C ?
Sol. Let the acceleration of the particle be a.