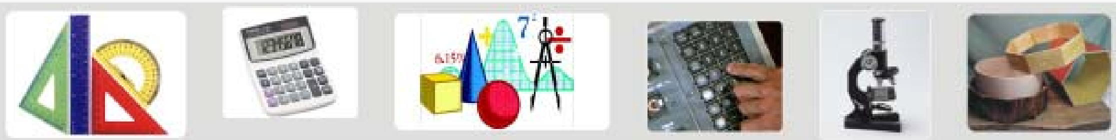
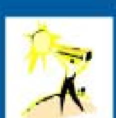
MATHEMATICS



THEORY

EXAMPLES

EXERCISES



SOLUTION & TIPS

TRIGONOMETRICAL RATIOS

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TRIGONOMETRICAL RATIOS

Preface

IIT-JEE Syllabus : Trigonometrical Ratios
Trigonometrical ratios of compound angles, Trigonometric ratios of multiple angles, sub multiple angles, conditional identities, greatest and the least value of the expression.

Trigonometry is the corner stone of the whole mathematics of which trigonometric ratio plays an important role. It is observed that there is a clear lack of problem solving aptitude which was an absolute prerequisite for an examination like IIT-JEE.

It is motivated us to compile the concepts, fundamentals to fulfill this vaccume but would be helpful to elevate the ordinary students to become extra ordinary. Before studying trigonometric ratio students are advised to clear the basic concept of trigonometry.

This material is exclusively designed by the CAREER POINT'S core members so that CPians need not refer to any other book or study material.

"Future belongs to those who are willing to work for it"

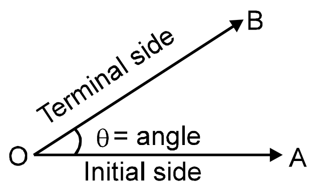
Total number of Questions in **Trigonometrical Ratios** are :
In chapter Examples 21

1. DEFINITION

Trigonometry is the branch of science in which we study about the angles and sides of a triangle.

1.1 ANGLE :

Consider a ray $\overrightarrow{OA}$. If this ray rotates about its end points O and takes the position OB, then the angle ∠AOB has been generated.



An angle is considered as the figure obtained by rotating a given ray about its end-point.

The initial position OA is called the initial side and the final position OB is called terminal side of the angle. The end point O about which the ray rotates is called the vertex of the angle.

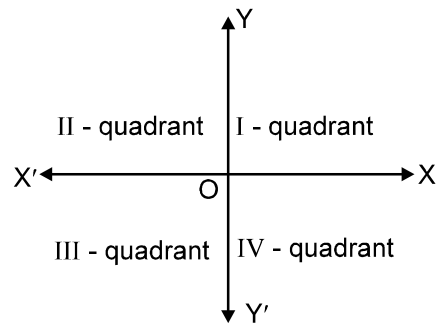
1.2 Sense of an Angle :

The sence of an angle is said to be positive or negative according as the initial side rotates in anticlockwise or clockwise direction to get to the terminal side.

1.3 Some Useful terms :

1.3.1 Quadrant :

Let XOX′ and YOY′ be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants.



The lines XOX′ and YOY′ are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY, YOX′, X′OY′ and Y′OX are known as the first, the second, the third and the fourth quadrant respectively.

1.3.2 Angle In Standard Position : An angle is said to be in standard position if its vertex coincides with the origin O and the initial side coincides with OX i.e. the positive direction of x-axis.

1.3.3 Co–terminal Angles : Two angles with different measures but having the same initial sides and the same terminal sides are known as co-terminal angles.

2. SYSTEM OF MEASUREMENT OF ANGLE

There are three system for measuring angles.

2.1 Sexagesimal or English system

2.2 Centesimal or French system

2.3 Circular system

2.1 Sexagesimal system :

The principal unit in this system is degree (°). One right angle is divided into 90 equal parts and each part is called one degree (1°). One degree is divided into 60 equal parts and each part is called one minute. Minute is denoted by (1′). One minute is equally divided into 60 equal parts and each part is called one second (1″).

In Mathematical form :

One right angle = 90°

$1^\circ = 60'$

$1' = 60''$

Examples based on Sexagesimal system

Ex.1 45° 15″ 30″ changes into degree

Sol. 60″ is equal to 1′

$1''$ is equal to $\left(\frac{1}{60}\right)'$

$30''$ is equal to $\left(\frac{1}{60}\times 30\right)' = \left(\frac{1}{2}\right)'$

Total minutes $\Rightarrow 15' + \left(\frac{1}{2}\right)' = \left(\frac{31}{2}\right)'$

60′ is equal to 1° and 1′ is equal to $\left(\frac{1}{60}\right)^\circ$

$\left(\frac{31}{2}\right)'$ is equal to $\left(\frac{1}{60}\times\frac{31}{2}\right)^\circ = \left(\frac{31}{120}\right)^\circ$

Total degrees $\Rightarrow 45^\circ +$

$\Rightarrow \qquad \Rightarrow$

2.2 Centesimal system :

The principal unit in system is grade and is denoted by (^{g}). One right angle is divided into 100 equal parts, called grades, and each grade is subdivided into 100 minutes, and each minutes into 100 seconds.

In Mathematical Form :

One right angle = 100^g

$1^g = 100'$

$1' = 100''$

Centesimal system

Ex.2 $50^g\ 30'\ 50''$ change into grade system.

Sol. We know that , $50' \Rightarrow \left(\frac{1}{2}\right)'$

Total minute $30' + \left(\frac{1}{2}\right)' = \left(\frac{61}{2}\right)'$

100′ is equal to 1^g

1′ is equal to $\left(\frac{1}{100}\right)^g$

$\left(\frac{61}{2}\right)'$ is equal to $\left(\frac{1}{100}\times\frac{61}{2}\right)^g =$

Total grade $\Rightarrow 50^g +$

$\Rightarrow \left(\frac{10000+61}{200}\right)^g \Rightarrow \left(\frac{10061}{200}\right)^g$

2.2.1 Relation between sexagesimal and centesimal systems :

One right angle = 90° (degree system) .... (1)

One right angle = 100^g (grade system) .... (2)

by (1) and (2),

$$90^\circ = 100^g$$

or, $$\frac{D}{90} = \frac{G}{100}$$

then we can say,

$$1^\circ = \left(\frac{100}{90}\right)^g$$

$$1^g = \left(\frac{9}{10}\right)^\circ$$

Examples based on Relation between sexagesimal and centesimal systems

Ex.3 63° 14′ 51″ change into grade system.

Sol. We know that in degree system

60″ equal to 1′

51″ is equals = $\left(\frac{51}{60}\right)' = (0.85)'$

(14.85)′ change into degree.

(14.85)′ is equals = $\left(\frac{14.85}{60}\right)^\circ$

$= (0.2475)^\circ$

So 63° 14′ 51″ = 63.2475°

63.2475° change into grade system.

63.2475° is equals = $\left(63.2475\times\frac{10}{9}\right)^g$

$= 70.2750^g$

$70.2750^g = 70^g\ 27'\ 50''$

finally we can say,

$63^o\ 14'\ 57' = 70^g\ 27'\ 50''$

2.3 Circular system :

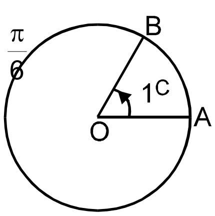
One radian, written as 1^C, is the measure of an angle subtended at the centre of a circle by an arc of length equal to the radius of the circle.

Consider a circle of radius r having centre at O. Let A be a point on the circle. Now cut off an arc AB whose length is equal to the radius r of the circle. Then by the definition the measure of $\angle AOB$ is 1 radian (1^C).

2.3.1 Some Important Conversion :

π Radian = 180^o

One radian = $\left(\frac{180}{\pi}\right)^o$



$\frac{\pi}{6}$ Radian = 30^o

$\frac{\pi}{4}$ Radian = 45^o

$\frac{\pi}{3}$ Radian = 60^o

$\frac{\pi}{2}$ Radian = 90^o

$\frac{2\pi}{3}$ Radian = 120^o

$\frac{3\pi}{4}$ Radian = 135^o

$\frac{5\pi}{6}$ Radian = 150^o

$\frac{7\pi}{6}$ Radian = 210^o

$\frac{5\pi}{4}$ Radian = 225^o

$\frac{5\pi}{3}$ Radian = 300^o

2.3.2 Relation between systems of measurement of angles :

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

Examples based on Relation between systems of measurement of angles

Ex.4 $\left(\frac{2\pi}{15}\right)^C$ change into degree system.

Sol. We know that, π radian = 180^o

$$1^C = \left(\frac{180}{\pi}\right)^o$$

$$\left(\frac{2\pi}{15}\right)^C = \left(\frac{2\pi}{15} \times \frac{180}{\pi}\right)^o = 24^o$$

Ex.5 Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15^o.

Sol. Let s be the length of the arc subtending an angle θ at the centre of a circle of radius r.

$$\text{then , } \theta = \frac{s}{r}$$

Here, r = 5 cm, and $\theta = 15^o = \left(15 \times \frac{\pi}{180}\right)^C$

$$\theta = \left(\frac{\pi}{12}\right)^C$$

$$\theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5}$$

$$s = \frac{5\pi}{12} \text{ cm.}$$

3.TRIGONOMETRICAL RATIOS OR FUNCTIONS

In the right angled triangle OMP , we have base (OM) = x, perpendicular (PM) = y and hypotenuse (OP) = r, then we define the following trigonometric ratios which are known as trigonometric function.

$\sin\theta = \frac{P}{H} = \frac{y}{r}$

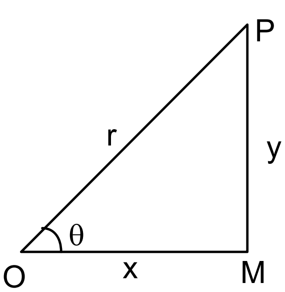
$\cos\theta = \frac{B}{H} = \frac{x}{r}$

$\tan\theta = \frac{P}{B} = \frac{y}{x}$

$\cot\theta = \frac{B}{P} = \frac{x}{y}$

$\sec\theta = \frac{H}{B} = \frac{r}{x}$

$\text{cosec}\theta = \frac{H}{P} = \frac{r}{y}$



Note :

(1) It should be noted that $\sin\theta$ does not mean the product of sin and θ. The $\sin\theta$ is correctly read sin of angle θ.

(2) These functions depend only on the value of the angle θ and not on the position of the point P chosen on the terminal side of the angle θ.

3.1 Fundamental Trigonometrical Identities :

(a) $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$

(b) $\cos\theta = \dfrac{1}{\sec\theta}$

(c) $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

(d) $1 + \tan^2\theta = \sec^2\theta$

or, $\sec^2\theta - \tan^2\theta = 1$

$(\sec\theta - \tan\theta) = \dfrac{1}{(\sec\theta + \tan\theta)}$

(e) $\sin^2\theta + \cos^2\theta = 1$

(f) $1 + \cot^2\theta = \text{cosec}^2\theta$

$(\text{cosec}\,\theta - \cot\theta) = \dfrac{1}{\text{cosec}\,\theta + \cot\theta}$

Examples based on Trigonometrical ratios or functions

Ex.6 Prove that, $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

Sol. L.H.S, $(\sin^8\theta - \cos^8\theta)$

or, $(\sin^4\theta)^2 - (\cos^4\theta)^2$

or, $(\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)[(\sin^4\theta + \cos^4\theta)]$

or, $(\sin^2\theta - \cos^2\theta)\,[(\sin^2\theta + \cos^2\theta) - 2\sin^2\theta\cos^2\theta]$

or, $(\sin^2\theta - \cos^2\theta)\,[(1 - 2\sin^2\theta\cos^2\theta)]$ = RHS

Ex.7 Prove the identity $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$

Sol. L.H.S $= \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$= \dfrac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$

[$\sec^2\theta - \tan^2\theta = 1$]

=

$= \dfrac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{\tan\theta - \sec\theta + 1}$

$= \sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$

= RHS

3.2 Signs of the trigonometrical ratios or functions:

Their signs depends on the quadrant in which the terminal side of the angle lies.

In First quadrant : $x > 0$, $y > 0 \Rightarrow \sin\theta = \dfrac{y}{r} > 0$, $\cos\theta = \dfrac{x}{r} > 0$, $\tan\theta = \dfrac{y}{x} > 0$, $\text{cosec}\,\theta = \dfrac{r}{y} > 0$, $\sec\theta = \dfrac{r}{x} > 0$ and $\cot\theta = \dfrac{x}{y} > 0$

Thus, in the first quadrant all trigonometry functions are positive.

In Second quadrant : $x < 0$, $y > 0 \Rightarrow \sin\theta = \dfrac{y}{r} > 0$, $\cos\theta = \dfrac{x}{r} < 0$, $\tan\theta = \dfrac{y}{x} < 0$, $\text{cosec}\,\theta = \dfrac{r}{y} > 0$, $\sec\theta = \dfrac{r}{x} < 0$ and $\cot\theta = \dfrac{x}{y} < 0$

Thus, in the second quadrant sin and cosec function are positive and all others are negative.

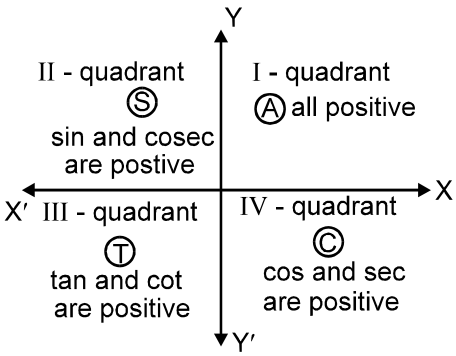
In Third quadrant : $x < 0$, $y < 0 \Rightarrow \sin\theta = \dfrac{y}{r} < 0$, $\cos\theta = \dfrac{x}{r} < 0$, $\tan\theta = \dfrac{y}{x} > 0$, $\text{cosec}\,\theta = \dfrac{r}{y} < 0$, $\sec\theta = \dfrac{r}{x} < 0$ and $\cot\theta = \dfrac{x}{y} > 0$

Thus, in the third quadrant all trigonometric functions are negative except tangent and cotangent.

In Fourth quadrant : $x > 0$, $y < 0 \Rightarrow \sin\theta = \dfrac{y}{r} < 0$, $\cos\theta = \dfrac{x}{r} > 0$, $\tan\theta = \dfrac{y}{x} < 0$, $\text{cosec}\,\theta = \dfrac{r}{y} < 0$, $\sec\theta = \dfrac{r}{x} > 0$ and $\cot\theta = \dfrac{x}{y} < 0$

Thus, in the fourth quadrant all trigonometric functions are negative except cos and sec.

To be Remember :



A crude aid to memorise the signs of trigonometrical ratio in different quadrant.

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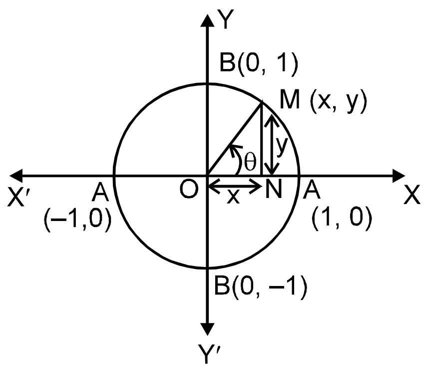
3.3 Variations in values of Trigonometrical Functions in Different Quadrants :



Let XOX′ and YOY′ be the coordinate axes. Draw a circle with centre at origin O and radius unity. Let M (x, y) be a point on the circle such that $\angle AOM = \theta$

then $x = \cos\theta$ and $y = \sin\theta$

$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ for all values of θ.

I – Quadrant

$\sin\theta \longrightarrow$ increases from 0 to 1

$\cos\theta \longrightarrow$ decreases from 1 to 0

$\tan\theta \longrightarrow$ increases from 0 to ∞

$\cot\theta \longrightarrow$ decreases from ∞ to 0

$\sec\theta \longrightarrow$ increases from 1 to ∞

$\text{cosec}\theta \longrightarrow$ decreases from ∞ to 1

II – Quadrant

$\sin\theta \longrightarrow$ decreases from 1 to 0

$\cos\theta \longrightarrow$ decreases from 0 to –1

$\tan\theta \longrightarrow$ increases from – ∞ to 0

$\cot\theta \longrightarrow$ decreases from 0 to – ∞

$\sec\theta \longrightarrow$ increases from –∞ to –1

$\text{cosec}\theta \longrightarrow$ increases from 1 to ∞

III – Quadrant

$\sin\theta \longrightarrow$ decreases from 0 to –1

$\cos\theta \longrightarrow$ increases from –1 to 0

$\tan\theta \longrightarrow$ increases from 0 to ∞

$\cot\theta \longrightarrow$ decreases from ∞ to 0

$\sec\theta \longrightarrow$ decreases from –1 to –∞

$\text{cosec}\theta \longrightarrow$ increases from –∞ to –1

IV – Quadrant

$\sin\theta \longrightarrow$ increases from –1 to 0

$\cos\theta \longrightarrow$ increases from 0 to 1

$\tan\theta \longrightarrow$ increases from –∞ to 0

$\cot\theta \longrightarrow$ decreases from 0 to –∞

$\sec\theta \longrightarrow$ decreases from ∞ to 1

$\text{cosec}\theta \longrightarrow$ decreases from –1 to – ∞

Remark:

+ ∞ and – ∞ are two symbols. These are not real number. When we say that tan θ increases from 0 to ∞ for as θ varies from 0 to $\frac{\pi}{2}$ it means that tan θ increases in the interval $\left(0, \frac{\pi}{2}\right)$ and it attains large positive values as θ tends to $\frac{\pi}{2}$. Similarly for other trigo. functions.

Examples based on **Signs of the trigonometrical ratios or functions**

Ex.8 If $\sec\theta = \sqrt{2}$, and $\frac{3\pi}{2} < \theta < 2\pi$. Find the value of $\frac{1+\tan\theta+\text{cosec}\theta}{1+\cot\theta-\text{cosec}\theta}$

Sol. If $\sec\theta = \sqrt{2}$

or, $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \pm\sqrt{1-\cos^2\theta}$

$= \pm\sqrt{1-\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

But θ lies in the fourth quadrant in which sinθ is negative.

$\sin\theta = -\frac{1}{\sqrt{2}}$, $\quad \text{cosec}\ \theta = -\sqrt{2}$

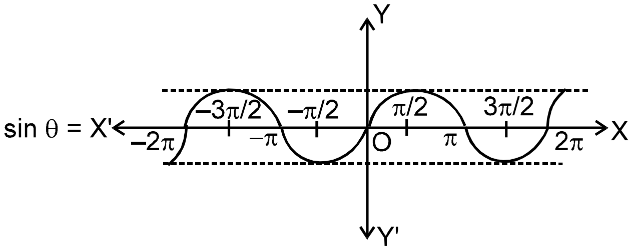
$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1}$

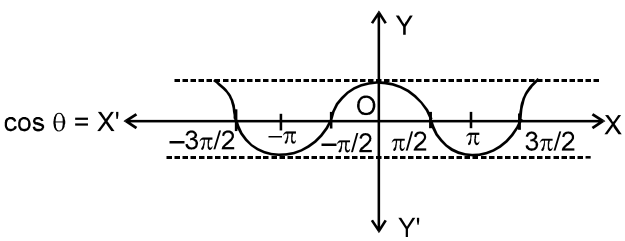
$\Rightarrow \tan\theta = -1$

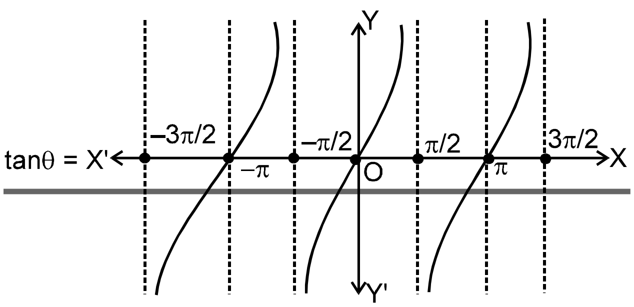
$\Rightarrow \cot\theta = -1$

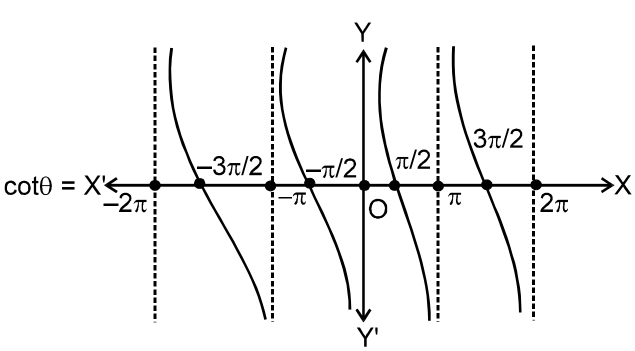
then, $\frac{1+\tan\theta+\text{cosec}\theta}{1+\cot\theta-\text{cosec}\theta} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} \Rightarrow -1$

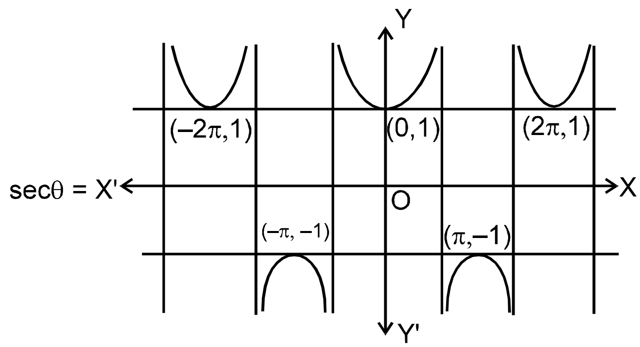
4. GRAPHS OF DIFFERENT TRIGONOMETRICAL RATIOS :



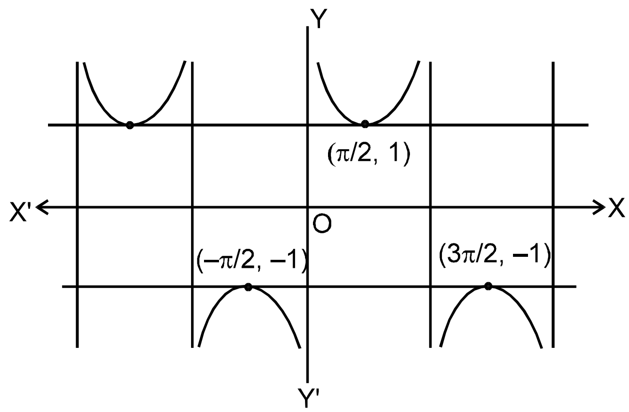








cosecθ =



4.1 Domain and Range of Trigonometrical Function

Trig. Function	Domain	Range
sin θ	R	[-1, 1]
cos θ	R	[-1, 1]
tan θ	R - {(2n+1) π/2, n ∈ z}	(–∞, ∞) or R
cosec θ	R - {nπ, n ∈ z}	(-∞, -1] ∪ [1, ∞)
sec θ	R - {(2n + 1) π/2, n ∈ z}	(-∞, -1] ∪ [1, ∞)
cot θ	R - {nπ, n ∈ z}	(–∞, ∞) = R

5. TRIGONOMETRICAL RATIOS OF ALLIED ANGLES

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90º.

5.1 Trigonometrical Ratios of (–θ) :

Let a revolving ray starting from its initial position OX, trace out an angle

$\angle$ XOA = θ. Let P(x, y) be a point on OA such that OP = r. Draw PM ⊥ from P on x-axis. angle $\angle$ XOA$'$ = –θ in the clockwise sense. Let P$'$ be a point on OA$'$ such that OP$'$ = OP. Clearly M and M$'$ coincide and ΔOMP is congruent to ΔOMP$'$. then P$'$ are (x, –y)

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$$\sin(-\theta) = \frac{-y}{r} \Rightarrow \frac{-y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta) = \frac{-y}{x} = -\tan\theta$$

Taking the reciprocal of these trigonometric ratios,

cosec (–θ)= –cosecθ

sec(–θ)= secθand cot(–θ) = – cotθ

Note : A function f(x) is said to be even function if f(–x) = f(x) for all x in its domain.

A function f(x) is an odd function if f(–x) = –f(x) for all x in its domain.

sinθ, tanθ, cotθ, cosecθ all odd functions and cosθ, secθ are even functions.

Examples based on **Allied angles**

Ex.9 Find the value of the following trigonometric ratios -

(a) cos (–45º) (b) sin (–30º)

(c) cot (–60º)

Sol.(a) cos (–45º) = cos45º [∵ cos(–θ) = cosθ]

= **Ans.**

(b) sin(–30º) = – sin 30º [∵ sin(–θ) = – sinθ]

= **Ans.**

(c) cot(–60º) = – cot 60º [∵ cot(–θ) = – cotθ]

= **Ans.**

5.2 Trigonometrical Functions of (90 – θ) :

Let the revolving line, starting from OA, trace out any acute angle AOP, equal to θ. From any point P on it draw PM ⊥ to OA. Three angles of a triangle are together equal to two right angles, and since OMP is a right angle, the sum of the two angles MOP and OPM is right angle.

$\angle$ OPM = 90º – θ.

[When the angle OPM is consider, the line PM is the ´base´ and MO is the ´perpendicular´]

$$\sin(90^\circ - \theta) = \sin MPO = \frac{MO}{PO} = \cos AOP = \cos\theta$$

$$\cos(90^\circ - \theta) = \cos MPO = \frac{PM}{PO} = \sin AOP = \sin\theta$$

$$\tan(90^\circ - \theta) = \tan MPO = \frac{MO}{PM} = \cot AOP = \cot\theta$$

$$\cot(90^\circ - \theta) = \cot MPO = \frac{PM}{MO} = \tan AOP = \tan\theta$$

$$\text{cosec}(90^\circ - \theta) = \text{cosec } MPO = \frac{PO}{MO} = \sec AOP = \sec\theta$$

and $\sec(90^\circ - \theta) = \sec MPO = \frac{PO}{PM} = \text{cosec } AOP = \text{cosec}\theta$

Trigo. ratio	$(-\theta)$	$90-\theta$ or $\left(\frac{\pi}{2}-\theta\right)$	$90+\theta$ or $\left(\frac{\pi}{2}+\theta\right)$	$180-\theta$ or $(\pi-\theta)$	$180+\theta$ or $(\pi+\theta)$	$270-\theta$ or $\left(\frac{3\pi}{2}-\theta\right)$	$270+\theta$ or $\left(\frac{3\pi}{2}+\theta\right)$	$360-\theta$ or $(2\pi-\theta)$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$

5.3 Trigonometrical Functions of $(90 + \theta)$:

Let a revolving ray OA starting from its initial position OX, trace out an angle $\angle XOA = \theta$ and let another revolving ray OA′ starting from the same initial position OX, first trace out an angle θ so as to coincide with OA and then it revolves through an angle of 90º in anticlockwise direction to form an angle $\angle XOA' = 90^\circ + \theta$.

Let P and P′ be points on OA and OA′ respectively such that OP = OP′ = r.

Draw perpendicular PM and P′M′ from P and P′ respectively on OX. Let the coordinates of P be (x, y). Then OM = x and PM = y clearly,

OM′ = PM = y and P′M′ = OM = x

so the coordinates of P′ are (–y, x)

$$\sin(90+\theta) = \frac{M'P'}{OP'} = \frac{x}{r} = \cos\theta$$

$$\cos(90+\theta) = \frac{OM'}{OP'} = \frac{-y}{r} = -\sin\theta$$

$$\tan(90+\theta) = \frac{M'P'}{OM'} = -\frac{x}{y} = -\cot\theta$$

similarly,

$\cot(90+\theta) = -\tan\theta$

$\sec(90+\theta) = -\operatorname{cosec}\theta$

$\operatorname{cosec}(90+\theta) = \sec\theta$

[where $-\pi/2 < \theta < \pi/2$]

5.4 Periodic Function :

All the trigonometric functions are periodic functions. They will repeat after a certain period

$$\left.\begin{aligned} \sin(2n\pi+\theta) &= \sin\theta \\ \cos(2n\pi+\theta) &= \cos\theta \\ \tan(2n\pi+\theta) &= \tan\theta \end{aligned}\right\} \text{ where } n \in I$$

Examples based on Trigonometric ratio of allied angles

Ex.10 Prove that, $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ = -1$

Sol. LHS = $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ$

$= \cos(360^\circ + 150^\circ)\cos(360^\circ - 30^\circ) + \sin(360^\circ + 30^\circ)\cos(90^\circ + 30^\circ)$

$= \cos 150^\circ \cos 30^\circ - \sin 30^\circ(-\sin 30^\circ)$

$= \cos(180^\circ - 30^\circ)\ \frac{3}{4} + \frac{1}{4}$

$= -\cos 30^\circ \left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}$

$= -\frac{3}{4} - \frac{1}{4} = -1 = \text{R.H.S}$

6. SUM OR DIFFERENCE OF THE ANGLE

The algebraic sums of two or more angles are generally called compound angles and the angles are known as the constituent angles.

For example : If A, B, C are three angles then A ± B, A + B + C, A – B + C etc. are compound angles.

6.1 (a) sin (A + B) =sin A cos B + cos A sin B

(b) sin(A – B) = sinA cosB – cosA sinB

$\frac{4}{5}$ (c) cos (A + B) = cos A cos B – sin A sin B

(d) cos (A – B) = cos A cos B + sin A sin B

(e) tan (A + B) = $\frac{\tan A + \tan B}{1 - \tan A \tan B}$

(f) tan (A – B) = $\frac{\tan A - \tan B}{1 + \tan A \tan B}$

(g) cot (A + B) = $\frac{\cot A \cot B - 1}{\cot B + \cot A}$

(h) cot (A – B) = $\frac{\cot A \cot B + 1}{\cot B - \cot A}$

6.2 Some More Results :

*(a) sin (A + B).sin (A – B) = $\sin^2$ A – $\sin^2$ B = $\cos^2$ B – $\cos^2$ A

*(b) cos (A + B).cos (A – B) = $\cos^2$ A – $\sin^2$ B = $\cos^2$ B – $\sin^2$ A

(c) sin (A + B + C) = sin A cos B cos C + cos A sin B sin C + cos A cos B sin C – sin A sin B sin C

(d) cos (A + B + C) = cos A cos B cos C – cos A. sin B sin C – sin A cos B sin C – sin A sinB cosC

(e) tan (A + B + C)

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

(Note : * Important)

Examples based on Sum or difference of the angle

Ex.11 If sin A = $\frac{3}{5}$ and cos B = $\frac{9}{41}$, 0 < A < $\frac{\pi}{2}$. 0 < B < $\frac{\pi}{2}$, find the values of the following -

(a) sin (A + B)

(b) cos (A – B)

Sol. (a) sin (A + B) ⇒ sin A cos B + cos A sin B

sin A = $\frac{3}{5}$

cos A = $\frac{4}{5}$

and cos B = $\frac{9}{41}$

sin B = $\frac{40}{41}$

sin (A + B)= $\frac{3}{5} \times \frac{9}{41} + \frac{4}{5} \times \frac{40}{41} = \frac{187}{205}$

(b) cos (A – B)= cosA cos B + sin A sin B

$= \times \frac{9}{41} + \frac{3}{5} \times \frac{40}{41} = \frac{156}{205}$

7. FORMULA TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

We know that,

sin A cos B + cos A sin B = sin (A + B) ....(i)

sin A cos B – cos A sin B = sin (A – B) ......(ii)

cos A cos B – sin A sin B = cos (A + B) .....(iii)

cos A cos B + sin A sin B = cos (A – B) .....(iv)

Adding (i) and (ii),

2 sin A cos B = sin (A + B) + sin (A – B)

Subtracting (ii) from (i),

2 cos A sin B = sin (A + B) – sin (A – B)

Adding (iii) and (iv),

2 cosA cos B = cos (A + B) + cos (A – B)

Subtraction (iii) from (iv).

2 sin A sin B = cos (A – B) – cos (A + B)

Formula :

(a) 2 sin A cos B = sin (A + B) + sin (A – B)

(b) 2 cos A sin B = sin (A + B) – sin (A – B)

(c) 2 cos A cos B = cos (A + B) + cos (A – B)

(d) 2 sin A sin B = cos (A – B) – cos (A + B)

Examples based on To transform the product into sum or difference

Ex.12 Prove that, cos(30º – A).cos(30º + A) + cos(45º + A).cos(45º – A) = cos2A + $\frac{1}{4}$

Sol. L.H.S. = cos(30º – A).cos(30º + A) + cos(45º + A).cos(45º – A)

= $\frac{1}{2}$[2 cos(30º – A).cos(30º + A) + 2 cos (45º + A).cos(45º – A)]

= $\frac{1}{2}$ cos 60º + cos 2A + cos 90º + cos 2A

= $\frac{1}{2}$[2cos2A + $\frac{1}{2}$]

= cos 2A + $\frac{1}{4}$ = R.H.S.

8. FORMULA TO TRANSFORM THE SUM OR DIFFERENCE INTO PRODUCT

We know that,

sin (A + B) + sin(A – B) = 2 sin A cos B ......(i)

Let A+ B = C and A – B = D

then A = $\frac{C+D}{2}$ and B = $\frac{C-D}{2}$

Substituting in (i),

(a) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

similarly other formula,

(b) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$

(c) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

☞ (d) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{D-C}{2}\right)$

Examples based on To Transform the sum of difference into product

Ex.13 Prove that, $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

Sol. L.H.S,

$(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$\left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2 +$

$\left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2$

$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right).\cos^2\left(\frac{\alpha-\beta}{2}\right) +$

$4\sin^2\left(\frac{\alpha+\beta}{2}\right).\cos^2\left(\frac{\alpha-\beta}{2}\right)$

$= 4\cos^2$

$\left(\frac{\alpha-\beta}{2}\right).\left[\cos^2\left(\frac{\alpha+\beta}{2}\right)+\sin^2\left(\frac{\alpha+\beta}{2}\right)\right]$

$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

9. TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES

Trigonometric ratios of an angle 2A in terms of an angle A :

(a) sin 2A= 2sin A cos A = $\frac{2\tan A}{1+\tan^2 A}$

(b) cos 2A= $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$

= 1– 2 $\sin^2$ A =

(c) tan 2A = $\frac{2\tan A}{1-\tan^2 A}$

(d) sin 3A = 3 sinA – 4 $\sin^3$A

(e) cos 3A = 4 $\cos^3$A – 3 cosA

(f) tan 3A =

(g) $\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$

(h) $\tan A = \frac{1-\cos 2A}{\sin 2A}$

(i) $\sqrt{1+\sin 2A}$ = |sin A + cos A|

$\frac{1-\tan^2 A}{1+\tan^2 A}$ $\frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

(j) $\sqrt{1-\sin 2A}$ = |sin A – cos A|

Examples based on Trigonometrical Ratios of Multiple angles

Ex.14 Prove that , $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\left(\frac{\theta}{2}\right)$

Sol. L.H.S = $\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$$= \frac{2\sin^2\left(\frac{\theta}{2}\right)+2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)+2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\theta}{2}\right)\left[\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right]}{2\cos\left(\frac{\theta}{2}\right)\left[\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right]} = \tan\left(\frac{\theta}{2}\right)$$

= R.H.S

Ex.15 Show that, $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$

where $\theta \in \left[-\frac{\pi}{16}, \frac{\pi}{16}\right]$

Sol. L.H.S., = $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$

$\left[1+\cos 8\theta = 2\cos^2\left(\frac{8\theta}{2}\right)\right]$

$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$

$= \sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$

$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} = \sqrt{2+2\cos 2\theta}$

$= \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)}$

= 2cosθ = R.H.S

10. CONDITIONAL TRIGONOMETRICAL IDENTITIES

We have certain trigonometric identities

like, $\sin^2\theta + \cos^2\theta = 1$

and $1 + \tan^2\theta = \sec^2\theta$ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If A, B, C denote the angle of a triangle ABC, then the relation A + B + C = π enables us to establish many important identities involving trigonometric ratios of these angles.

(I) If A + B + C = π, then A + B = π – C, B + C = π – A and C + A = π – B

(II) If A + B + C = π, then sin(A + B) = sin(π – C) = sinC

similarly, sin (B + C) = sin(π – A) = sin A

and sin (C + A) = sin (π – B) = sin B

(III) If A + B + C = π, then cos(A + B) = cos(π – C) = – cos C

similarly, cos (B + C) = cos(π – A) = – cos A

and cos (C + A) = cos (π – B) = – cos B

(IV) If A + B + C = π, then tan(A + B) = tan(π – C) = – tan C

similarly, tan (B + C) = tan(π – A) = – tan A

and, tan (C + A) = tan (π – B) = – tan B

(V) If A + B + C = π, then $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$ and

$\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$ and $\frac{C+A}{2} = \frac{\pi}{2} - \frac{B}{2}$

$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$

$$\cos\left(\frac{A+B}{2}\right)=\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)=\sin\left(\frac{C}{2}\right)$$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\left(\frac{C}{2}\right)$$

All problems on conditional identities are broadly divided into the following four types :

(I) Identities involving sines and cosines of the multiple or sub-multiples of the angles involved.

(II) Identities involving squares of sines and cosines of the multiple or sub-multiples of the angles involved.

(III) Identities involving tangents and cotangents of the multiples or sub-multiples of the angles involved.

(IV) Identities involving cubes and higher powers of sines and cosines and some mixed identities.

10.1 TYPE I : Identities involving sines and cosines of the multiple or sub-multiple of the angles involved.

Working Methods :

Step – 1 Express of the sum of first two terms as product by using C & D formulae.

Step – 2 In the product obtained in step II replace the sum of two angles in terms of the third by using the given relation.

Step – 3 Expand the third term by using formulae (Double angle change into single angle or change into half angle).

Step – 4 Taking common factor.

Step – 5 Express the trigonometric ratio of the single angle in terms of the remaining angles.

Step – 6 Use the one of the formulae given in the step I to convert the sum into product.

Examples based on Conditional trigonometrical identities type I

Ex.16 If A + B + C = π, prove that , cosA + cosB + cosC =1 + 4 $\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$

Sol. L.H.S.= cosA+ cosB+cosC

$$= 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$$

$$= 2\cos\left(\frac{\pi}{2}-\frac{C}{2}\right).\cos\left(\frac{A}{2}-\frac{B}{2}\right)+\cos C$$

$$= 2\sin\left(\frac{C}{2}\right).\cos\left(\frac{A}{2}-\frac{B}{2}\right)+1-2\sin^2\left(\frac{C}{2}\right)$$

$$= 2\sin\left(\frac{C}{2}\right).\cos\left(\frac{A}{2}-\frac{B}{2}\right)-2\sin^2\left(\frac{C}{2}\right)+1$$

$$= 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]+1$$

$$= 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right]+1$$

$$= 2\sin\left(\frac{C}{2}\right)\left[\cos\left(\frac{A}{2}-\frac{B}{2}\right)-\cos\left(\frac{A}{2}+\frac{B}{2}\right)\right]+1$$

$$= 2\sin\left(\frac{C}{2}\right)\left[2\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right)\right]+1$$

$$= 1+4\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right) \quad = \text{R.H.S.}$$

Ex.17 If A + B + C =π, Prove that

$$\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)$$

$$= 1+4\sin\left(\frac{\pi-A}{4}\right).\sin\left(\frac{\pi-B}{4}\right).\sin\left(\frac{\pi-C}{4}\right)$$

$$= 1+4\sin\left(\frac{B+C}{4}\right).\sin\left(\frac{C+A}{4}\right).\sin\left(\frac{A+B}{4}\right)$$

Sol. L.H.S.= $\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right)$

$$= 2\sin\left(\frac{A+B}{4}\right).\cos\left(\frac{A-B}{4}\right)+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$= 2\sin\left(\frac{\pi-C}{4}\right).\cos\left(\frac{A-B}{4}\right)+1-2\sin^2\left(\frac{\pi-C}{4}\right)$$

$$= 2\sin\left(\frac{\pi-C}{4}\right)\left[\cos\left(\frac{A-B}{4}\right)-\sin\left(\frac{\pi-C}{4}\right)\right]+1$$

$$= 2\sin\left(\frac{\pi-C}{4}\right)\left[\cos\left(\frac{A-B}{4}\right)-\cos\left\{\frac{\pi}{2}-\left(\frac{\pi-C}{4}\right)\right\}\right]+1$$

$$= 2\sin\left(\frac{\pi-C}{4}\right).\left[\cos\left(\frac{A-B}{4}\right)-\cos\left(\frac{\pi+C}{4}\right)\right]+1$$

$$=2\sin\left(\frac{\pi-C}{4}\right)\left[2\sin\left(\frac{A-B+\pi+C}{8}\right)\sin\left(\frac{\pi+C-A+B}{8}\right)\right]+1$$

$$=2\sin\left(\frac{\pi-C}{4}\right)\left[2\sin\left(\frac{A+C+\pi-B}{8}\right)\sin\left(\frac{\pi+C-A+B}{8}\right)\right]+1$$

$$=2\sin\left(\frac{\pi-C}{4}\right)$$

$$\left[2\sin\left(\frac{\pi+B+\pi-B}{8}\right)\sin\left(\frac{\pi-A+\pi-A}{8}\right)\right]+1$$

$$= 2\sin\left(\frac{\pi-C}{4}\right)\left[2\sin\left(\frac{\pi-B}{4}\right).\sin\left(\frac{\pi-A}{4}\right)\right]+1$$

$$= 1+4\sin\left(\frac{\pi-A}{4}\right).\sin\left(\frac{\pi-B}{4}\right).\sin\left(\frac{\pi-A}{4}\right)$$

$$= 1+4\sin\left(\frac{B+C}{4}\right).\sin\left(\frac{C+A}{4}\right).\sin\left(\frac{A+B}{4}\right)$$

= R.H.S

10.2 TYPE II :Identities involving squares of sines and cosines of multiple or sub-multiples of the angles involved.

Working step :

(I) Arrange the terms on the L.H.S of the identity so that either $\sin^2A - \sin^2B = \sin(A+B).\sin(A-B)$ or $\cos^2A - \sin^2B = \cos(A+B).\cos(A-B)$ can be used.

(II) Take the common factor outside.

(III) Express the trigonometric ratio of a single angle inside the bracket into that of the sum of the angles.

(IV) Use the formulaes to convert the sum into product.

$\frac{1}{2}$

Examples based on Conditional trigonometrical identities type II

Ex.18 If $A + B + C = \pi$

Prove that, $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$

Sol. **I Method**

L.H.S. $\cos^2 A + \cos^2 B + \cos^2 C$

$= \cos^2 A + (1 - \sin^2 B) + \cos^2 C$

$= (\cos^2 A - \sin^2 B) + \cos^2 C + 1$

[$\because A + B = \pi - C$, $\cos(A + B) = -\cos C$]

$= \cos(A + B).\cos(A - B) + \cos^2 C + 1$

$= -\cos C.\cos(A - B) + \cos^2 C + 1$

$= -\cos C[\cos(A - B) - \cos C] + 1$

$= -\cos C[\cos(A - B) + \cos(A + B)] + 1$

[$\cos C = -\cos(A + B)$]

$= -\cos C[2\cos A\cos B) + 1$

$= 1 - 2\cos A\cos B\cos C$ = R.H.S.

II Method

$\cos^2 A + \cos^2 B + \cos^2 C$

$= [2\cos^2 A + 2\cos^2 B + 2\cos^2 C]$

[$\because \cos 2A = 2\cos^2A - 1$]

$= [(1 + \cos 2A) + (1 + \cos 2B) + (1 + \cos 2C)]$

$= \frac{1}{2}[3 + \cos 2A + \cos 2B + \cos 2C]$

$= \frac{3}{2} + \frac{1}{2}[\cos 2A + \cos 2B + \cos 2C]$

$= \frac{3}{2} + \frac{1}{2}[2\cos(A + B).\cos(A - B) + 2\cos^2 C - 1]$

$= \frac{3}{2} + \frac{1}{2}[-2\cos C\cos(A - B) + 2\cos^2 C - 1]$

$= \frac{3}{2} - \frac{1}{2} + \frac{1}{2}[-2\cos C\ \{\cos(A - B) - \cos C\}]$

$= 1 - \cos C[\cos(A - B) - \cos C]$

$= 1 - \cos C[\cos(A - B) + \cos(A + B)]$

[$\cos C = -\cos(A + B)$]

$= 1 - \cos C[2\cos A.\cos B]$

$= 1 - 2\cos A.\cos B\cos C$

10.3 Type III :Identities for tan and cot of the angles

Working step :

(I) Express the sum of the two angles in terms of third angle by using the given relation.

(II) Taking tan from both the sides.

(III) Expand the L.H.S in step II by using the formula for the tangent of the compound angles.

(IV) Use cross multiplication in the expression obtained in the step III.

(V) Arrange the terms as per the requirement in the sum.

Examples based on Conditional trigonometrical identities type III

Ex.19 If $x + y + z = xyz$

Prove that,

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

Sol. Let $x = \tan A$, $y = \tan B$, $z = \tan C$

then $x + y + z = xyz$

$\tan A + \tan B + \tan C = \tan A.\tan B.\tan C$

$\Rightarrow \tan A + \tan B + \tan C - \tan A\tan B\tan C = 0$

Dividing by $[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$ both the sides

$$\Rightarrow \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = 0$$

$\Rightarrow$ tan(A + B + C) = 0

$\Rightarrow$ A + B + C = $n\pi$ [$n \in z$]

Now, A + B + C = $n\pi$

2A + 2B + 2C = $2n\pi$

$\Rightarrow$ tan(2A + 2B + 2C) = tan $2n\pi$

$$\Rightarrow \frac{\tan 2A + \tan 2B + \tan 2C - \tan 2A \tan 2B \tan 2C}{1 - \tan 2A \tan 2B - \tan 2B \tan 2C - \tan 2C \tan 2A} = 0$$

$\Rightarrow$ tan2A + tan2B + tan2C – tan2A tan2B tan2C = 0

$\Rightarrow$ tan2A + tan2B + tan2C – tan2A tan2B tan2C

$$\Rightarrow \frac{2\tan A}{1-\tan^2 A} + \frac{2\tan B}{1-\tan^2 B} + \frac{2\tan C}{1-\tan^2 C}$$

$$= \frac{2\tan A}{1-\tan^2 A} \cdot \frac{2\tan B}{1-\tan^2 B} \cdot \frac{2\tan C}{1-\tan^2 C}$$

$$\Rightarrow \frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2}$$

$$= \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2} = \frac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

11. TO FIND THE GREATEST AND LEAST VALUE OF THE EXPRESSION [a sinθ + b cosθ]

Let $a = r\cos\alpha$ ....(1)

and $b = r\sin\alpha$ ....(2)

Squaring and adding (1) and (2)

then $a^2 + b^2 = r^2$

or, $r = \sqrt{a^2 + b^2}$

$\therefore$ a sinθ + b cosθ

= r (sinθ cosα + cosθ sinα)

= r sin(θ + α)

But $-1 \le \sin\theta \le 1$

so $-1 \le \sin(\theta + \alpha) \le 1$

then $-r \le r\sin(\theta + \alpha) \le r$

hence,

$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

then the greatest and least values of a sinθ + b cosθ are respectively $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

Examples based on To find the greatest and least value of the expression

Ex.20 Prove that 5 cosθ+ 3 cos$\left(\theta + \frac{\pi}{3}\right)$ + 3. lies between – 4 and 10.

Sol. The given expression is,

5 cosθ+ 3 cos$\left(\theta + \frac{\pi}{3}\right)$ + 3

$\Rightarrow$ 5 cosθ + 3[cosθ cos 60º – sinθ sin 60º] + 3

$\Rightarrow$ 5 cosθ + 3 $[\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta]$ + 3

$\Rightarrow \frac{1}{2}[13\cos\theta - 3\sqrt{3}\sin\theta] + 3$

Put 13 = r cosα, $3\sqrt{3}$ = r sinα

r = $\sqrt{169 + 27}$ = 14

$\Rightarrow \frac{1}{2}[r\cos(\theta + \alpha)] + 3$

$\Rightarrow \frac{14}{2}[\cos(\theta + \alpha)] + 3$

$\Rightarrow$ 7 cos(θ + α) + 3

Hence maximum and minimum values of expression are (7+ 3) and (–7 +3) i.e., 10 and – 4 respectively.

12. MISCELLANEOUS POINTS

(1) Some useful Identities :

(a) $\tan(A + B + C) = \frac{\sum \tan A - \tan A \tan B \tan C}{1 - \sum \tan A . \tan B}$

(b) cotθ – tanθ = 2 cot 2θ

(c) $\frac{1}{4}\sin 3\theta = \sin\theta.\sin(60 - \theta).\sin(60 + \theta)$

(d) $\frac{1}{4}\cos 3\theta = \cos\theta.\cos(60 - \theta).\cos(60 + \theta)$

(e) tan 3θ = tanθ.tan(60 – θ).tan (60 + θ)

(f) tan(A + B) – tanA – tanB = tanA.tanB.tan(A + B)

(2) Some useful result :

(a) ver sinθ = 1– cosθ

(b) cover sinθ = 1 – sinθ

(3) Some useful series :

(a) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots.$ + to n terms

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)} \;;\; \beta \neq 2n\pi$$

(b) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots..$

$$+ \text{ to n terms} = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)} \;;$$

$\beta \neq 2n\pi$

Examples based on Series

Ex.21 Prove that $\cos\left(\frac{\pi}{14}\right) + \cos\left(\frac{3\pi}{14}\right) + \cos\left(\frac{5\pi}{14}\right) = \frac{1}{2}\cot\left(\frac{\pi}{14}\right)$

Sol. Here $\alpha = \frac{\pi}{14}, \beta = \frac{2\pi}{14}$ and n = 3.

$$S = \frac{\cos\left[\frac{\pi}{14} + \left(\frac{3-1}{2}\right)\left(\frac{2\pi}{14}\right)\right]\sin\left(\frac{2\pi}{14}\times\frac{3}{2}\right)}{\sin\left(\frac{2\pi}{14}\times\frac{1}{2}\right)}$$

$$S = \frac{2\cos\left(\frac{3\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)}{2\sin\left(\frac{\pi}{14}\right)}$$

$$S = \frac{\sin\left(\frac{6\pi}{14}\right)}{2\sin\left(\frac{\pi}{14}\right)} = \frac{\frac{1}{2}\sin\left(\frac{\pi}{2} - \frac{\pi}{14}\right)}{\sin\left(\frac{\pi}{14}\right)}$$

$$S = \frac{1}{2}\cot\left(\frac{\pi}{14}\right)$$

(4) An Increasing Product series :

(a) $p = \cos\alpha . \cos 2\alpha . \cos 2^2\alpha \ldots \cos(2^{n-1}\alpha)$

$$\begin{cases} \frac{\sin 2^n\alpha}{2^n \sin\alpha}, \text{if } \alpha \neq n\pi \\ 1, \text{ if } \alpha = 2k\pi \\ -1, \text{ if } \alpha = (2k+1)\pi \end{cases}$$

(5) sine, cosine and tangent of some angle less than 90º.

	15º	18º	22½º	36º
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

(6) Conversion 1 radian = 180º/π = 57º 17/ 45// (approximately)

and $1^\circ = \frac{\pi}{180}$ = 0.01475 radians (approximately)

(7) Basic right angled triangle are (pythogerian Triplets)

3, 4, 5 ; 5, 12, 13; 7, 24, 25; 8, 15, 17; 9, 40, 41; 11, 60, 61; 12, 35, 37; 20, 21, 29 etc.

(8) Each interior angle of a regular polygon of n sides

$$= \frac{n-2}{n}\times 180 \text{ degrees}$$