1. RESULTANT OF A FORCE SYSTEM

1.1 Introduction

Engineering Mechanics may be defined as science of force. It is a study of various effects force as it acts on a body. It is further divided into Statics and Dynamics. Statics is the study of effects of force on stationary bodies and Dynamics is the study of effects of force on moving bodies.

1.2 Force

Force is that physical quantity which changes or tends to change the state of the body on which it acts. This change in state may be w.r.t. motion of body or w.r.t. its form & dimensions (as in case of a deforming force). Force is a vector quantity for which magnitude, direction as well as line of action exist. If we change the line of action of force, its effect on the body is entirely different (Fig.1-A). For force, however, point of action is not an important aspect. Keeping magnitude, direction and line of action same, if we change the point of action of force, its effect on the body remains same (Fig.1-B).

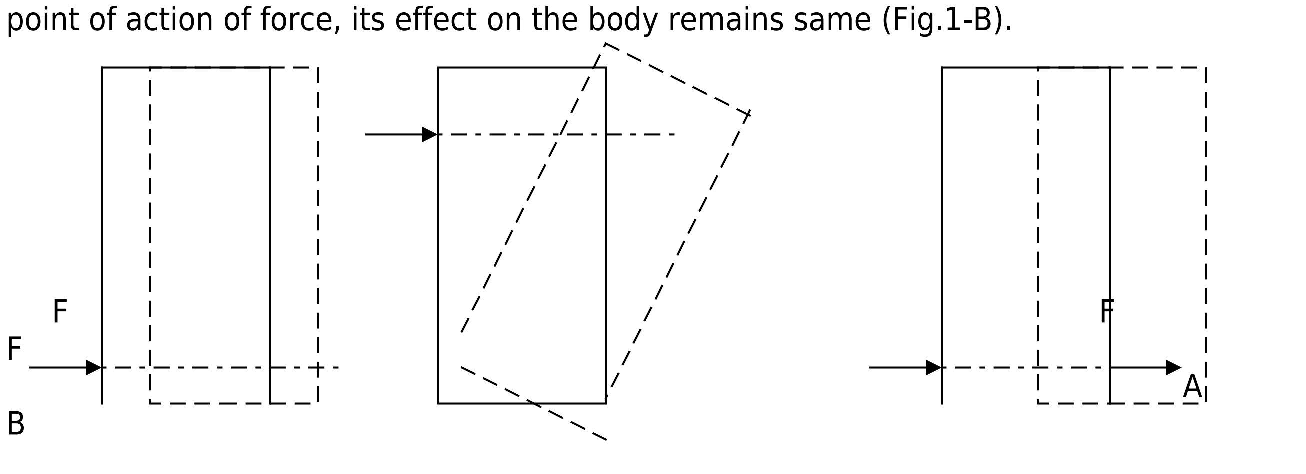
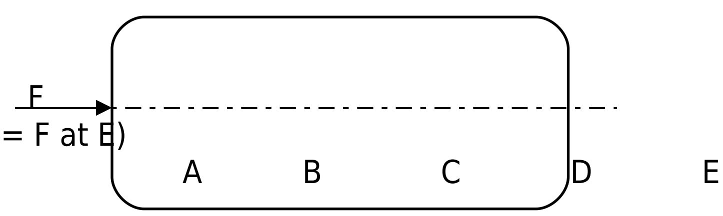


Fig.1-A

Fig.1-B

1.3 Theorem of Transmissibility of Force

The theorem states that if given force is applied at any point (physically connected with the body) along its line of action, its effect on the body remains unchanged so that, for the sake of analysis, given force may be transmitted to any suitable point along its line of action.



(F at A = F at B = F at C = F at D = F at E)

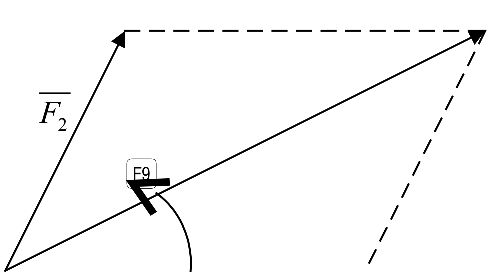
Fig.1-C

1.4 Composition of Forces

It is a process of finding resultant of given forces. Resultant is that single force which has same effect, as the combined effect of all given forces.

1.4.1 Parallelogram Law

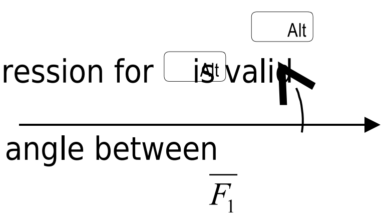
If two forces acting at a point are represented in magnitude and direction by two sides of a parallelogram, diagonal passing through the common point of action of two forces gives the resultant.



$\overline{R}$

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$\tan\alpha = \frac{F_2\sin\theta}{F_1 + F_2\cos\theta}$$

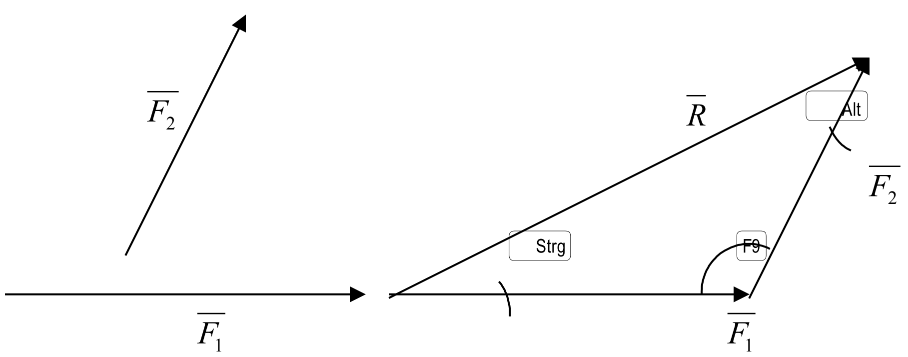


[It should be noted that above expression for θ is valid as it is, only when θ is defined as the angle between $\overline{R}$ and $\overline{F_1}$].

Fig.1-D

1.4.2 Triangle Law

If two given forces are represented in magnitude and direction by two sides of a triangle taken in an order, the third side taken in opposite order gives their resultant.



$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{R}{\sin\theta}$$

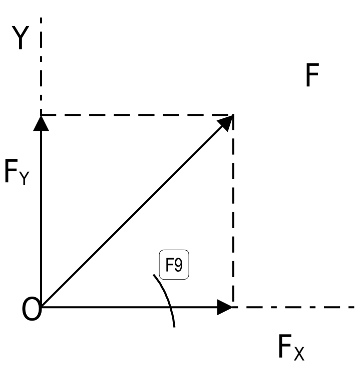
Fig.1-E

1.4.3 Polygon Law

If several forces acting on a body are represented in magnitude and direction by sides of a polygon taken an order, the closing side of the polygon, taken in opposite order gives their resultant.

1.5 Resolution of Force

It is a process of splitting a given force into several forces (called Components) such that the combined effect of these components is same as that of the original force. Every component is a part of original force effective in a particular direction. Thus, a force is resolved to explore its effect in that direction. Two mutually perpendicular components of a force are called as rectangular components.



$$F = \sqrt{F_X^2 + F_Y^2}$$

$$\tan\theta = \frac{F_Y}{F_X}$$

Further, $F_X = F\cos\theta$

X $F_Y = F\sin\theta$

Fig.1-F

Note that the expressions $F_X = F\cos\theta$ and $F_Y = F\sin\theta$ are applicable only when angle θ is expressed w.r.t. X-axis. If angle θ is expressed w.r.t Y-axis, these expressions will be $F_X = F\sin\theta$ and $F_Y = F\cos\theta$. In short, angle θ w.r.t. whichever direction, X or Y, is known, that directional component is cosine component and the other component is sine component. If the original force terminates into a point, then the components are shown terminating into that point. If the given force originates from a point, then the components are shown originating from that point.

1.6 Classification of Force Systems

(a) Coplanar Force System: All the forces in the system lie in one plane.

(b) Non-coplanar Force System: All the forces in the system do not lie in one plane but are scattered in space.
(c) Concurrent System: Lines of action of all forces pass through a common point.
(d) Like-parallel System: Lines of action of all forces are parallel to each other and all forces have same directions.
(e) Unlike-parallel System: Lines of action of all forces are parallel to each other but some forces have direction
opposite of the remaining ones.
(f) General / Non-concurrent, non-parallel System: Lines of action of forces in the system are neither concurrent
nor parallel.

1.7 Method of finding Resultant of Concurrent Force System

Step I: Resolving all the given forces into X and Y components.
Step II: Taking independent vector summations ΣF_X (sum of force components along X) and ΣF_Y (sum of force components along Y) assuming a suitable sign convention, say, (→) : +ve, (←) : -ve, (↑) : +ve, (↓) : -ve.
Step III: Finding magnitude of resultant $R = \sqrt{(\Sigma F_X)^2 + (\Sigma F_Y)^2}$ and direction of resultant defined by angle θ
w.r.t. X axis as, $\tan\theta = \Sigma F_Y / \Sigma F_X$. Line of action of resultant of concurrent system naturally passes through the
common point of action of all the forces in the system.

1.8 Moment of a Force

Force quantity is associated with translational motion of the body. But under specific conditions, force may produce rotational motion, which is an entirely different type of motion. This ability of force to
M
produce rotation in the body is called as moment of force. The magnitude of moment is called as torque or moment itself and
d
is calculated as the product of magnitude of force and moment arm.
i.e. Moment @ point P: M = F × d (↺)
The point P about which the moment of force is being considered is called as moment center and the perpendicular distance d between moment center and line of action of force is called as moment arm.

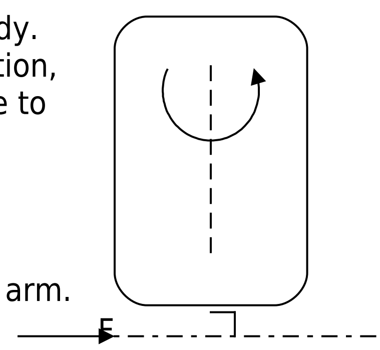


Fig.1-G

1.9 Couple

Two equal and opposite forces (magnitudes equal, lines of action parallel and directions opposite) separated by a fixed distance constitute a couple.
F
The ability of a couple to rotate a body is called its moment and the magnitude of this moment is called torque or moment itself. Torque is
P
calculated as product of magnitude of the force in the couple and the
d
distance separating the two forces.
Torque of couple @ point P: Γ = F × d (↺).

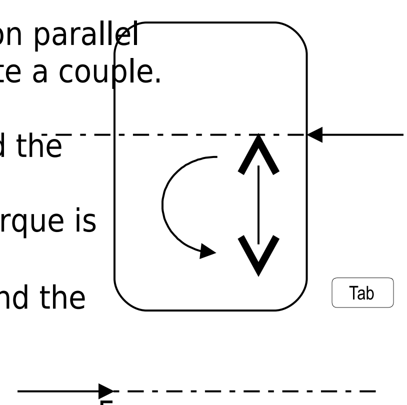
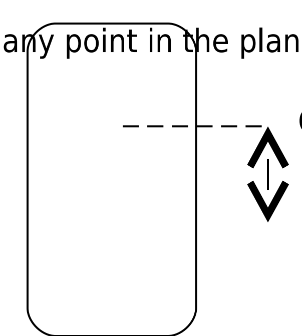


Fig.1-H

1.9.1 Properties of Couple

(a) A couple produces pure rotation in the body without any translation.
(b) A couple can be balanced by an equal and opposite couple only.
(c) Moment of given couple is a constant quantity in magnitude and sense, irrespective of the choice of moment
center. A given couple will have same rotational effect about any point in the plane.
Moment of couple @ P = Fd (↺)
Moment of couple @ Q = Σ(Moment of each force @ Q)



= Fy + [-F(d+y)]

y

= - Fd

= Fd(↻)

[assuming sign convention (↺) : +ve and (↻) : -ve]

d

∴ Moment of couple @ P = Moment of couple @ Q.

F

Fig.1-I

1.10 Varignon's Theorem

It states that moment of resultant of a force system about any point in the plane is equal to vector sum of moments of all the forces and couples in that system about the same point. In other words, the combined rotational effect of all the forces and couples in the system is fully represented by the rotational effect of the resultant alone about the same moment center.

1.11 Resultant of Parallel Force System

Step I: Vector sum of all the given forces assuming a suitable sign convention. This will give magnitude and direction of the resultant.

Step II: Assuming the found resultant at an unknown position x from the chosen moment center and then applying Varignon's Theorem to calculate x, which gives position of line of action of resultant.

1.12 Equivalent Force Systems

Two force systems, one derived from the other, which are different apparently, but which have absolutely same

effects on the body are called as Equivalent systems. Using concept of equivalent system, it is possible to determine effect of a force at a point not on its line of action or effect of entire given force system at any particular point. In Fig.1-J, force systems FS-1 and FS-2 are equivalent systems.

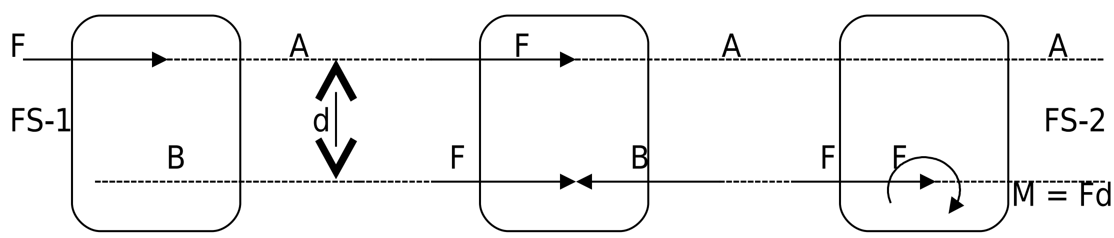


Fig.1-J

1.13 Resultant of General Force System

Step I: Resolving the given forces into X and Y components at their points of action.

Step II: Independent vector summations ΣF_X and ΣF_Y assuming a suitable sign convention (→) : +ve, (←) : -ve, (↑) : +ve, (↓) : -ve.

Step III: Finding magnitude of resultant $R = \sqrt{\left(\Sigma F_X\right)^2 + \left(\Sigma F_Y\right)^2}$ and direction of resultant defined by

angle θ w.r.t. X axis as, $\tan\theta = \frac{\Sigma F_Y}{\Sigma F_X}$.

Step IV: Assuming the found resultant at an unknown distance x from the chosen moment center and then applying Varignon's Theorem to calculate x, which gives position of line of action of resultant.

1.14 Distributed Loads

Force acting over an extremely small area is called as Point load or Concentrated load. Practically, however, loads are not always acting at a point but they may be acting over a finite area. Such loads are called as distributed loads. Since our present study is restricted to coplanar systems, we shall consider distributed loads acting over a finite length.

1.14.1 Uniformly Distributed Load (u.d.l.)

When total given load is made to act, by distributing it uniformly, over a finite length, it is called as Uniformly Distributed Load. Magnitude of u.d.l. is represented by its intensity w i.e. the amount of load acting per unit length of the loading span. To analyze a u.d.l., it is converted into resultant point load as shown in Fig.1-K. Magnitude of resultant is calculated as R = intensity × span, its line of action passing through the centroid of the loading rectangle.

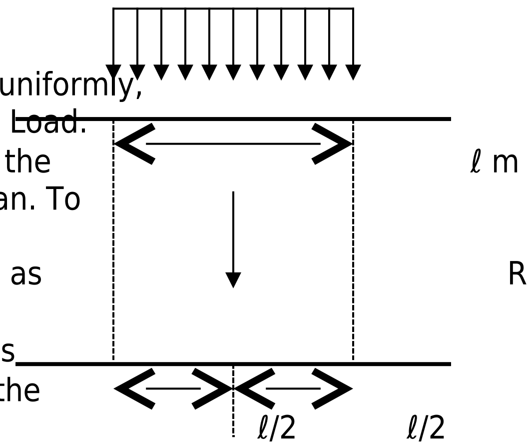


Fig.1-K

1.14.2 Uniformly Varying Load (u.v.l.)

When the total load is made to act in such a way that variation in the loading is uniform or constant over a finite length, it is called as Uniformly Varying Load. Magnitude of u.v.l. is represented by its intensity w. To analyse a u.v.l., it is converted into a resultant point load with magnitude R = ½ (intensity × span), line of action passing through the centroid of loading triangle as shown in Fig.1-L.

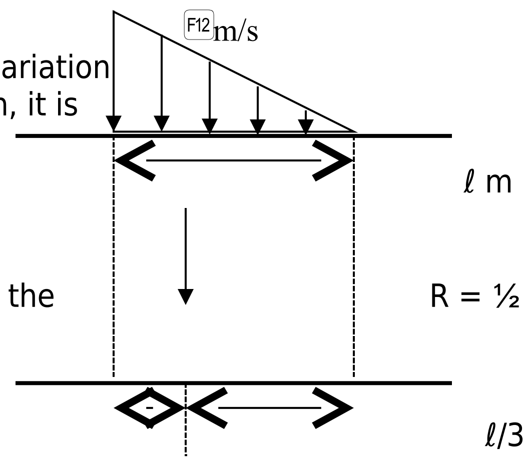


Fig.1-L

1.14.3 Trapezoidal Loading

This is a particular case of u.v.l. in which the variation in the load is from some intensity w_1 to some other intensity w_2. A convenient way of analyzing a trapezoidal loading is to convert it into a combination of one u.d.l. and one u.v.l. and then finding their resultants, thereby converting a trapezoidal load into two resultant point loads. This is shown in Fig.1-M.

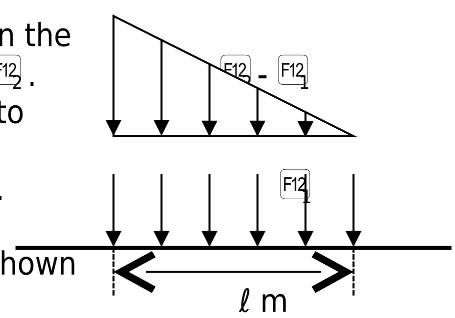


Fig.1-M

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2. EQUILIBRIUM

2.1 Introduction

Equilibrium simply means balance. The forces acting in a system have such magnitudes, directions and lines of action, that they completely nullify each other's effect, leaving no resultant force to translate the body, neither a resultant couple moment to rotate it. In Statics, such a force balance necessarily implies a complete state of rest of the body.

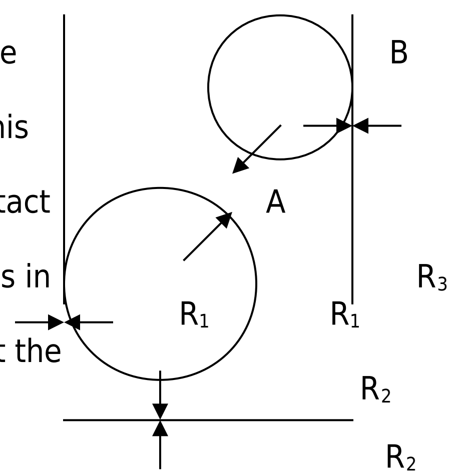
2.2 Analytical Conditions of Equilibrium

Parallel and General force systems may reduce to a resultant force or a resultant couple moment. For equilibrium, resultant force as well as resultant couple moment, both should be zero. This gives us three necessary & sufficient conditions of equilibrium for Parallel and General force systems as indicated in the table below. Concurrent force system, however, can never reduce to a couple. As all the forces in Concurrent system pass through one point, couple forming is impossible. So a Concurrent system may reduce to a resultant force only. For equilibrium, this resultant force should be zero, which gives two necessary and sufficient conditions of equilibrium (see the table).

Force System	Resultant	Calculation	Equilibrium Conditions
Concurrent System	Resultant Force	$R = \sqrt{(\Sigma F_X)^2 + (\Sigma F_Y)^2}$	$\Sigma F_X = 0$ $\Sigma F_Y = 0$
Parallel System & General System	Resultant Force	$R = \sqrt{(\Sigma F_X)^2 + (\Sigma F_Y)^2}$	$\Sigma F_X = 0$ $\Sigma F_Y = 0$
	Resultant Couple Moment	$R_M = \Sigma M$ @ any point	ΣM @ any point = 0

2.3 Normal Reaction

When two bodies are in physical contact with each other, there may exist a give and take of force between the two bodies. This give and take between the two bodies across the point of contact is called as Reaction between the two bodies. When the bodies in contact are assumed to be ideally smooth (frictionless), this reaction acts along the common normal to the two surfaces at the point of contact and is called as Normal Reaction.



2.4 Free – Body – Diagram (FBD)

Fig.2-A

Equilibrium principle states that when the entire arrangement is in equilibrium, every element, every constituent of that arrangement is in equilibrium. Thus, to analyze any equilibrium case, we first isolate each body from the whole arrangement and consider equilibrium of each body separately. The diagram showing an isolated body with all the forces acting on that body from external (i.e. forces received by that body and not applied by it) is called as Free-Body-Diagram (FBD). Fig.2-A shows reactions at points of contact between bodies A and B and Fig.2-B shows FBD of A.

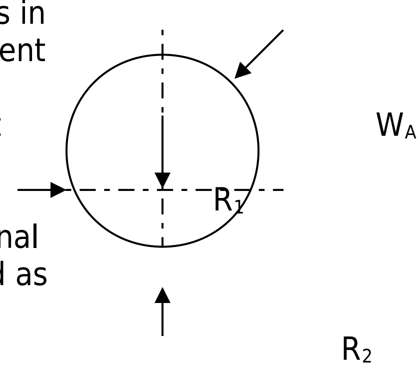


Fig.2-B

2.5 Lami's Theorem

It states that when three concurrent forces are in equilibrium, magnitude of each force is directly proportional to the sine of the angle between the other two forces.

i.e. $F_1 \propto \sin\alpha$, $F_2 \propto \sin\beta$, $F_3 \propto \sin\gamma$

or $\dfrac{F_1}{\sin\alpha} = \dfrac{F_2}{\sin\beta} = \dfrac{F_3}{\sin\gamma} = K$ (constant)

It should be remembered that angle between two forces is defined only when either their heads or tails are matching.

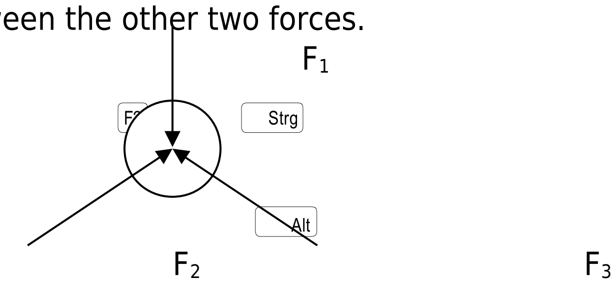
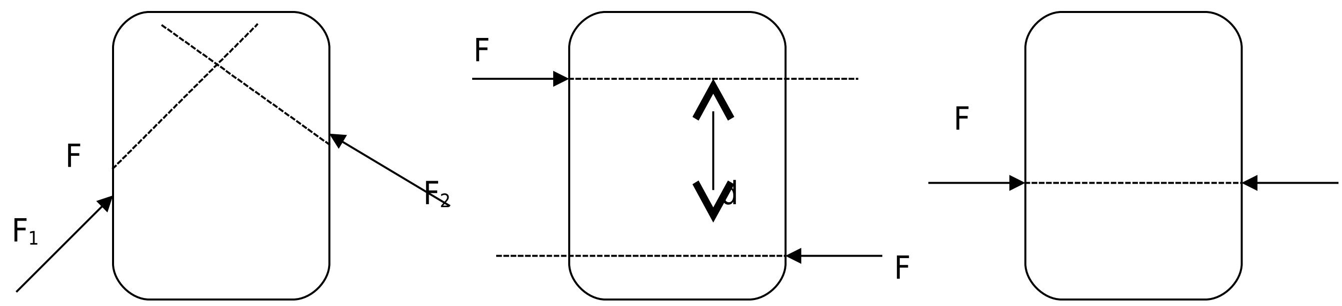


Fig.2-C

2.6 Two – Force System

A force system consisting of only two forces is called as Two-Force system. A 2-Force system will be in equilibrium if and only if, the two forces are equal, opposite and have same line of action. This is clear from Fig.2-D.



$\Sigma F_X \neq 0, \Sigma F_Y \neq 0$ (Translation) $\Sigma F_X = 0, \Sigma F_Y = 0, \Sigma M \neq 0$ (Rotation) $\Sigma F_X = 0, \Sigma F_Y = 0, \Sigma M = 0$ (Equilibrium)

Fig.2-D

2.7 Two – Force Member

A linear member subjected effectively to only two forces acting at its ends is called as a 2-Force member. Consider a linear member subjected to force systems concurrent at its two ends as shown. Each system will have a single resultant force so that the member is subjected to only two forces (resultants), which makes it a 2-Force member. Now for equilibrium of this 2-Force member, it is essential that these two resultants should be equal, opposite and have same line of action. Thus a 2-Force member in equilibrium is subjected to two forces, which act along the longitudinal axis of the member. Such forces deform the 2-Force member only longitudinally (i.e. there is no bending of the member), which results into simple longitudinal stresses, either compressive or tensile. These internal forces are equal and opposite to the external deforming action and are indicated as shown. Thus, when a 2-Force member is in compression, its internal forces are pointing outwards and when the member is in tension, the internal forces are pointing inwards.

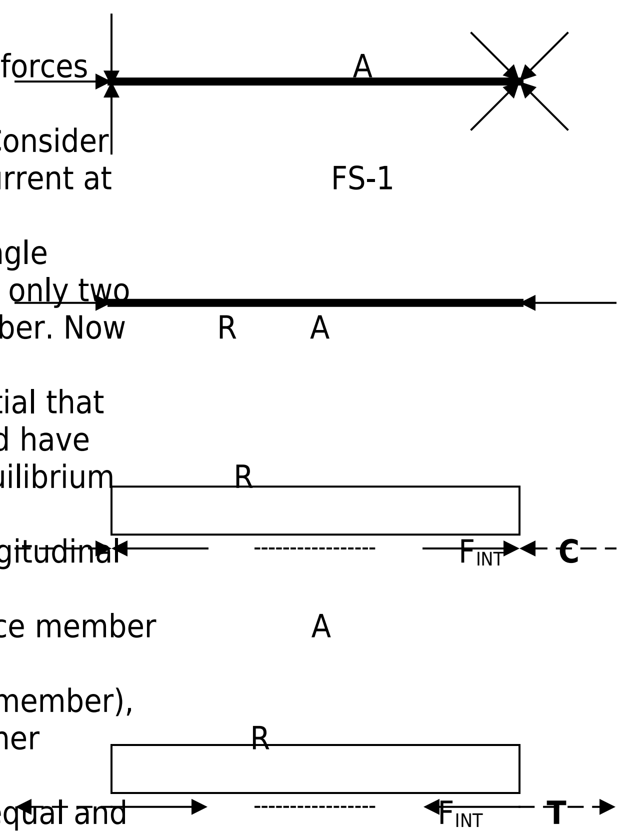
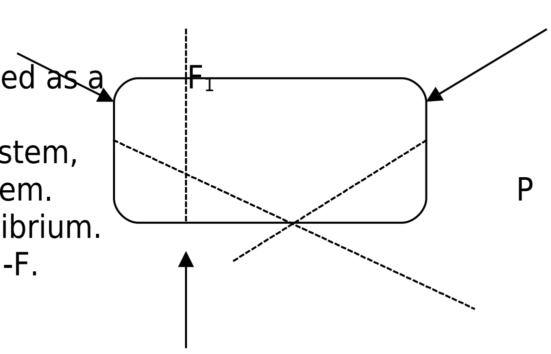


Fig.2-E

2.8 Three – Force System

A force system consisting of only three forces is called as a Three-Force System. For equilibrium of a 3-Force system, the system has to be a Concurrent or a Parallel system. A 3-Force General system can never remain in equilibrium. Consider the 3-Force General system shown in Fig.2-F.



Using transmissibility, F_1 and F_2 are drawn at P. R is their resultant. Thus 3-Force system is reduced to a 2-Force system of R and F_3. Now this system cannot be in equilibrium if R and F_3 have different lines of action. In short, in case of a 3-Force General system, any one force can never balance resultant of remaining two forces. Hence a 3-Force General system can never be in equilibrium.

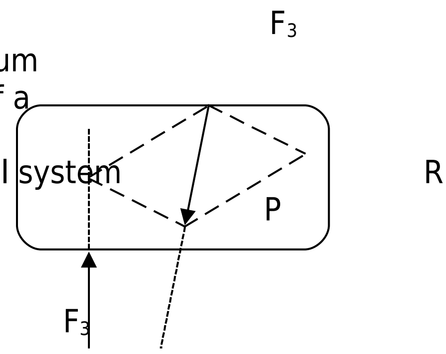


Fig.2-F

2.9 Types of Joints

A member may be connected to another member in different ways. The connection between two members is called as Joint. Through the medium of connection, there is a give & take of forces between the two members, which is called as Reaction at the Joint.

2.9.1 Cantilever Joint

The joint is so made that there is complete fixity of the member on one side with another member, other end freely hanging. To any horizontal action F_X, joint offers horizontal reaction H_A, to any vertical action F_Y, joint offers vertical reaction V_A and to any couple moment action , joint offers moment reaction M_A, thereby resisting completely any translation and rotation in the member. Thus, a cantilever joint offers all the three reactions viz. H_A, V_A and M_A (Fig.2-G).

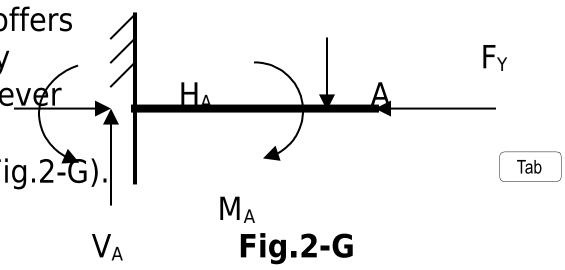


Fig.2-G

2.9.2 Hinge Joint

The joint consists of a hinge element fixed to a member and a cylindrical pin for connection with the other member. To any applied action F_X and F_Y, hinge joint can offer equal and opposite reactions H_A and V_A, thereby completely resisting any kind of translation. But if a couple moment is applied to the member, there will be free rotation of the member about the smooth frictionless pin of the hinge joint. As the hinge joint is incapable of resisting rotation of the member, we say that moment reaction M_A is absent and it offers only two reactions viz. H_A and V_A (Fig.2-H).

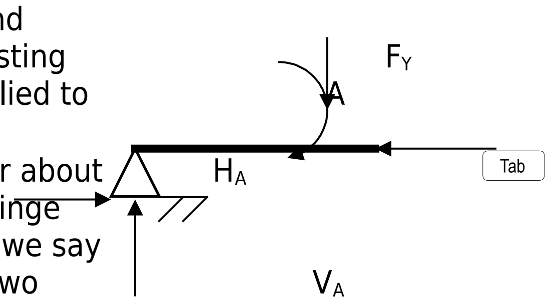


Fig.2-H

2.9.3 Roller Joint

The same hinge when mounted over rollers makes it a roller joint. As seen earlier, due to the frictionless pin, moment reaction M_A is absent. Further, due to rollers, reaction H_A is absent and the joint is incapable of resisting tangential translation of member over the surface of the other member. Thus a roller joint offers only one reaction V_A or R_A, which is the usual normal reaction between frictionless wheels and the supporting surface.

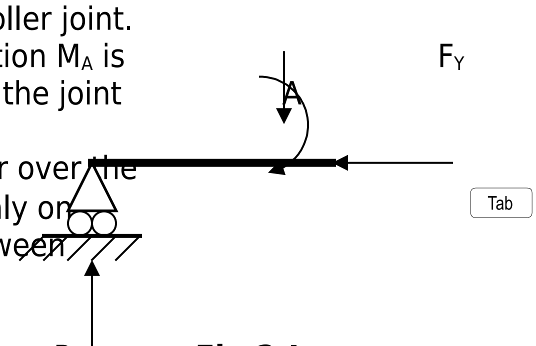


Fig.2-I

2.10 Equilibrant

Equilibrant is that single force, which when added to the given force system, brings the unbalanced system into equilibrium. As the equilibrant nullifies the net effect of all forces, in other words, as it balances the resultant of the given force system, it is equal, opposite and along the line of action of resultant of the force system.

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3. ANALYSIS OF STRUCTURES

3.1 Introduction

Structure is an arrangement of members, joints and supports assembled to carry loads coming at various elevations to earth safely. Multistoried buildings, manufacturing plants, bridges, roadways, railways and airports are all examples of structure.

(a) Member: The element of structure, which is actively involved in the process of force transfer e.g. slab, beam,
column, etc. is called as member.

(b) Joint: Connection between two members is called as joint, e.g. hinge joint, roller joint, cantilever joint, etc.

(c) Support: Connection between entire structure and earth i.e. foundation is called as support.

3.2 Beam

Beam is a linear member of structure, which receives loads in transverse direction from slab. Due to the transverse loading, beam undergoes bending and develops complicated internal
Beam
stresses, partly tensile and partly compressive. This is unlike a 2-Force member, which never bends but undergoes longitudinal deformation and develops simple longitudinal stresses, either tensile or compressive. Beams are usually horizontal.
Column

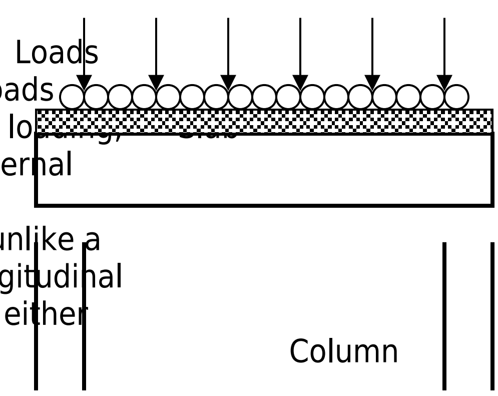

Fig.3-A

3.3 Determinate & Indeterminate Beams

(a) Determinate Beam: A beam that can be analyzed for its support reactions by concept of equilibrium alone is called as Statically Determinate beam. Number of unknown support reactions for a determinate beam is
not more than 3.

(b) Indeterminate Beam: If number of unknown support reactions is greater than 3, all the reactions cannot be
found by concept of equilibrium, in which, we have but only three equations. Such a beam is called as Statically Indeterminate beam. Such beams do exist but advanced methods are required to analyze such beams.

3.4 Compound Beam
A beam made up of two or more beams connected to each other using internal hinges or pins is called as Compound beam. To analyze a compound beam, FBD of each constituent beam is considered and all the support reactions are found. Like a hinge joint, an internal hinge is also capable of offering two reactions H_P and V_P, which are found during analysis of isolated constituent beams.

3.5 Truss
Truss is a framed structure made up of members, joints and supports assembled to carry the loads to earth safely. A truss like structure is particularly required in bridges or in roof supporting system of large auditorium because a truss may run over long column-less span, giving a safe and economical design. A truss is so designed that all the joints are necessarily internal hinges or pins and the load from slab are made to act only at the joints i.e. no member of the truss directly receives the load transversely. As the loads are acting only at the ends of the members, each member of truss is a 2-Force member, which does not undergo any bending, but deforms only longitudinally and develops simple tensile or compressive stresses. While analyzing the truss, self weight of the member is ignored.

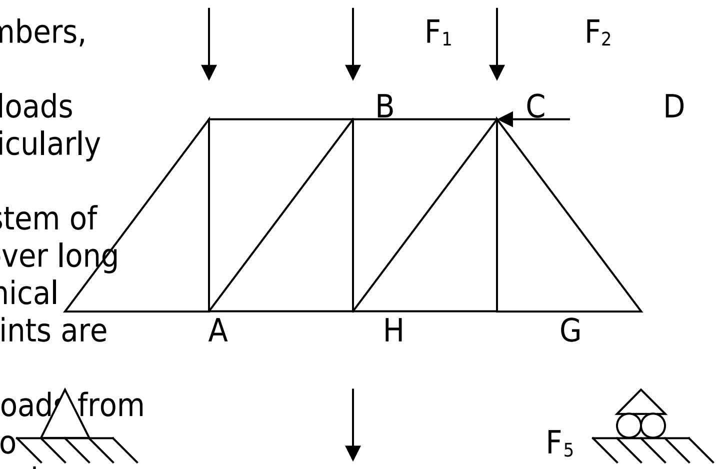

Fig.3-B

3.6 Analysis of Truss
It involves finding support reactions and nature and magnitude of member internal forces. Support reactions can be found by simply considering equilibrium of entire truss under the action of external known forces and unknown reactions. Member internal forces can be found by either method of joints or method of sections, which are again extended applications of equilibrium methods only.

3.6.1 Method of Joints
In method of joints, each joint of truss is isolated, its FBD is drawn and its equilibrium is considered. FBD of any joint shows the external loads acting at that joint and internal forces of the members meeting at that joint. Every such FBD is a Concurrent force system in equilibrium, which can be solved for two unknown member forces.

3.6.2 Method of Sections
In method of sections, a suitable section is considered and truss is imagined to be cut into two parts by that section. Any one part is isolated, its FBD is drawn and its equilibrium is considered. The FBD of isolated part shows all the external loads acting on that part and the internal forces of the members cut by the imagined section. Every such FBD is a General force system in equilibrium, which can be analyzed for three unknown member forces.
Following rules are to be followed while imagining a section in this method;
(1) Section should cut those members for which we desire to find internal forces.
(2) Section should start from outside and end on outside of the truss.
(3) As far as possible, section should cut at the most three members.

3.7 Stability Configuration for Simply Supported Truss
A truss supported using one hinge support and one roller support is called as Simply Supported Truss.
If 'j' represents number of joints and 'm' represents number of members, then
(1) When $m = 2j - 3$, truss is stable and determinate i.e. all internal member forces can be found by equilibrium methods alone. Such a truss is called as a Perfect Frame.
(2) When $m > 2j - 3$, truss is stable, in fact over-safe but is indeterminate i.e. all the member internal forces cannot be found by simple equilibrium methods. It is called as Redundant frame. Such trusses do exist but advanced methods are required for their analysis.

(3) When m < 2j – 3, truss is unstable and unsafe. It is not a structure at all and is, in fact, a mechanism. It is called as Imperfect frame of Deficient frame.

3.8 Cantilever Truss

It is possible to design a safe and determinate truss with both the supports as hinge supports and stability configuration m = 2j – 4. Such a truss is called as Cantilever Truss. If m < 2j – 4, the cantilever truss is unstable and if m > 2j – 4, it is indeterminate.

3.9 Methods of Inspection for Zero Force Member

A member that carries no force is called as Zero-Force Member. Following are inspection methods to identify a zero-force member without any actual calculations;

(1) If there are only three members meeting at a joint, two of them collinear and no external force acts at that particular joint, then the two collinear members develop equal internal force and the non-collinear third member is a zero-force member.
(2) If only two members meet at a joint with no external force at that joint, then both the members are zero-force members.

3.10 Frames

Frames are also structures with similar engineering applications as those of a truss, but frames are more complex in design than a truss. Unlike truss members, frame members may be bent up bars. Further, a frame member may be directly acted upon by transverse loads and couple moments from slab, so that it undergoes bending like a beam and develops complicated internal forces. Thus analysis for internal force of frame member is beyond the scope of equilibrium. Frames can be analyzed by equilibrium methods for support reactions and pin joint reactions. To find unknown pin reactions, FBD of each member is drawn and equilibrium of each member is considered. This step is called as dismembering of frame. A systematic analysis easily gives answers to all unknown support reactions and pin reactions.

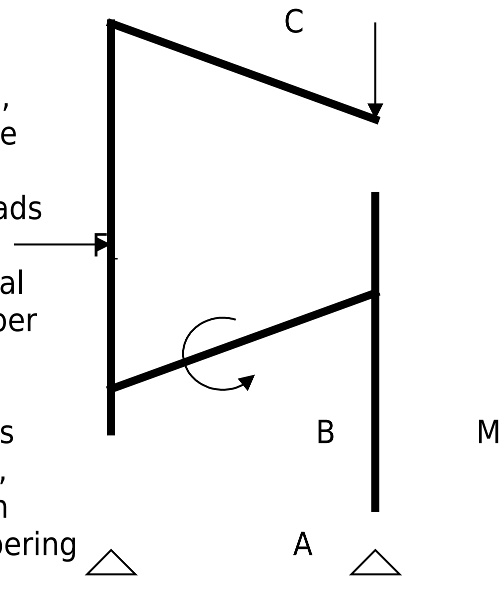

Fig.3-C

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4. FRICTION

4.1 Introduction

The opposition offered by two surfaces in physical contact, to any tendency of relative tangential motion or actual tangential motion between them, is termed as friction. Force of friction is simply a tangential reaction between two bodies across the contact. Being a reaction, it is naturally a self-adjusting force i.e. it changes its direction and up to a certain extent, its magnitude, according to the change in applied force trying to cause the relative motion. Mechanical interlocking of the irregularities over two surfaces in contact is believed to be the major cause of frictional opposition.

4.2 Types of Friction

(a) Static Friction: When two bodies in contact are not actually undergoing any relative slide but due to external force, there is a tendency developed of such a relative motion, the friction between two surfaces opposing this tendency is called as Static Friction.

(b) Dynamic Friction: When one body is actually sliding over the other body, the opposition offered by two surfaces in contact is called as Dynamic or Kinetic or Sliding Friction.

4.3 Behavior of Frictional Force

A graph of applied force v/s frictional force is plotted.
limiting
As the applied force is gradually increased, frictional
equilibrium
force also increases up to a certain limit called as Limiting Frictional Force. This is the maximum frictional force that can be developed between given two surfaces. Body is on the verge of sliding over the supporting surface
kinetic friction
and is said to be in the state of limiting equilibrium or of impending motion. If external force is further increased, the frictional opposition suddenly reduces and body actually starts sliding over the support. Here onwards, the frictional
motion
opposition more or less remains constant.
P

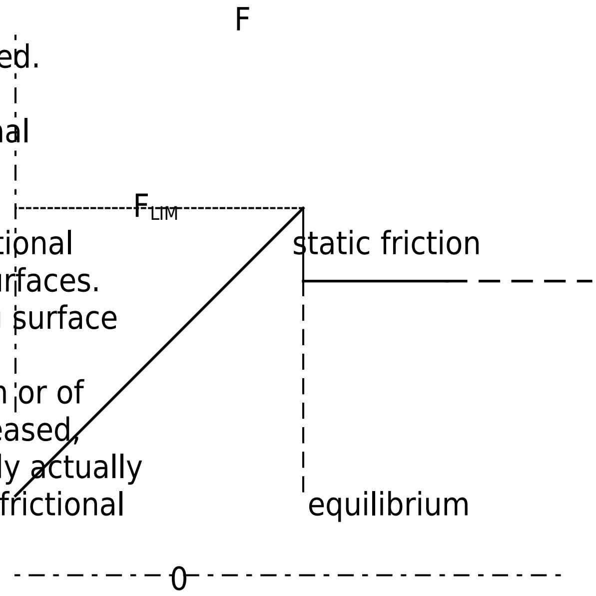


Fig.4-A

4.4 Newton's Laws of Friction

(a) Limiting frictional force is proportional to the normal reaction between two surfaces in contact.
i.e. $F_{LIM} \propto N$ ∴ $F_{LIM} = \mu_s N$, or $F_{LIM}/N = \mu_s$ where μ_s is a constant, called as Coefficient of static friction.

(b) When one body is actually sliding over another body, kinetic frictional force is proportional to the normal reaction between two bodies.
i.e. $F_K \propto N$ ∴ $F_K = \mu_K N$, or $F_K/N = \mu_K$ where μ_K is a constant, called as Coefficient of kinetic friction.

4.5 Angle of Friction

Resultant of normal reaction N and frictional force F is called as Total Reaction R. Angle ø made by total reaction R with common normal to the two surfaces in limiting equilibrium is called as Angle of Friction.
We have;
$\tan ø = F_{LIM}/N = \mu_s$ or $ø = \tan^{-1}(\mu_s)$.

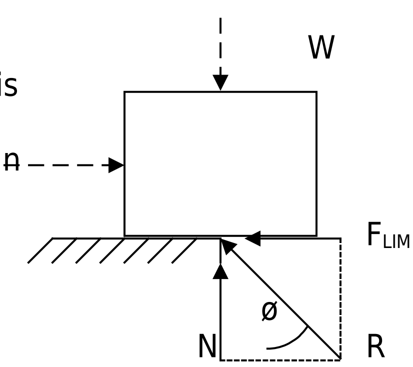


Fig.4-B

4.6 Angle of Repose

It is the angle made by an inclined plane with horizontal for which, the body kept on that plane is on the verge of sliding down, under the action of its own weight. Angle of repose is numerically equal to angle of friction i.e. [illegible] = ø.

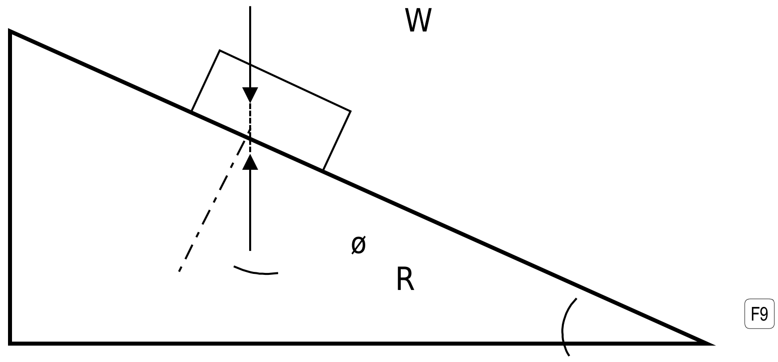
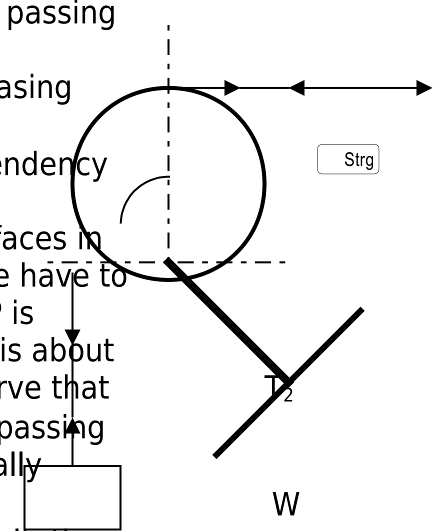


Fig.4-C



4.7 Belt Friction

Let us imagine that weight W is to be pulled up using a belt passing over a pulley or drum as shown. As we gradually start increasing force applied to the other end of the belt, belt develops a tendency to slide relative to the pulley surface to which, both the surfaces in contact oppose by friction. Thus to actually lift weight W, we have to overcome not just W itself, but also this force of friction. If P is the force applied at the other end of the belt when the belt is about to slide over the pulley and W is about to be lifted, we observe that P $>$ W. If T_1 and T_2 are the tensions in the parts of the belt passing over pulley as shown, T_1 = P and T_2 = W. Then it automatically follows that, $T_1 > T_2$. T_1 and T_2 are, respectively, called as Tight-side tension and Slack-side tension. In case of a flat belt, they are related to each other as;

Fig.4-D

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

In case of a V-belt and Rope, they are related to each other as;

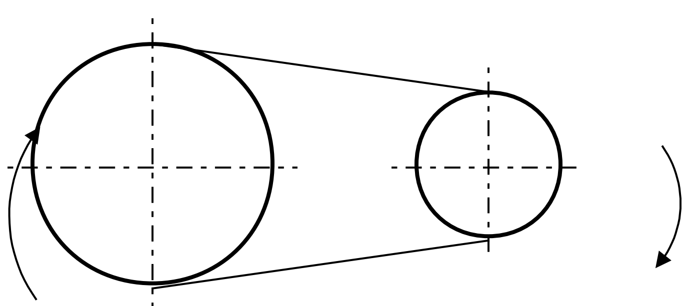
$$\frac{T_1}{T_2} = e^{\mu\theta \operatorname{cosec}\alpha}$$

where μ is the coefficient of static friction between belt and pulley, θ is the contact angle or lap angle and α is the semi-angle of the groove in case of a V-belt or Rope.

In general, tension in the part of belt that is about to leave the pulley is more than the part of belt, which is about to enter the pulley.

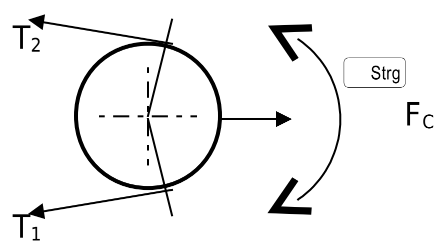
4.8 Power Transmission Using Belts

Using belt or rope drives, it is possible to transmit power generated at a wheel (called as driver pulley) to a far away wheel (called as follower or driven pulley).



Driver Pulley Driven Pulley

Fig.4-E



Driven Pulley

Fig.4-F

Let R be the radius of the driven pulley.
Moment acting about the center of pulley M = $(T_1 - T_2) \times R$.
If the pulley turns through angle θ, workdone W = Mθ = $(T_1 - T_2)R \times \theta$.

$$Power = \frac{dW}{dt} = (T_1 - T_2)R\frac{d\theta}{dt}$$

$$\therefore Power = (T_1 - T_2)R\omega = (T_1 - T_2)v$$

Here, ω is the angular speed of the driven wheel and v is linear speed of the belt (such that v = Rω).

When the belt moves around the driven pulley, due to inertia force (also known as centrifugal force), additional tension is induced in the belt element, which is called as centrifugal tension (Fig.4-F).
It may be proved that, this centrifugal tension is;

$T_C = \frac{wv^2}{g}$, where w = Weight of belt per unit length or ($\frac{w}{g}$) is the mass of belt per unit length.
Thus maximum tension, the given belt element can withstand will be, T_{perm} or $T_{max} = T_1 + T_C$.

It may further be proved that for maximum power transmission, $T_C = \frac{T_{perm}}{3}$.

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5. CENTROID, CG & MOMENT OF INERTIA

5.1 Introduction

To analyze a distributed force system, it is convenient to find its resultant, so that effectively all the forces may be assumed to be concentrated at a point. Similarly, distributed quantities like length, area, volume, mass and weight may be assumed to be concentrated at a point for the ease in the analysis. Such a point in case of length, area and volume is called as Centroid, in case of mass, is called as Center of Mass and in case of weight force, is called as Center of Gravity.

5.2 Calculation of Centroid Coordinates

Following are the expressions using which centroid of any given irregular shape may be found. These expressions have been derived from Varignon's Theorem.

Centroid of a solid w.r.t to its volume is calculated as;

$$\overline{X} = \frac{v_1x_1 + v_2x_2 + ... + v_nx_n}{v_1 + v_2 + ... + v_n} = \frac{\sum vx}{\sum v}, \quad \overline{Y} = \frac{v_1y_1 + v_2y_2 + ... + v_ny_n}{v_1 + v_2 + ... + v_n} = \frac{\sum vy}{\sum v}$$

where $v_1, v_2, ..., v_n$ are the volumes of regular solid parts (into which given irregular solid is divided) for which centroid coordinates (x_1,y_1), (x_2,y_2),.....,(x_n,y_n) are known.

Similarly, centroid of a plane irregular lamina may be found using expression;

$$\overline{X} = \frac{a_1x_1 + a_2x_2 + ... + a_nx_n}{a_1 + a_2 + ... + a_n} = \frac{\sum ax}{\sum a}, \quad \overline{Y} = \frac{a_1y_1 + a_2y_2 + ... + a_ny_n}{a_1 + a_2 + ... + a_n} = \frac{\sum ay}{\sum a}$$

where $a_1, a_2, ..., a_n$ are the areas of regular geometric figures (into which given irregular lamina is divided) for which centroid coordinates (x_1,y_1), (x_2,y_2),.....,(x_n,y_n) are known.

Centroid of a linear element like wire, bent into some shape may be found using expression;

$$\overline{X} = \frac{l_1x_1 + l_2x_2 + ... + l_nx_n}{l_1 + l_2 + ... + l_n} = \frac{\sum lx}{\sum l}, \quad \overline{Y} = \frac{l_1y_1 + l_2y_2 + ... + l_ny_n}{l_1 + l_2 + ... + l_n} = \frac{\sum ly}{\sum l}$$

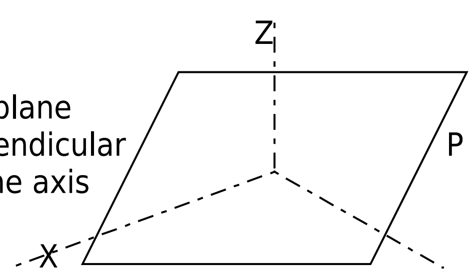
where $l_1, l_2, ..., l_n$ are the lengths of such regular line elements for which centroid coordinates (x_1,y_1), (x_2,y_2),....,
(x_n,y_n) are known.

5.3 Area Moment of Inertia

In the section design of beams, columns, shafts and other machine parts, an expression of the form $\sum ar^2$ or $\int r^2 da$ frequently occurs which is called as Second Moment of Area or Moment of Inertia w.r.t. Area. It is denoted by I. Strength of the section depends upon M.I. and choosing a section with appropriate M.I. ensures a safe as well as economical design.

5.4 Perpendicular Axes Theorem

M.I. of a lamina about an axis perpendicular to its plane is equal to sum of its M.I. about two mutually perpendicular axes, in the plane of lamina and concurrent with the axis perpendicular to the plane of lamina i.e. $I_Z = I_X + I_Y$



Y

Fig.5-A

5.5 Parallel Axes Theorem

M.I. of a lamina about any axis is equal to sum of its M.I. about a parallel axis through its centroid and product of area of lamina and square of the distance between two parallel axes i.e. $I_P = I_G + Ah^2$.

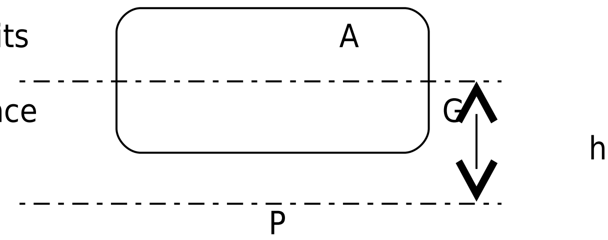
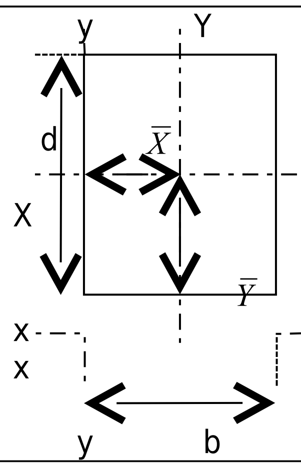
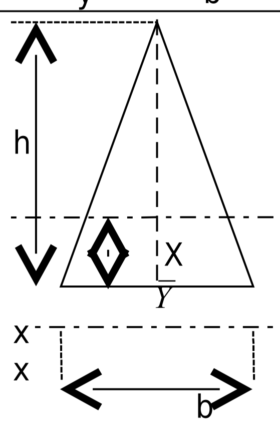
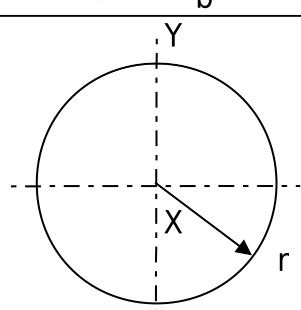
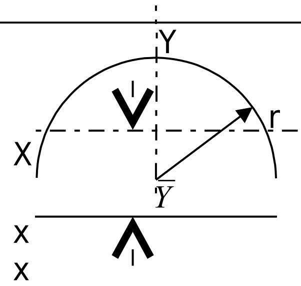
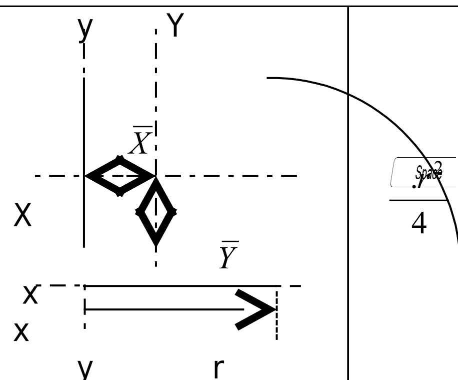
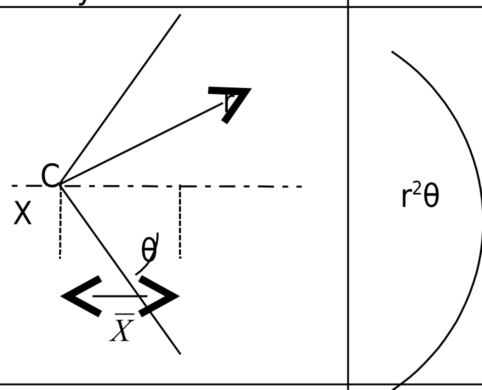
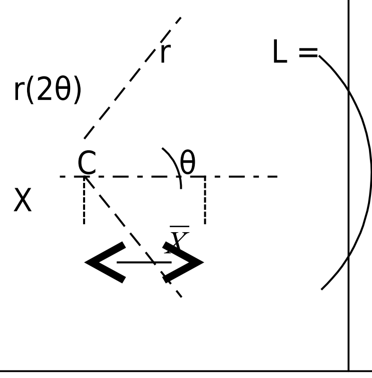


Fig.5-B

Sr.No.	SHAPE	AREA	CENTROID	II MOMENT OF AREA
1		$b \times d$	$\bar{X} = \frac{b}{2},\ \bar{Y} = \frac{d}{2}$	$I_X = \frac{bd^3}{12},\ I_Y = \frac{db^3}{12}$
2		$\frac{1}{2}bh$	$\bar{Y} = \frac{h}{3}$	$I_X = \frac{bh^3}{36},\ I_{xx} = \frac{bh^3}{12}$
3		πr^2	$\bar{X} = 0, \bar{Y} = 0$	$I_X = I_Y = \frac{\pi r^4}{4}$
4		$\frac{\pi r^2}{2}$	$\bar{X} = 0, \bar{Y} = \frac{4r}{3\pi}$	$I_X = 0.11r^4,$ $I_{xx} = I_Y = \frac{\pi r^4}{8}$

5		$\frac{\pi r^2}{4}$	$\overline{X} = \frac{4r}{3\pi}$, $\overline{Y} = \frac{4r}{3\pi}$	$I_X = I_Y = 0.0545r^4$ $I_{xx} = I_{yy} = \frac{\pi r^4}{16}$
6		$r^2\theta$	$\overline{X} = \frac{2r\sin\theta}{3\theta}$ (from center, along the axis of symmetry)	-
7		-	$\overline{X} = \frac{r\sin\theta}{\theta}$ (from center, along the axis of symmetry)	-

6. RECTILINEAR MOTION

6.1 Introduction

Dynamics is the study of various effects of force on moving bodies. It is divided into Kinematics and Kinetics.

Kinematics is a study of pure geometry of motion, without taking into account mass of the moving body, forces acting on the body and energy of the system. Kinetics is an advanced study of motion, taking into account mass of the body, forces acting on the body and energy of the system.

6.2 Particle Dynamics & Rigid Body Dynamics

Dynamics is further divided into Particle Dynamics and Rigid Body Dynamics. When a body undergoes pure translational motion, its overall dimensions and mass distribution may be ignored and the entire mass may be assumed to be concentrated at a point. Moving body is idealized as a moving point mass and is called as particle.

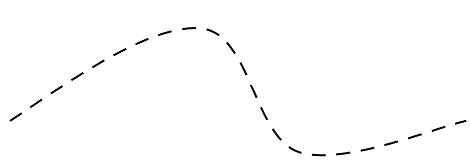
When motion of a body involves rotational motion, its mass distribution cannot be ignored. Body cannot be idealized as a point mass or particle and has to be considered as a rigid body. In short, study of translational motion comes under particle dynamics and study of motion involving rotations comes under rigid body dynamics.

6.3 Motion

Whenever a body changes its position w.r.t. an observer, the observer has experience or feeling of motion of that body. Motion is a relative term and has to be defined w.r.t. some observer.

6.3.1 Translational Motion

A body is said to be in pure translation when all the constituents of the body undergo equal displacements. When path traced by the body is a straight line, it is said to be in Rectilinear Translation (Fig.8-A). When the body traces
a curve, it is said to be in Curvilinear Translation (Fig.8-B)



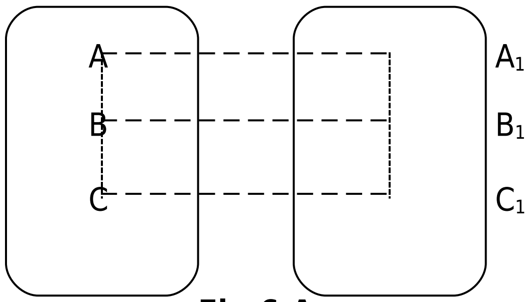


Fig.6-A

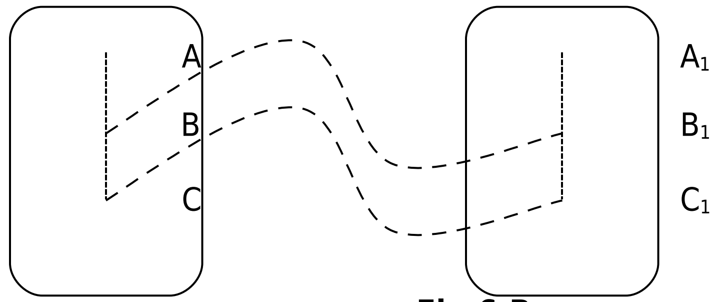


Fig.6-B

6.3.2 Rotational Motion

A body is said to be in pure rotation when all the constituents of the body trace concentric circles, the centers of which lie on a common axis of rotation. The axis of rotation may pass through the body or may be outside the body.

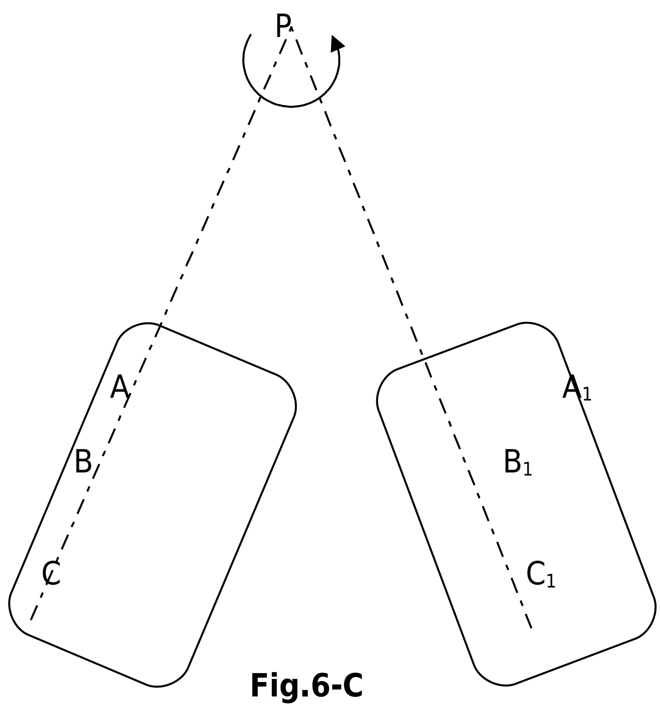


Fig.6-C

6.4 Kinematics of Rectilinear Motion

Consider a particle moving along a straight line. Let at time instant t, the particle be at point A (position x) and at a latter instant t+Δt, at point B (position x+Δx), both time and position measured from origin O (t = 0, x = 0).

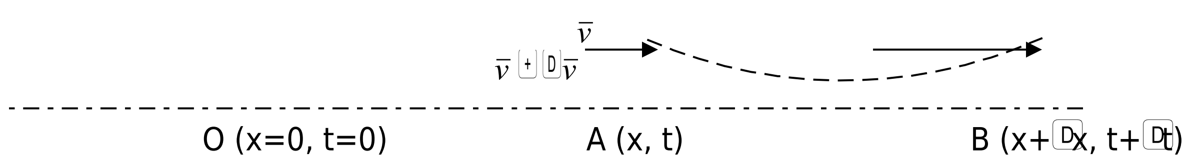


Fig.6-D

Displacement over journey AB = change in position in a specific direction = (x+Δx) – (x)
$\Rightarrow$ Displacement = Δx (→) or $+\Delta\bar{x}$. Distance traveled over journey AB = Δx (scalar).
Average velocity of particle is defined as the ratio of displacement and the time interval over which the displacement takes place. Similarly average speed is the ratio of distance traveled to the required time interval.
Displacement $\Delta\bar{x}$ takes place in time interval Δt so that,

Average velocity over journey AB; $\bar{v}_{avg} = \frac{\Delta\bar{x}}{\Delta t}$, Average speed over journey AB; $v_{avg} = \frac{\Delta x}{\Delta t}$.

$\Rightarrow$ Instantaneous velocity may then be found as $\bar{v}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\bar{x}}{\Delta t} = \frac{d\bar{x}}{dt}$

Thus, if position $\bar{x} = f(t)$ is known, its first ordered derivative w.r.t. time gives instantaneous velocity.

Let $\bar{v}$ and $\bar{v} + \Delta\bar{v}$ be the instantaneous velocities of particle at points A and B.
Change in velocity over journey AB = $\Delta\bar{v}$. This change takes place over time interval Δt so that average acceleration of particle over journey AB; $\bar{a}_{avg} = \dfrac{\Delta\bar{v}}{\Delta t}$

$$\therefore \text{ Instantaneous acceleration; } \bar{a}_{inst} = \lim_{\Delta t \to 0} \frac{\Delta\bar{v}}{\Delta t} = \frac{d\bar{v}}{dt}$$

Thus, if velocity $\bar{v} = f(t)$ is known, its first ordered derivative w.r.t. time gives instantaneous acceleration.

6.5 Types of Rectilinear Motion

A particle may undergo rectilinear motion in three different ways; viz. a = 0 i.e. Uniform Motion, a = K(constant) i.e. Uniformly Accelerated Motion and a = f(t) i.e. Variable Acceleration Motion.

6.5.1 Uniform Motion

As acceleration of particle is zero, its speed remains constant throughout the motion. We have only one equation of analysis; distance = speed $\times$ time i.e. s = ut.

6.5.2 Uniformly Accelerated Motion

When acceleration 'a' of the particle is constant, the motion is said to be Uniformly Accelerated Motion. Such a motion may be analyzed using three kinematical equations;

$$v = u + at,$$
$$v^2 = u^2 + 2as,$$
$$s = ut + \tfrac{1}{2}at^2.$$

These are vector equations, in which, u → initial velocity, v → final velocity, s → displacement and t → time interval.
A particular case of uniformly accelerated motion is Gravity Motion in which, acceleration a = g = 9.81m/s^2(↓).

6.5.3 Variable Acceleration Motion

In this type of motion, acceleration of particle changes as the time progresses. Acceleration is known as a = f(t) or f(v) or f(x). Using integration and known initial conditions of the motion, it is possible to derive the functions; velocity
v = g(t) and position x = ø(t), so that motion of particle is fully known.

6.6 Motion Curves (a-t, v-t, x-t Curves)

A graph showing instantaneous acceleration of particle v/s time is called as a-t curve. Area under a-t curve in the interval $t = t_1$ to $t = t_2$ represents change in velocity of particle over that interval. Similarly a graph showing instantaneous velocity v/s time is called as v-t curve. Slope of tangent to v-t curve at any point represents acceleration of particle at that instant. Also, area under v-t curve, in the interval $t = t_1$ to $t = t_2$ gives change in position i.e. displacement of particle over that interval. A graph showing position of particle v/s time is called as x-t curve. Slope of tangent at any point to x-t curve gives instantaneous velocity of the particle.

6.7 Absolute Motion & Relative Motion

Motion of a particle defined w.r.t. a stationary observer i.e. the motion, as experienced by an observer who is stationary at a fixed position on ground, is called as Absolute motion of that particle. Motion of a particle defined w.r.t. a moving observer i.e. motion of particle, as experienced or felt by an observer who himself is in some kind of motion is called as Relative motion of the particle w.r.t. that observer.

Let A and B be two particles in some kinds of motion as shown. Vectors $\bar{r}_A$ and $\bar{r}_B$ are the position vectors of A and B w.r.t. a fixed X-O-Y frame of reference and are called as absolute positions of A and B. Now let us define a vector $\overline{AB}$, which represents position of B relative to A (i.e. position of B as observed by an observer associated with moving particle A). It is denoted by $\bar{r}_{B/A}$.

Using Δ law, we can write,

$\bar{r}_A + \bar{r}_{B/A} = \bar{r}_B$

$\therefore \bar{r}_{B/A} = \bar{r}_B - \bar{r}_A$

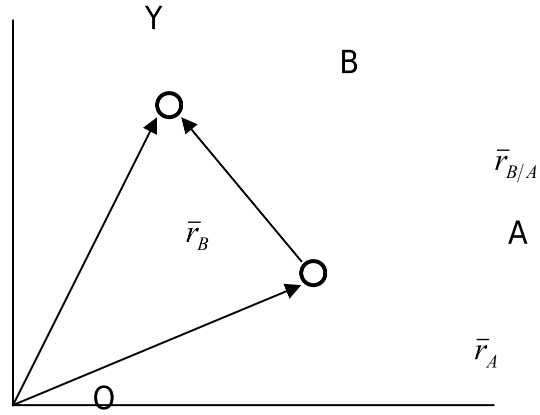
Differentiating w.r.t. time, we have,

$$\frac{d\bar{r}_{B/A}}{dt} = \frac{d\bar{r}_B}{dt} - \frac{d\bar{r}_A}{dt}$$

$\therefore \bar{v}_{B/A} = \bar{v}_B - \bar{v}_A$.

Differentiating once again w.r.t. time, we get,

$\therefore \bar{a}_{B/A} = \bar{a}_B - \bar{a}_A$



X

Fig.6-E

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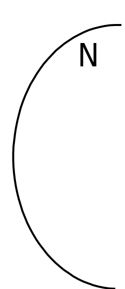
7. CURVILINEAR MOTION

7.1 Introduction

Whenever a particle travels along some curve other than a



straight line, it is said to be in curvilinear motion. For any curve, it is possible define center of curvature C and radius of curvature for an elemental segment, by drawing normals at the ends of that segment as shown (Fig.7-A).

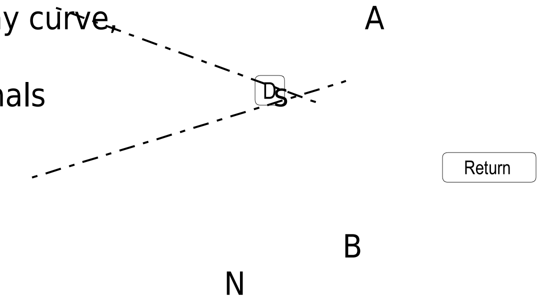
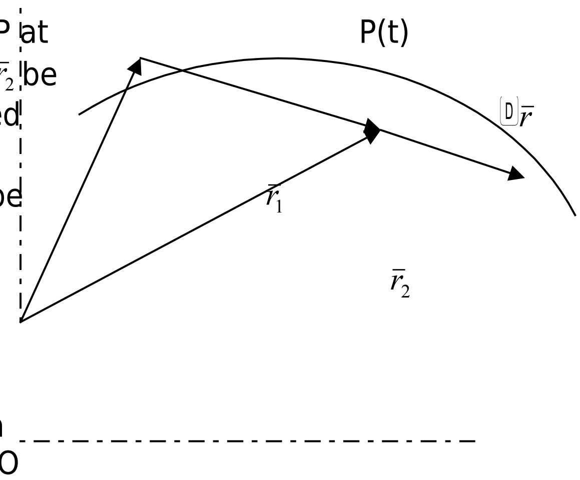


Fig.7-A

7.2 Kinematics of Curvilinear Motion

Consider a particle tracing a curve. Let it be at point P at instant t and at Q at a latter instant t+Δt. Let $\bar{r}_1$ and $\bar{r}_2$ be the position vectors of points P and Q w.r.t. some fixed origin. Displacement of particle over journey PQ will be given by vector $\overline{PQ}$. Using triangle law, we have,



$\bar{r}_1 + \overline{PQ} = \bar{r}_2 \;\therefore\; \overline{PQ} = \bar{r}_2 - \bar{r}_1 = \Delta\bar{r}$ (say).

Thus, change in position vector gives displacement in curvilinear motion.

Fig.7-B

This displacement takes place over time interval Δt so that we have;

Average velocity over journey PQ, $\bar{v}_{avg} = \dfrac{\Delta\bar{r}}{\Delta t}$.

$\bar{v}_{avg}$ will have same direction as that of $\Delta\bar{r}$ (Fig.7-B).

Instantaneous velocity may then be found as;

$$\bar{v}_{inst} = \lim_{\Delta t \to 0}\left(\frac{\Delta\bar{r}}{\Delta t}\right) = \frac{d\bar{r}}{dt}.$$

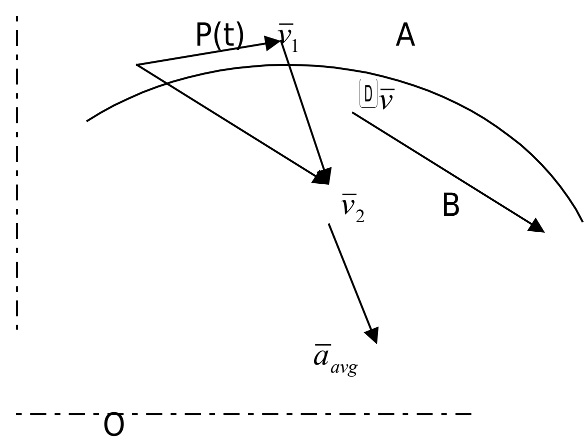


Fig.7-C

Thus, if position $\bar{r} = f(t)$ is known, its first ordered derivative w.r.t. time gives instantaneous velocity.

As Δt → 0, point Q approaches point P along the curve and almost coincides with P in limiting case. $\overline{PQ}$, which is a chord of the curve, tends to become tangent to the curve at P. Velocity $\bar{v}_{avg}$, which always lies along $\overline{PQ}$ and which is now called as $\bar{v}_{inst}$ is thus tangential to the curve at every point.

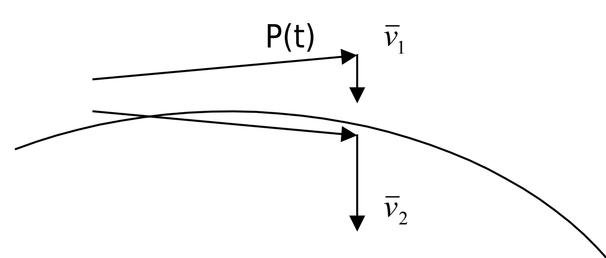


Fig.7-D

Let $\bar{v}_1$ and $\bar{v}_2$ be the velocities of particle at points P(t) and Q(t+Δt). Vector $\bar{v}_2$ is moved parallel to itself and drawn from point P and vector $\overline{AB}$ is developed.

Using triangle law, we have; $\bar{v}_1 + \overline{AB} = \bar{v}_2$

∴ $\overline{AB} = \bar{v}_2 - \bar{v}_1 = \Delta\bar{v}$ i.e. change in velocity of particle over journey PQ. This change takes place over time interval Δt so that average acceleration of particle over journey PQ will be;

$\bar{a}_{avg} = \frac{\Delta\bar{v}}{\Delta t}$.

Again, $\bar{a}_{avg}$ will have same direction as that of $\Delta\bar{v}$ (Fig.7-C).

Instantaneous acceleration may then be found as; $\bar{a}_{inst} = \lim_{\Delta t \to 0}\left(\frac{\Delta\bar{v}}{\Delta t}\right) = \frac{d\bar{v}}{dt}$

Thus, if velocity $\bar{v} = f(t)$ is known, its first ordered derivative w.r.t. time gives instantaneous acceleration.

As $\Delta t \to 0$, difference between $\bar{v}_2$ and $\bar{v}_1$ reduces and in limiting case, $\bar{v}_2$ is almost identical to $\bar{v}_1$. If we plot both the velocity vectors from common originating point P and develop $\overline{AB}$ = $\Delta\bar{v}$, we observe that its direction is towards the concavity of the curve. $\bar{a}_{avg}$, which always lies along $\Delta\bar{v}$ and which is now called as $\bar{a}_{inst}$, is thus always inclined inwards i.e. towards concavity of the curve (Fig.7-D).

7.3 Normal & Tangential Components of Acceleration

Instantaneous acceleration, which is an inwardly inclined vector, may be resolved into two components, along tangent and along normal to the curve at any point. We have;

$\bar{a} = \bar{a}_N + \bar{a}_T$

$a = \sqrt{(a_N)^2 + (a_T)^2}$ and $\tan\theta = \frac{a_N}{a_T}$.

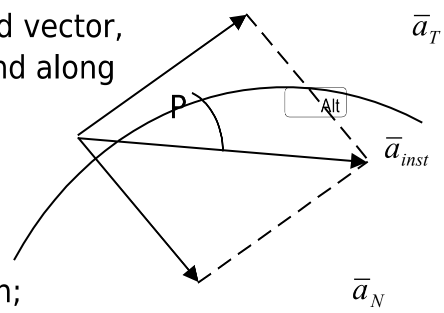


Fig.7-E

It may be proved that, magnitude of normal acceleration;

$a_N = \frac{v^2}{\rho}$, where v is the instantaneous speed of the particle and ρ is the radius of curvature at that point.

Magnitude of tangential acceleration, $a_T = \frac{dv}{dt}$, i.e. the time rate at which speed of the particle changes, irrespective of the direction aspect of the velocity vector. Thus, in curvilinear motion, if speed v = f(t) is known, its first ordered derivative w.r.t. time gives magnitude of tangential acceleration a_T.

7.4 Kinematical Equations in Curvilinear Motion

When magnitude of tangential acceleration a_T is constant, kinematical equations in scalar form may be used to analyze curvilinear motion. These equations may be written as;

$v = u + (a_T)t$,

$v^2 = u^2 + 2(a_T)s$,

$s = ut + \frac{1}{2}(a_T)t^2$

In these equations, u ≡ initial speed, v ≡ final speed, t ≡ time interval, a_T ≡ magnitude of tangential acceleration (which is constant), s ≡ distance traveled i.e. curve length traced by particle.

7.5 Radius of Curvature

For a particle moving in curvilinear motion in X-Y plane, if equation of path y = f(x) is known, then radius of curvature of path at any point may be calculated as;

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

If horizontal and vertical positions are known as functions of time, i.e. x = f(t) and y = f(t), then it can be proved that,

radius of curvature at any instant;

$\rho = \frac{\left(v_X^2 + v_Y^2\right)^{3/2}}{v_X a_Y - a_X v_Y}$, where $v_X = \frac{dx}{dt}$, $a_X = \frac{d^2x}{dt^2}$, $v_Y = \frac{dy}{dt}$, $a_Y = \frac{d^2y}{dt^2}$.

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8. PROJECTILE MOTION

8.1 Introduction

Any particle projected in air in inclined manner (e.g. a ball struck by a bat, a stone thrown or a bullet fired) is called as Projectile. The typical motion it undergoes is called its Projectile Motion. Initial velocity 'u' imparted to a projectile while projecting is called as Velocity of Projection. Angle ' ' made by this velocity with horizontal is called as Angle of Projection.

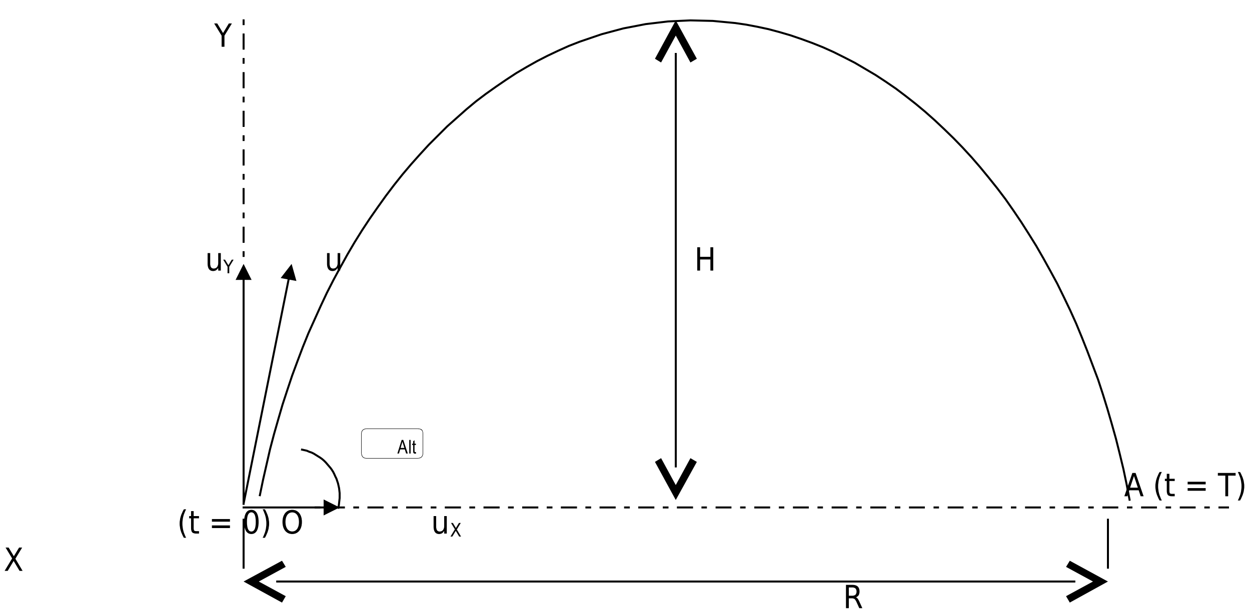

Fig.8-A

8.2 Component Motion

Projectile motion is best analyzed as component motion. Projection velocity u is resolved into two components,
u_x = ucos and u_y = usin . Actual curvilinear motion of projectile is imagined as a combination of two simultaneous motions, along X and along Y. Neglecting air resistance, component u_x is assumed to remain constant throughout the motion. Thus X directional motion of projectile is uniform motion with constant speed. Vertical component u_y is subjected to downwards gravitational acceleration so that it reduces during ascend, at the rate 9.81m/s^2, becomes zero at the maxima of path and increases at the same rate while descending. Thus Y directional motion of projectile is completely governed by gravity and is simply gravity motion. Composition of uniform motion along X and uniformly accelerated (gravity) motion along Y gives the actual projectile motion.

8.3 Equation of Path

The typical path traced by projectile is called as Trajectory. It may be proved that equation of this trajectory is;

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}.$$

Quantities u and α, being the initial parameters, are constants so that this equation is of the form $y = ax^2 + bx + c$, which represents a parabola. Hence trajectory of a projectile is always parabolic.

Analyzing projectile motion as component motion, following standard results can be proved;

(1) Time of Flight (time interval for which projectile remains in air), $T = \frac{2u\sin\alpha}{g}$.

(2) Horizontal Range (distance between point of projection and point of striking), $R = \frac{u^2\sin 2\alpha}{g}$.

(3) Maximum Height (distance of maxima of trajectory from X axis), $H = \frac{u^2\sin^2\alpha}{2g}$

We can further prove that, for a given velocity of projection u,

(1) Horizontal range is maximum at angle of projection $\alpha = 45°$ and the corresponding $R_{max} = \frac{u^2}{g}$

(2) For two complementary angles of projection α and $(90° - \alpha)$, horizontal range R is same.

It may be noted that all the above standard results are applicable only when point of projection and point of striking are on the same plane. When they are not, detailed analysis as component motion has to be carried out.

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9. KINETICS OF PARTICLES

9.1 Newton's Second Law

Newton's Second Law states that acceleration produced in a particle due to an external effective force is such that its magnitude is proportional to magnitude of force and direction same as that of the force.

$a \propto \Sigma F$, which may also be written as, $\Sigma F \propto a$

$\therefore \Sigma F = ma$ or vectorially, $\Sigma \bar{F} = m\bar{a}$.

While analyzing rectilinear motion, it is convenient to assume one of the reference axes, say X, in the direction of acceleration, so that the Newton's II Law statement may be written in component form as;

$\Sigma F_X = ma_X$ and $\Sigma F_Y = 0$ (as there is no acceleration in Y direction).

In case of curvilinear motion, acceleration may be resolved into normal and tangential components so that the Newton's II Law in component form may be expressed as;

$\Sigma F_N = ma_N = m\frac{v^2}{r}$ and $\Sigma F_T = ma_T = m\frac{dv}{dt}$

9.2 D'Alembert's Principle

From Newton's II law, we have, $\Sigma \bar{F} = m\bar{a}$, which can be written as $\Sigma \bar{F} - m\bar{a} = 0$. For this equation to be dimensionally correct, quantity $m\bar{a}$ must have same nature as that of the force $\Sigma \bar{F}$. Thus, we observe that, LHS of this equation is vector sum of two forces ($\Sigma \bar{F}$) & ($-m\bar{a}$) and RHS is zero, which implies equilibrium. But particle is actually accelerating. Hence D'Alembert has suggested that a vector ($-m\bar{a}$), called as Inertia Vector, may be added to given force system acting on the particle and the system may be brought into apparent equilibrium, called as Dynamic Equilibrium. Analysis of an accelerating particle may then be carried out like equilibrium analysis, using equations ΣF_x (including inertia vector) = 0 and $\sum F_y$ (including inertia vector) = 0. D'Alembert's Principle is just an alternative way of expressing Newton's II Law.

9.3.1 Work by a Constant Force

Whenever there is displacement in the particle under the action of external force, work is said to be done by force on the particle. When force $\overline{F}$ and displacement $\overline{S}$ are in the same direction,

$\overline{S}$

work is calculated as product of magnitude of force and magnitude of displacement i.e. W = FS. In a more general case when vectors $\overline{F}$

$\overline{F}$

and $\overline{S}$ subtend angle θ with each other, work is calculated as
W = FScosθ (Fig.9-A).
W is +ve for 0 ≤ θ < 90°,

$\overline{S}$

W is –ve for 90° < θ ≤ 180°,
W is zero for θ = 90°.

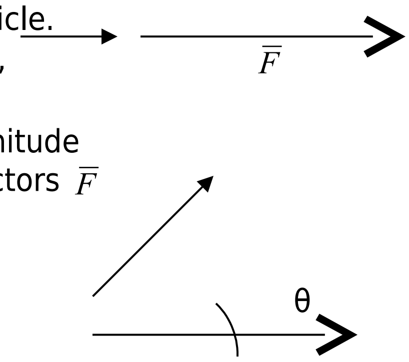

Fig.9-A

9.3.2 Work by a Variable Force

Consider a spring deformed through x_1. Let the externally applied force be gradually increased so that there is total final deformation of x_2 in the spring. It is observed that the magnitude of internal force f is proportional to deformation x at any instant in the spring.

$f \propto x$

$\therefore f = kx$, where k is called as spring constant or stiffness of the material.

Let there be an elemental deformation dx in the spring over and above x. Elemental work dW by internal force
dx
over elemental deformation dx will be;
dW = - f dx = - (kx)dx.
$x_2 - x_1$
Total work by elastic force over deformation x_1 to x_2 (or over additional deformation $x_2 - x_1$) will be;
B

$$W_E = \int_{x_1}^{x_2} dW = \int_{x_1}^{x_2} -kx\,dx$$

x_2

$$\therefore W_E = -\tfrac{1}{2}k\left(x_2^2 - x_1^2\right)$$

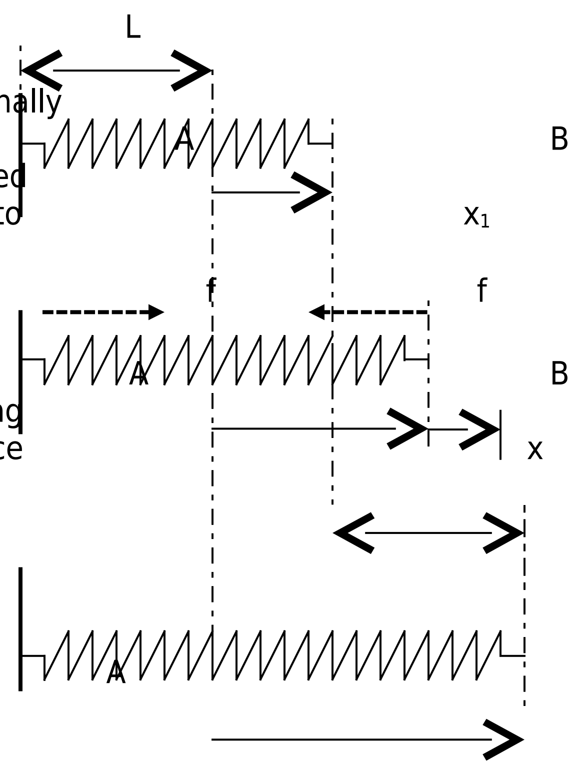

Fig.9-
B

Thus, during deformation, workdone by elastic forces is negative.
If the same spring restitutes from x_2 to x_1, work by elastic forces will be;

$$W_E = \int_{x_2}^{x_1} dW = \int_{x_2}^{x_1} -kx\,dx$$

$$\therefore W_E = +\tfrac{1}{2}k\left(x_2^2 - x_1^2\right)$$

Thus, during restitution, workdone by elastic forces is positive.
In case, the spring is originally in its natural state (deformation $x_1 = 0$) and then deforms through x ($x_2 = x$) or restitutes through x to its natural state, the above expressions will change to;

$W_E = -\tfrac{1}{2}kx^2$ . . . during deformation and

$W_E = +\tfrac{1}{2}kx^2$ . . . during restitution.

9.4 Work-Energy Principle

Total work by all the forces acting on a particle is equal to change in kinetic energy of the particle.

$\sum W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$, where u and v are initial and final speeds of the particle.

9.5 Potential Energy

If a particle of mass m is above a chosen datum level by height h, the function mgh is defined as Gravity Potential Energy of the particle. When particle is at datum level itself, h = 0 and hence, gravity potential energy, $PE_G = 0$.

If an elastic material or spring has a deformation of x, the function ½ kx^2 is called as Elastic Potential Energy of the system. When the spring is in natural state, deformation x = 0 and hence elastic potential energy, $PE_E = 0$.

9.6 Conservative & Non-conservative Forces

Gravity force and elastic restoring force are called as Conservative forces because when only these two forces are working during the motion, total mechanical energy (KE + PE) of the particle remains constant. All other forces like frictional force, air resistance, etc are called as Non-conservative forces, since mechanical energy is not conserved when these forces are working, but a part of it gets converted into other forms like heat, light, sound, etc.

9.7 Power

The time rate at which the work is accomplished is called as Power.

Power = $\frac{dW}{dt} = \frac{d(F.s)}{dt} = F\frac{ds}{dt} = F.v$

Thus Power may be calculated as product of magnitude of force exerted by engine and the speed it develops in the particle.

9.8 Impulse - Momentum Principle

Impulse is a vector quantity, which is product of force and time interval for which it acts. Linear momentum is also a vector, which is product of mass of particle and its instantaneous velocity. The Impulse – Momentum Principle states that total impulse of an external force on a particle is equal to change in its linear momentum. We have;

$$\int_{t_1}^{t_2} \overline{F} dt = m\overline{v} - m\overline{u}$$

Instead of a single particle, if a system of particles with masses m_1, m_2,.....m_n , moving with initial velocities $\overline{u}_1, \overline{u}_2, ..., \overline{u}_n$ is acted upon by an external force $\overline{F}$ so that their final velocities are $\overline{v}_1, \overline{v}_2, ..., \overline{v}_n$, then the above equation changes to;

$$\int_{t_1}^{t_2} \overline{F} dt = \sum m\overline{v} - \sum m\overline{u}$$

If impulse of external force is zero and yet the velocities of the particles change (this is possible when the cause of the change in velocity lies within the system),

$\int_{t_1}^{t_2} \overline{F} dt = 0$ and the above equation reduces to $\sum m\overline{u} = \sum m\overline{v}$, which is one of the very important principles in Physics, the momentum conservation principle.

9.9 Principle of Conservation of Momentum

It states that, in absence of impulse of an external force, linear momentum of the entire system of particles remains constant.

$$m_1\overline{u}_1 + m_2\overline{u}_2 + ... + m_n\overline{u}_n = m_1\overline{v}_1 + m_2\overline{v}_2 + ... + m_n\overline{v}_n .$$

9.10 Impact

Impact is the collision between two particles in which, actual interval of physical contact is extremely short, but over that short duration, a very large give-and-take of force takes place between two colliding particles. A common normal drawn to the two colliding surfaces at the point of contact is called as line of impact. When center of mass of each colliding particle lies on the line of impact, the impact is called as Central Impact, otherwise Eccentric Impact. Central impact is further classified into Direct Central Impact and Oblique Central Impact.

9.10.1 Direct Central Impact

When in a central impact, velocities of both colliding particles, before collision and after collision, lie on the line of impact, it is said to be a Direct Central Impact (Fig.9-C).

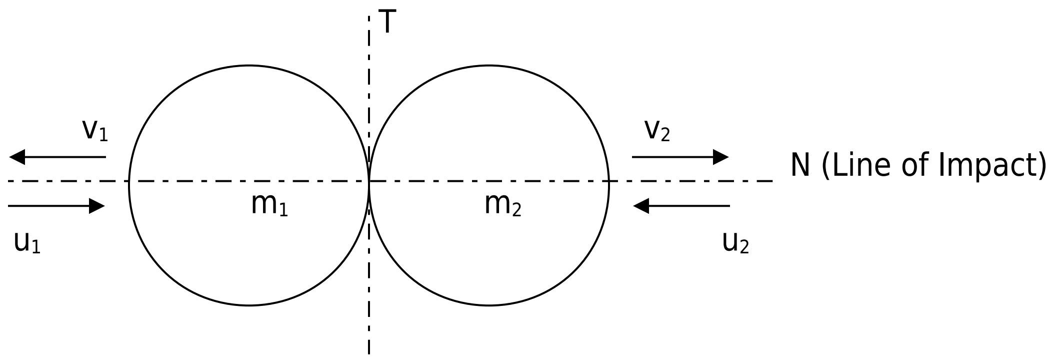

Fig.9-C

Consider two particles with masses m_1 and m_2, moving with velocities $\bar{u}_1$ and $\bar{u}_2$. If they undergo a direct central impact, over short duration of collision, they are observed to travel with a common velocity. Let $\bar{v}_1$ and $\bar{v}_2$ be their velocities just after impact. During collision, particles exert large forces over each other, but as far as system of two colliding particles is concerned, external impulse is absent. Hence linear momentum of the entire system is conserved during the impact.

$$\therefore m_1\bar{u}_1 + m_2\bar{u}_2 = m_1\bar{v}_1 + m_2\bar{v}_2$$

If actual impact is closely observed, we find that impact duration Δt is divided into two intervals, period of deformation Δt_1, in which, colliding particles exert large forces on each other and undergo deformation, followed by period of restitution Δt_2, in which, forces exerted by particles reduce, particles try to regain their original size & shape and separation occurs. The ratio of impulse of internal force during restitution to that during deformation is a constant and is defined as Coefficient of Restitution, denoted by e.

We have; $e = \dfrac{\int \bar{F}_2\, \Delta t_2}{\int \bar{F}_1\, \Delta t_1}$.

It can be further proved that, $e = \dfrac{\bar{v}_2 - \bar{v}_1}{\bar{u}_1 - \bar{u}_2}$.

9.10.2 Oblique Central Impact

When velocities of the colliding particles before and after impact do not lie along the line of impact, the impact is said to be Oblique Central Impact. To analyze an oblique impact, velocities $\bar{u}_1$, $\bar{u}_2$, $\bar{v}_1$ and $\bar{v}_2$ are resolved into components, along normal and tangent (Fig.9-D). Actual oblique impact is imagined to be a combination of two simultaneous impacts, along normal and along tangent. Considering part-impact along normal and ignoring all vectors along tangent, we observe that this part-impact is exactly identical to a direct central impact (Fig.9-E).

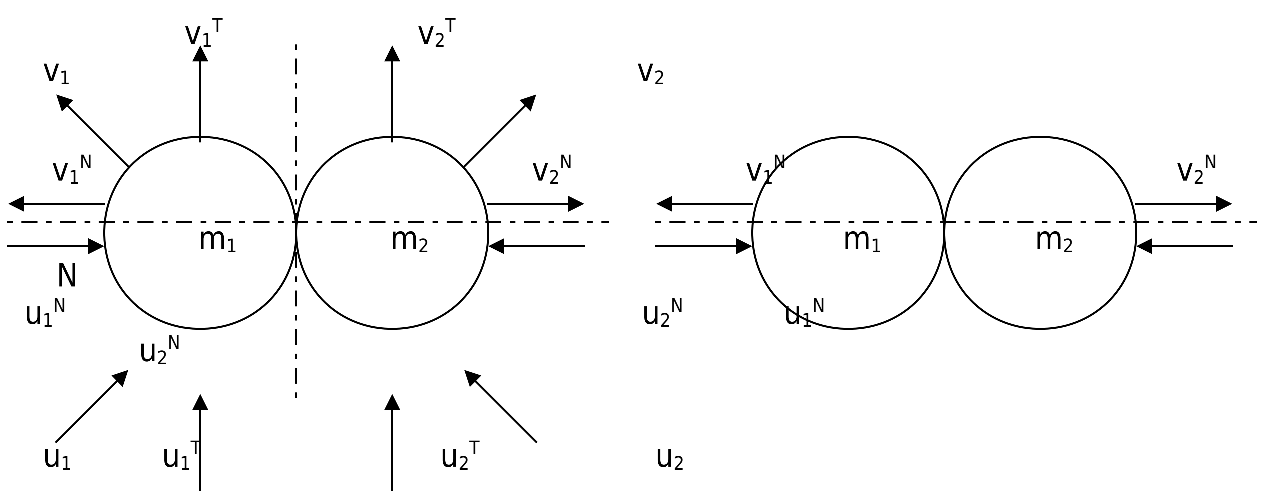

Fig.9-D

Fig.9-E

Impact along normal may, thus, be analyzed using equations;

$m_1\bar{u}_1^N + m_2\bar{u}_2^N = m_1\bar{v}_1^N + m_2\bar{v}_2^N$ and $e = \dfrac{\bar{v}_2^N - \bar{v}_1^N}{\bar{u}_1^N - \bar{u}_2^N}$

Now considering part-impact along tangent and ignoring all vectors along normal (Fig.9-F), conservation of momentum principle in tangential direction for the entire system may be written as;

$$m_1\bar{u}_1^T + m_2\bar{u}_2^T = m_1\bar{v}_1^T + m_2\bar{v}_2^T$$

Consider FBD of any one particle, say of m_1 (Fig.9-G). If $u_1^T > u_2^T$, over short duration of impact, m_1 will try to slide tangentially over m_2 and the two surfaces will develop frictional opposition to this relative sliding. Using Impulse-Momentum Principle for m_1 alone, we can write;

$$\int_{t_1}^{t_2} \bar{F}dt = m\bar{v}_1^T - m_1\bar{u}_1^T$$

However, we assume the colliding particles to be ideally smooth, so that the frictional forces are absent.

$\therefore\ m_1\bar{v}_1^T = m_1\bar{u}_1^T \Rightarrow\ \bar{v}_1^T = \bar{u}_1^T$.

Similarly, by considering FBD of m_2 and ignoring friction, it may be proved that,

$m_2\bar{v}_2^T = m_2\bar{u}_2^T \Rightarrow\ \bar{v}_2^T = \bar{u}_2^T$.

Thus we see that, in tangential impact, momentum of not only the entire system, but of each individual colliding particle is conserved.

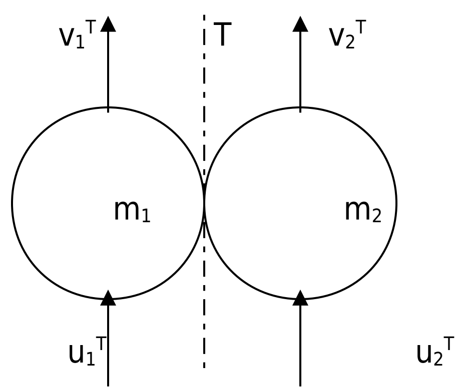

Fig.9-F

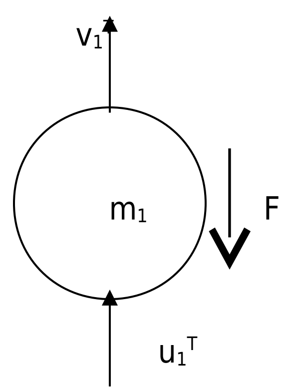

Fig.9-G

9.11.1 Perfectly Plastic Impact

When in an impact, two particles coalesce and onwards travel together with a common velocity, the impact is said to be Perfectly Plastic Impact. In perfectly plastic impact, there is a tremendous loss in KE of the system. Stages of restitution and separation do not exist and the colliding particles undergo permanent deformations.

Final velocities, $\bar{v}_1 = \bar{v}_2 = \bar{v}$ (say). The momentum conservation principle may be written as;

$m_1\bar{u}_1 + m_2\bar{u}_2 = (m_1 + m_2)\bar{v}$

and coefficient of restitution e = 0.

9.11.2 Perfectly Elastic Impact

When during an impact, not only the linear momentum but also KE of the system is conserved, it is said to be a Perfectly Elastic Impact. In perfectly elastic impact, we have;

$m_1\bar{u}_1 + m_2\bar{u}_2 = m_1\bar{v}_1 + m_2\bar{v}_2$ (conservation of momentum)

$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$ (conservation of KE)

Solving these two equations, we get, $\bar{v}_2 - \bar{v}_1 = \bar{u}_1 - \bar{u}_2$ i.e. e = 1.

This is a theoretical idea and not observed practically.

9.11.3 Semi-Elastic / Semi-Plastic Impact

More frequently observed collisions are of this type in which 0 < e < 1. Linear momentum is conserved during this impact but KE of this system is partly lost. Colliding particles undergo very small amount of permanent deformation and most of the original size and shape is regained during the stage of restitution.

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10. KINEMATICS OF RIGID BODIES

10.1 Translational Motion

A rigid slab is said to be in pure translation, when any imaginary straight line A-B-C on its surface remains parallel to itself throughout the motion. When the slab moves along a straight-line path, it is said to be in rectilinear translation (Fig.6-A). When this path is a curve, it is said to be in curvilinear translation (Fig.6-B). In translational motion, all constituents of the body undergo same displacements over any interval of time and have same instantaneous velocities and accelerations, i.e. $\overline{AA_1} = \overline{BB_1} = \overline{CC_1}$, $\bar{v}_A = \bar{v}_B = \bar{v}_C$, $\bar{a}_A = \bar{a}_B = \bar{a}_C$.

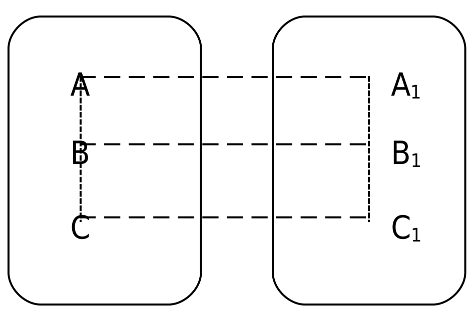


Fig.6-A (repeated)

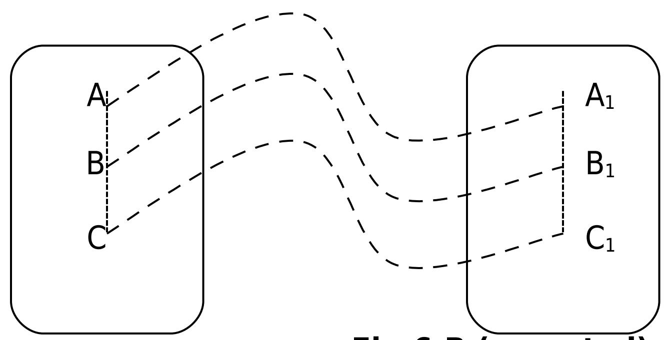


Fig.6-B (repeated)

10.2 Rotational Motion

A rigid slab is said to be undergoing pure rotational motion when its constituents trace concentric circles with a common center (Fig.10-A). This center may lie within or without the slab and it is a perfectly stationary point. Linear velocity of every constituent is such that its direction is perpendicular to the radius drawn from the center of rotation to that point i.e. if P is the center and A, B, C, etc. are constituents, then $v_P = 0, \bar{v}_A \perp PA, \bar{v}_B \perp PB, \bar{v}_C \perp PC$, etc.

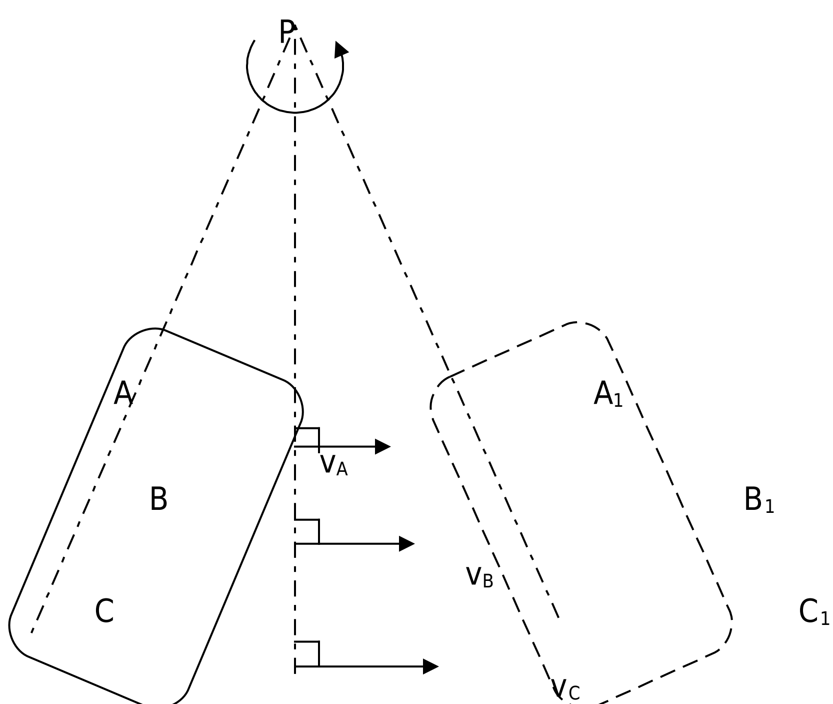


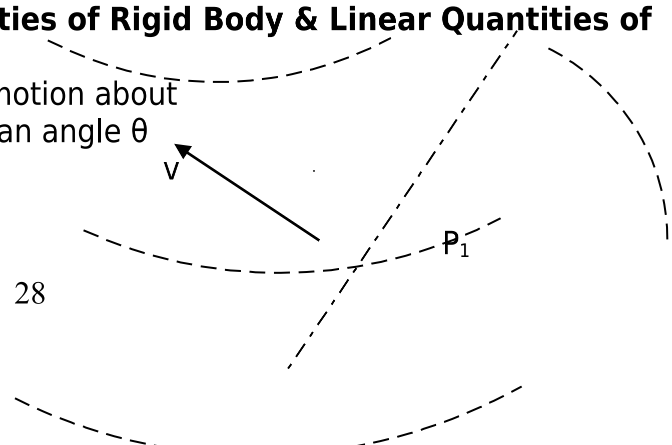
Fig.10-A

10.3 Relations between Angular Quantities of Rigid Body & Linear Quantities of its Constituent

Consider a rigid slab undergoing rotational motion about a fixed point O. Let radius OP turns through an angle θ in time t. We have;

Curve length s = r θ

Differentiating w.r.t. time,



$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

$\Rightarrow$ v = rω

where v is the linear speed of point P and ω is the angular speed of slab.

Differentiating again w.r.t. time,

$$\frac{dv}{dt} = r\frac{d\omega}{dt}$$

$\Rightarrow$ $a_T = r\alpha$

where a_T is the tangential acceleration of point P and α is angular acceleration of slab.

Normal acceleration of point P will be;

$$a_N = \frac{v^2}{r} = r\omega^2$$

It is to be remembered that all these relations are scalar relations.

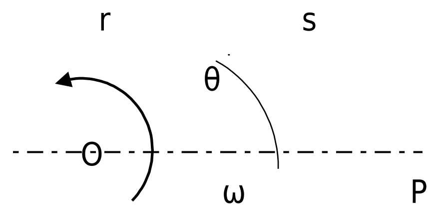


Fig.10-B

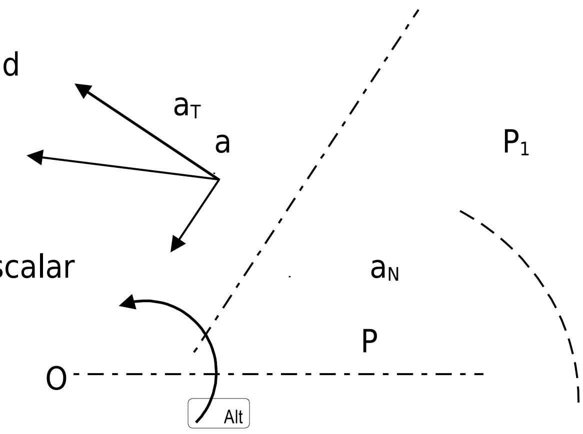


Fig.10-C

10.4 General Plane Motion

When a rigid body moves in such a way that its motion is neither purely translational nor purely rotational, it is said to be in General Plane Motion, e.g. sliding of a rod against wall and floor (Fig.10-D), rolling of a wheel over stationary surface (Fig.10-E), etc. It is possible to analyze every general plane motion as a combination of pure translation and pure rotation as shown.

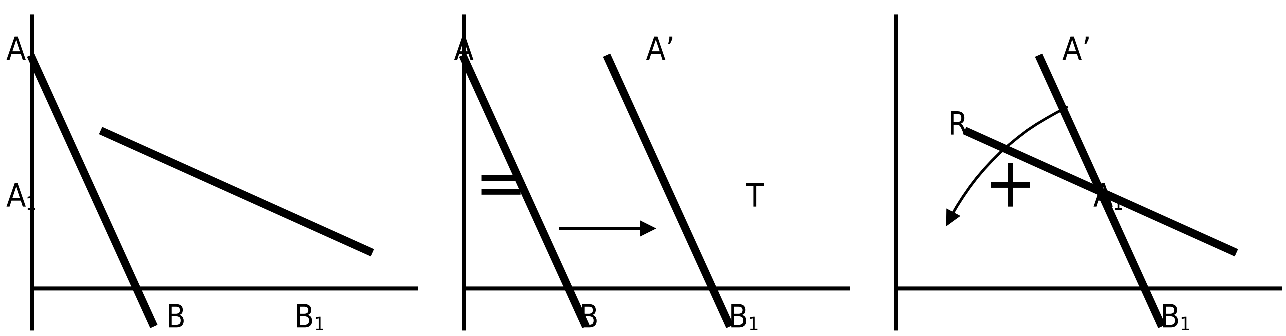


Fig.10-D

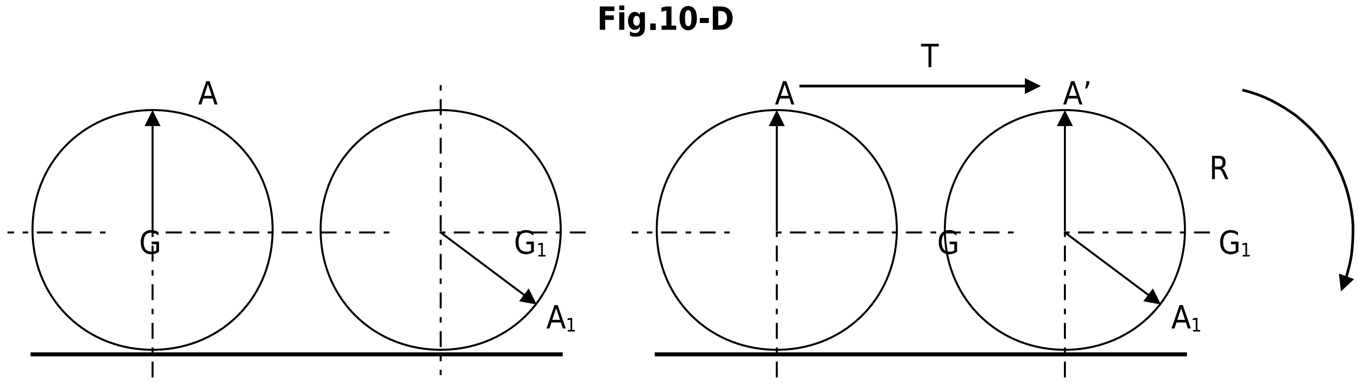


Fig.10-E

10.5 Instantaneous Center of Rotation (ICR)

Analyzing GPM as a combination of pure translation and pure rotation proves to be lengthy & time consuming. There is, however, a very efficient method to find velocity of any constituent of the body at any given instant. For a body in GPM, it is possible to locate, at every instant, one such point I in the plane such that, I itself is stationary and all the constituents of the body A, B, C, . . . etc. have velocities in the directions perpendicular to the segments drawn from I to those points, i.e. $v_I = 0$ and $\bar{v}_A \perp IA, \bar{v}_B \perp IB, \bar{v}_C \perp IC$ . . . etc, which is a condition observed in pure rotational motion. Thus body in actual GPM, appears to be undergoing pure rotation about point I at the defined instant. Point I is called as Instantaneous Center of Rotation (ICR). ICR may or may not be a constituent of the body.

Further, position of point I itself is not fixed and keeps changing every instant as the body moves. Thus, GPM of any body may be analyzed as pure rotational motion of the body at the given instant about ICR.

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