Test Paper-1
Engineering Mathematics

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Q. No. 1 - 10 Carry One Mark Each

MCQ 1.1 Every diagonal elements of a Hermitian matrix is

(A) Purely real (B) 0

(C) Purely imaginary (D) 1

SOL 1.1 A square matrix $\mathbf{A}$ is said to Hermitian if $\mathbf{A}^{Q} = \mathbf{A}$. So $a_{ij} = \overline{a}_{ji}$. If $i = j$ then $a_{ii} = \overline{a}_{ii}$ i.e. conjugate of an element is the element itself and a_{ii} is purely real.

Hence (A) is correct option.

MCQ 1.2 If $\mathbf{A}$ is a 3-rowed square matrix, then adj(adj $\mathbf{A}$) is equal to

(A) $|\mathbf{A}|^6$ (B) $|\mathbf{A}|^3$

(C) $|\mathbf{A}|^4$ (D) $|\mathbf{A}|^2$

SOL 1.2 We have $|\text{adj}(\text{adj}\mathbf{A})| = |\mathbf{A}|^{(n-1)^2}$

Putting $n = 3$, we get $|\text{adj}(\text{adj}\mathbf{A})| = |\mathbf{A}|^4$

Hence (C) is correct option.

MCQ 1.3 If $z = xyf\left(\frac{y}{x}\right)$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ is equal to

(A) z (B) $2z$

(C) xz (D) yz

SOL 1.3 The given function is homogeneous of degree 2.

Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

Hence (B) is correct option.

MCQ 1.4 $\int e^x\{f(x) + f'(x)\}dx$ is equal to

(A) $e^x f'(x)$ (B) $e^x f(x)$

(C) $e^x + f(x)$ (D) None of these

SOL 1.4

Let $I = \int e^x\{f(x) + f'(x)\}dx$

$$= \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$

$$= \left\{f(x)\,e^x - \int f'(x)\,e^x\,dx\right\} + \int e^x f'(x)\,dx = f(x).e^x$$

Hence (B) is correct option.

MCQ 1.5 The solution of the differential equation $2x\frac{dy}{dx} = 2 - y$ is

(A) $y = 2 - \sqrt{\frac{c}{x}}$ (B) $y = 2 + \sqrt{\frac{c}{x}}$

(C) $y = 2 - c\sqrt{x}$ (D) $y = 2 + c\sqrt{x}$

SOL 1.5 This is separable and written as $\frac{2dy}{2-y} = \frac{dx}{x}$

Integrating $-2\ln(2-y) = \ln x - \ln c$

$\Rightarrow \ln c(2-y)^{-2} = \ln x \Rightarrow y = 2 - \sqrt{\frac{c}{x}}$

Hence (A) is correct option.

MCQ 1.6 If $v = 2xy$, then the analytic function $f(z) = u + iv$ is

(A) $z^2 + c$ (B) $z^{-2} + c$

(C) $z^3 + c$ (D) $z^{-3} + c$

SOL 1.6

$$\frac{\partial v}{\partial x} = 2y = h(x,y), \frac{\partial v}{\partial y} = 2x = g(x,y)$$

by Milne's Method $f'(z) = g(z,0) + ih(z,0) = 2z + i0 = 2z$

On integrating $f(z) = z^2 + c$

Hence (A) is correct option.

MCQ 1.7 The following is the data of wages per day: 5, 4, 7, 5, 8, 8, 8, 5, 7, 9, 5, 7, 9, 5, 7, 9, 10, 8 The mode of the data is

(A) 5 (B) 7

(C) 8 (D) 10

SOL 1.7 Since 8 occurs most often, mode $= 8$

Hence (C) is correct option.

MCQ 1.8 If the probabilities that A and B will die within a year are p and q respectively, then the probability that only one of them will be alive at the end of the year is

(A) pq (B) $p(1-q)$

(C) $q(1-p)$ (D) $p + q - 2pq$

SOL 1.8 Required probability

$$= P[(A \text{ dies and } B \text{ is alive}) \text{ or } (A \text{ is alive and } B \text{ dies})]$$

$$= p(1-q) + (1-p)q = p + q - 2pq$$

Hence (D) is correct option.

MCQ 1.9 If $\text{cov}(X, Y) = 10$, $\text{var}(X) = 6.25$ and $\text{var}(Y) = 31.36$, then $\rho(X, Y)$ is

(A) $\frac{5}{7}$ (B) $\frac{4}{5}$

(C) $\frac{3}{4}$ (D) 0.256

SOL 1.9

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \frac{10}{\sqrt{6.25 \times 31.36}} = \frac{5}{7}$$

Hence (A) is correct option.

MCQ 1.10 Let r be the correlation coefficient between x and y and b_{yx}, b_{xy} be the regression coefficients of y on x and x on y respectively then

(A) $r = b_{xy} + b_{yx}$ (B) $r = b_{xy} \times b_{yx}$

(C) $r = \sqrt{b_{xy} \times b_{yx}}$ (D) $r = \frac{1}{2}(b_{xy} + b_{yx})$

SOL 1.10

$$b_{yx} = r.\frac{\sigma y}{\sigma x} \text{ and } b_{xy} = r.\frac{\sigma x}{\sigma y}$$

$$r^2 = b_{xy} \times b_{yx} \Rightarrow r = \sqrt{b_{xy} \times b_{yx}}$$

Hence (C) is correct option.

Q. No. 11- 21 Carry Two Mark Each

MCQ 1.11 The root of equation $2x - \log_{10} x = 7$ by regular false method correct to three places of decimal is

(A) 3.683 (B) 3.789

(C) 3.788 (D) 3.790

SOL 1.11

Let $f(x) = 2x - \log_{10} x - 7$

Taking $x_0 = 3.5$, $x_1 = 4$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 3.5 - \frac{0.5}{0.3979 + 0.5441}(-0.5441)$$

$$= 3.7888$$

Since $f(3.7888) = -0.0009$ and $f(4) = 0.3979$, therefore the root lies between 3.7888 and 4.

Taking $x_0 = 3.7888, x_1 = 4$, we obtain

$$x_3 = 3.7888 - \frac{0.2112}{0.3988}(-.009) = 3.7893$$

Hence the required root correct to three places of decimal is 3.789.

Hence (B) is correct option.

MCQ 1.12 For the differential equation $\frac{dy}{dx} = x - y^2$ given that

x:	0	0.2	0.4	0.6
y:	0	0.02	0.0795	0.1762

Using Milne predictor-correction method, the y at next value of x is

(A) 0.2498 (B) 0.3046

(C) 0.4648 (D) 0.5114

SOL 1.12

x: 0 0.2 0.4 0.6

On calculation we get

$$f(x) = x - y^2$$
$$f_1(x) = 0.1996$$
$$f_2(x) = 0.3937$$
$$f_3(x) = 0.5689$$

Using predictor formula

$$y_4^{(p)} = y_0 + \frac{4}{3}h(2f_1 - f_2 + 2f_3)$$

Here $h = 0.2$

$$y_4^{(p)} = 0 + \frac{0.8}{3}[2(.1996) - (.3937) + 2(.5689)]$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 - 4f_3 + f^*_3),$$

$$f^*_4 = f(x_4, y_4^{(p)}) = f(.8, 0.3049) = .07070$$

$$y_4^{(c)} = .0795 + \frac{2}{30}[.3937 + 4(.5689) + .7070] = .3046$$

Hence (B) is correct option.

MCQ 1.13 The ranks obtained by 10 students in Mathematics and Physics in a class test are a follows

Rank in Maths	Rank in Chem.
1	3
2	10
3	5
4	1
5	2
6	9

7	4
8	8
9	7
10	6

The coefficient of correlation between their ranks is

(A) 0.15 (B) 0.224

(C) 0.625 (D) None

SOL 1.13

$$r = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x).\text{var}(y)}} = \frac{-16.5}{\sqrt{2.89 \times 100}} = -0.97$$

Hence (D) is correct option.

MCQ 1.14 A can solve 90% of the problems given in a book and B can solve 70%. What is the probability that at least one of them will solve a problem, selected at random from the book?

(A) 0.16 (B) 0.63

(C) 0.97 (D) 0.20

SOL 1.14

Let $E =$ the event that A solves the problem, and

$F =$ the event that B solves the problem.

Clearly E and F are independent events.

$$P(E) = \frac{90}{100} = 0.9,$$

$$P(F) = \frac{70}{100} = 0.7,$$

$$P(E \equiv F) = P(E).P(F) = 0.9 \times 0.7 = 0.63$$

Required probability $= P(E \cup F)$

$= P(E) + P(F) - P(E \equiv F)$

$= (0.9 + 0.7 - 0.63) = 0.97$

Hence (C) is correct option.

MCQ 1.15 $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = ?$

(A) $\frac{\pi ab}{a + b}$ (B) $\frac{\pi(a + b)}{ab}$

(C) $\frac{\pi}{a + b}$ (D) $\pi(a + b)$

SOL 1.15

$$I = \int_c \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz = \int_c f(z)\,dz$$

where c is be semi circle r with segment on real axis from $-R$ to R.

The poles are $z = \pm ia, z = \pm ib$. Here only $z = ia$ and $z = ib$ lie within the contour c

$$\int_c f(z)\,dz = 2\pi i$$

(sum of residuces at $z = ia$ and $z = ib$)

Residue at $z = ia$, is

$$= \lim_{z \to b} (z - ib) \frac{z^2}{(z-ia)(z-ia)(z^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Residuce at $z = ib$ is

$$= \lim_{z \to b} (z - ib) \frac{z^2}{(z-ib)(z+ia)(z+ib)(z-ib)}$$

$$= \frac{-b}{2i(a^2-b^2)}$$

$$\int_c f(z)\,dz = \int_r f(z)\,dz + \int_{-R}^{R} f(z)\,dz$$

$$= \frac{2\pi i}{2i(a^2-b^2)}(a-b) = \frac{\pi}{a+b}$$

Now $\int_r f(z)\,dz = \int_0^{\pi} \frac{ie^{2i\theta}\, i\mathrm{R}e^{\iota\theta}\,d\theta}{(R^2 e^{2i\theta}+a^2)(R^2 e^{2i\theta}+b^2)}$

$$= \int_0^{\pi} \frac{\frac{e^{3i\theta}}{R}\,d\theta}{\left(e^{2i\theta}+\frac{a^2}{R^2}\right)\left(e^{2i\theta}+\frac{b^2}{R^2}\right)}$$

Now when $R \to \infty$, $\int_r b(z)\,dz = 0$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dz = \frac{\pi}{a+b}$$

Hence (C) is correct option.

MCQ 1.16 $\int_c z^2 e^{1/z}\,dz = ?$ where c is $|z| = 1$

(A) $i3\pi$

(B) $-i3\pi$

(C) $\frac{i\pi}{3}$

(D) None of the above

SOL 1.16

$$f(z) = z^2 e^{\frac{1}{z}} = z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right)$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \dots$$

The only pole of $f(z)$ is at $z=0$, which lies within the circle $|z|=1$

$$\int_c f(z)\,dz = 2\pi i \text{ (residue at } z=0)$$

Now, residue of $f(z)$ at $z=0$ is the coefficient of $\frac{1}{z}$ i.e. $\frac{1}{6}$

$$\int_c f(z)\,dz = 2\pi i \times \frac{1}{6} = \frac{1}{3}\pi i$$

Hence (C) is correct option.

MCQ 1.17 The family of conic represented by the solution of the DE $(4x+3y+1)\,dx+(3x+2y+1)\,dy=0$ is

(A) Circles (B) Parabolas

(C) Hyperbolas (D) Ellipses

SOL 1.17 In the given equation $M=4x+3y+1$ and $N=3x+2y+1$,

$$\frac{\partial M}{\partial y}=3,\quad \frac{\partial N}{\partial x}=3$$

Thus $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$,

Hence the given equation is exact. The solution is

$$\int (4x+3y+1)\,dx+\int (2y+1)\,dy = c$$

$$2x^2+3xy+x+y^2=c \qquad \text{...(i)}$$

Here, $a=2,\ b=1,\ h=\frac{3}{2}$

$$h^2-ab=\frac{9}{4}-2\times 1=\frac{1}{4}>0$$

Hence, (i) represents a family of hyperbolas.
Hence (C) is correct option.

MCQ 1.18 The equation of the curve, for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2=A\sin 2\theta$. This satisfies the differential equation

(A) $r\frac{dr}{d\theta}=\tan 2\theta$ (B) $r\frac{d\theta}{dr}=\tan 2\theta$

(C) $r\frac{dr}{d\theta}=\cos 2\theta$ (D) $r\frac{d\theta}{dr}=\cos 2\theta$

SOL 1.18

Given that $r^2=A\sin 2\theta$ ...(i)

Differentiating (i) with respect to θ, we get

$$2r\frac{dr}{d\theta}=2A\cos 2\theta$$

or, $$r\frac{dr}{d\theta}=A\cos 2\theta \qquad \text{...(ii)}$$

Eliminating 'A' between (i) and (ii), we get

$$r\frac{dr}{d\theta} = \frac{r^2}{\sin 2\theta}\cos 2\theta$$

or, $$r\frac{d\theta}{dr} = \tan 2\theta$$

Which is the required differential equation.
Hence (B) is correct option.

MCQ 1.19 If $F(a) = \frac{1}{\log a}, a > 1$ and $F(x) = \int a^x\,dx + K$ is equal to

(A) $\frac{1}{\log a}(a^x - a^a + 1)$

(B) $\frac{1}{\log a}(a^x - a^a)$

(C) $\frac{1}{\log a}(a^x + a^a + 1)$

(D) $\frac{1}{\log a}(a^x + a^a - 1)$

SOL 1.19

$$F(x) = \int a^x\,dx + K = \frac{a^x}{\log a} + K$$

$$\Rightarrow F(a) = \frac{a^a}{\log a} + K$$

$$K = \frac{1}{\log a} - \frac{a^a}{\log a} = \frac{1 - a^a}{\log a}$$

$$F(x) = \frac{a^x}{\log a} + \frac{1 - a^n}{\log a} = \frac{1}{\log a}[a^x - a^a + 1]$$

Hence (A) is correct option.

MCQ 1.20 For what value of $x\left(0 \le x \le \frac{\pi}{2}\right)$, the function $y = \frac{x}{(1 + \tan x)}$ has a maxima ?

(A) $\tan x$

(B) 0

(C) $\cot x$

(D) $\cos x$

SOL 1.20

Let $$z = \frac{1 + \tan x}{x} = \frac{1}{x} + \frac{\tan x}{x}$$

Then, $$\frac{dz}{dx} = -\frac{1}{x^2} + \sec^2 x \text{ and } \frac{d^2z}{dx^2} = \frac{2}{x^3} + 2\sec^2 x\tan x$$

$$\frac{dz}{dx} = 0 \Rightarrow -\frac{1}{x^2} + \sec^2 x = 0 \Rightarrow x = \cos x$$

$$\left[\frac{d^2z}{dx^2}\right]_{x=\cos x} = 2\cos^3 x + 2\sec^2 x\tan x > 0$$

Thus z has a minima and therefore y has a maxima at $x = \cos x$.
Hence (D) is correct option.

MCQ 1.21 If $\mathbf{A}_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then consider the following statements :

I. $\mathbf{A}_\alpha \cdot \mathbf{A}_\beta = \mathbf{A}_{\alpha\beta}$

II. $\mathbf{A}_\alpha \cdot \mathbf{A}_\beta = \mathbf{A}_{(\alpha+\beta)}$

III. $(\mathbf{A}_\alpha)^n = \begin{bmatrix} \cos^n\alpha & \sin^n\alpha \\ -\sin^n\alpha & \cos^n\alpha \end{bmatrix}$

IV. $(\mathbf{A}_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

Which of the above statements are true ?

(A) I and II
(B) I and IV
(C) II and III
(D) II and IV

SOL 1.21

$$\mathbf{A}_\alpha \cdot \mathbf{A}_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$
$$= \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix} = A_{\alpha+\beta}$$

Also, it is easy to prove by induction that

$$(\mathbf{A}_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

Hence (D) is correct option.

Common data for Question 22 to Q. 23

$$f(z_0) = \int_c \frac{3z^2+7z+1}{(z-z_0)}\,dz, \text{ where } c \text{ is the circle } x^2+y^2=4$$

MCQ 1.22 The value of $f(3)$ is

(A) 6
(B) $4i$
(C) $-4i$
(D) 0

SOL 1.22

$$f(3) = \int_c \frac{3z^2+7z+1}{z-3}\,dz, \text{ since } z_0 = 3 \text{ is the only singular}$$

point of $\frac{3z^2+7z+1}{z-3}dz$, and it lies outside the circle $x^2+y^2=4$ i.e.,

$|z| = 2$,therefore $\frac{3z^2+7z+1}{z-3}$ is analytic everywhere within c.

Hence by Cauchy's theorem -

$$f(3) = \int_c \frac{3z^2+7z+1}{z-3}\,dz = 0$$

Hence (D) is correct option.

MCQ 1.23 The value of $f'(1-i)$ is

(A) $7(\pi+i2)$
(B) $6(2+i\pi)$
(C) $2\pi(5+i13)$
(D) 0

SOL 1.23 The point $(1-i)$ lies within circle $|z| = 2$ (... the distance of $1-i$ i.e., (1,1) from

the origin is $\sqrt{2}$ which is less than 2, the radius of the circle).
Let $\phi(z) = 3z^2 + 7z + 1$ then by Cauchy's integral formula

$$\int_c \frac{3z^2 + 7z + 1}{z - z_0} dz = 2\pi i\phi(z_0)$$

$\Rightarrow \quad f(z_0) = 2\pi i\phi(z_0) \Rightarrow f'(z_0) = 2\pi i\phi'(z_0)$

and $\quad f''(z_0) = 2\pi i\phi''(z_0)$

since, $\quad \phi(z) = 3z^2 + 7z + 1$

$\Rightarrow \quad \phi'(z) = 6z + 7$ and $\phi''(z) = 6$

$$f'(1 - i) = 2\pi i[6(1 - i) + 7] = 2\pi(5 + 13i)$$

Hence (C) is correct option.

Common data for Question 24 to Q. 25

Expand the function $\frac{1}{(z-1)(z-2)}$ in Laurent's series for the condition given in question.

MCQ 1.24 $1 < |z| < 2$

(A) $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + ...$

(B) $... - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{18}z^3 - ...$

(C) $\frac{1}{z^2} + \frac{3}{z^2} + \frac{7}{z^4} ...$

(D) None of the above

SOL 1.24

Here $\quad f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ ...(i)

Since, $|z| > 1 \Rightarrow \frac{1}{|z|} < 1$ and $|z| < 2 \Rightarrow \frac{|z|}{2} < 1$

$$\frac{1}{z-1} = \frac{1}{z\left(1 - \frac{1}{z}\right)} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

and $$\frac{1}{z-2} = \frac{-1}{2}\left(1 - \frac{z}{2}\right)^{-1} = -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{9} + ...\right]$$

equation (1) gives -

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{9} + ...\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right)$$

or $$f(z) = ... - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{18}z^3 - ...$$

Hence (B) is correct option.

MCQ 1.25 $|z| > 2$

(A) $\frac{6}{z} + \frac{13}{z^2} + \frac{20}{z^3} + \dots$

(B) $\frac{1}{z} + \frac{8}{z^2} + \frac{13}{z^3} + \dots$

(C) $\frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \dots$

(D) $\frac{2}{z^2} - \frac{3}{z^3} + \frac{4}{z^4} \dots$

SOL 1.25

$$\frac{2}{|z|} < 1 \Rightarrow \frac{1}{|z|} < \frac{1}{2} < 1 \Rightarrow \frac{1}{|z|} < 1$$

$$\frac{1}{z-1} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{1}{2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right)$$

and $$\frac{1}{z-2} = \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots\right)$$

Laurent's series is given by

$$f(z) = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{98}{z^3} + \dots\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right)$$

$$= \frac{1}{z}\left(\frac{1}{z} + \frac{3}{z^2} + \frac{7}{z^3} + \dots\right)$$

$$\Rightarrow \quad f(z) = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \dots$$

Hence (C) is correct option.

Answer Sheet									
1.	(A)	6.	(A)	11.	(B)	16.	(C)	21.	(D)
2.	(C)	7.	(C)	12.	(B)	17.	(C)	22.	(D)
3.	(D)	8.	(D)	13.	(D)	18.	(B)	23.	(C)
4.	(B)	9.	(A)	14.	(C)	19.	(A)	24.	(B)
5.	(A)	10.	(C)	15.	(C)	20.	(D)	25.	(C)