HSC - BOARD - 2015

Date: 28.02.2015 **MATHEMATICS (40) - SOLUTIONS**

SECTION - I

Q.1 (A)

(i) (d) 32 A (2)

$$A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$$

$$A=2I$$

$$A^6=2^6\ I^6=64\ I=32(2I)=32A$$

$$A^6=32A$$

(ii) (c) $\left(\frac{2\pi}{3}\right)$ (2)

$$\cos^{-1}\left(-\frac{1}{2}\right)=\pi-\cos^{-1}\left(\frac{1}{2}\right) \qquad \left[\because \cos^{-1}(-x)=\pi-\cos x\right]$$

$$=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

(iii) (a) $fg=ch$ (2)

$hxy+gx+fy+c=0$ ....(i)

Comparing with $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$, we get

$$A=0,\ B=0,\ C=c\ ,\ H=\frac{h}{2},\ G=\frac{g}{2},\ F=\frac{f}{2}$$

If equation (i) represents a pairs of straight line then $\begin{vmatrix}A&H&G\\H&B&F\\G&F&C\end{vmatrix}=0$

$$\begin{vmatrix}0&\frac{h}{2}&\frac{g}{2}\\\frac{h}{2}&0&\frac{f}{2}\\\frac{g}{2}&\frac{f}{2}&c\end{vmatrix}=0$$

$$-\frac{h}{2}\left[\frac{ch}{2}-\frac{fg}{4}\right]+\frac{g}{2}\left[\frac{f}{4}\right]=0$$

$$-\frac{ch^2}{4}+\frac{fgh}{8}+\frac{fgh}{8}=0$$

$$-\frac{ch^2}{4}+\frac{fgh}{4}=0$$

$$ch^2=fgh$$

$$ch=fg$$

Q.1 (B)

(i) **Converse :** If the areas of two triangles are equal then they are congruent. (1)

Contrapositive : If the area of two triangles are not equal then they are not congruent. (1)

(ii) Comparing the equation $x^2+kxy-3y^2=0$ with $ax^2+2hxy+by^2=0$,

we get, $a=1,\ 2h=k,\ b=-3$. (1)

Let m_1, and m_2 be the slopes of the lines represented by $x^3+kxy-3y^2=0$

$$\therefore\ m_1+m_2=\frac{-2h}{b}=-\frac{k}{(-3)}=\frac{k}{3}$$

and $m_1m_2=\frac{a}{b}=\frac{1}{(-3)}=-\frac{1}{3}$

Now, $m_1+m_2=2(m_1\ m_2)$

$$\therefore\frac{k}{3}=2\left(-\frac{1}{3}\right)\qquad\therefore\ k=-2$$ (1)

(iii) The angle θ between the planes $\bar{r}\cdot\bar{n}_1=p_1$ and $\bar{r}\cdot\bar{n}_2=p_2$ is given by

$$\cos\theta=\frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}$$ .....(i)

Here, $\bar{n}_1=2\hat{i}+\hat{j}-\hat{k}$ and $\bar{n}_2=\hat{i}+2\hat{j}+\hat{k}$ (1)

$$\therefore\ \bar{n}_1\cdot\bar{n}_2=\left(2\hat{i}+\hat{j}-\hat{k}\right)\cdot\left(\hat{i}+2\hat{j}+\hat{k}\right)$$

$$=2(1)=1(2)+(-1)(1)$$

$$=2+2-1=3$$

Also, $|\bar{n}_1|=\sqrt{2^2+1^2+(-1)^2}=\sqrt{4+1+1}=\sqrt{6}$

$$|\bar{n}_2|=\sqrt{1^2+2^2+(-1)^2}=\sqrt{1+4+1}=\sqrt{6}$$

$\therefore$ from (1) we get, $\cos\theta = \dfrac{3}{\sqrt{6}\cdot\sqrt{6}} = \dfrac{1}{2}$

$\therefore \theta = \dfrac{\pi}{3}$ (1)

(iv) $3x - 1 = 6y + 2 = 1 - z$

$3\left(x - \dfrac{1}{3}\right) = 6\left(y + \dfrac{1}{3}\right) = -(z-1)$

$\dfrac{x - \frac{1}{3}}{\frac{1}{3}} = \dfrac{y + \frac{1}{3}}{\frac{1}{6}} = \dfrac{z-1}{-1}$ (1)

Line is passing through the point $\left(\dfrac{1}{3}, \dfrac{-1}{3}, 1\right)$ with dr^s $\dfrac{1}{3}, \dfrac{1}{6}, -1$ *i.e.* $2, 1, -6$

$\therefore$ Vector equation of line is $\bar{r} = \left(\dfrac{1}{3}\bar{i} - \dfrac{1}{3}\bar{j} + \bar{k}\right) + \lambda\left(2\bar{i} + \bar{j} - 6\bar{k}\right)$ (1)

(v) $\vec{c} = x\vec{a} + y\vec{b}$

$4\hat{i} + 3\hat{j} = x\left(\hat{i} + 2\hat{j}\right) + y\left(-2\hat{i} + \hat{j}\right)$ (1)

$4\hat{i} + 3j = (x - 2y)\hat{i} + (2x + y)\hat{j}$

Comparing on both side

$4 = x - 2y$ & $3 = 2x + y$

$x - 2y = 4$ (i)

$2x + y = 3$ (ii)

Solving equation (i) and (ii) we get

$x = 2$ and $y = -1$ (1)

Q.2 (A)

(i) $A(1, 1, 1)$, $B(2, 1, 3), C(3, 2, 2), D(3, 3, 4)$

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ at the position vectors of the point A, B, C, D respectively w.r.t. O, then

$\bar{a} = \hat{i} + \hat{j} + \hat{k}$

$\bar{b} = 2\hat{i} + \hat{j} + 3\hat{k}$

$\bar{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

$\bar{d} = 3\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \overrightarrow{AB} = \hat{i} + 2\hat{k}$

$\overrightarrow{AC} = 2\hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{AD} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ (1)

Volume of parallelopiped $= \overline{AB} \cdot \left[\overline{AC} \times \overline{AD}\right]$

$$= \left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix} \qquad (1)$$

$$= 1(3-2) + 2\,(4-2)$$

$$= 1 + 4$$

$$= 5 \text{ cubic units} \qquad (1)$$

(ii) $\sim(\sim p \wedge \sim q) \vee q$

(1) p	(2) q	(3) $\sim p$	(4) $\sim q$	(5) $\sim p \wedge \sim q$	(6) $\sim(\sim p \wedge \sim q)$	(7) $\sim(\sim p \wedge \sim q) \vee q$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

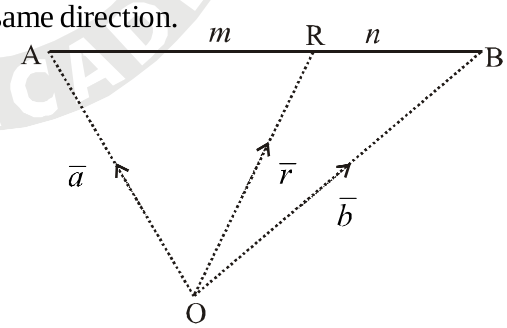
Column No. 5 (1 Mark) Column No. 7 (1 Mark)

The above statement is contigency. (1)

(iii) Let $\bar{a}, \bar{b}$ & $\bar{c}$ be the position vector of the points A, B, C respectively w.r.t. O.

As R is a point on the line segment $AB(A-R-B)$ and $\dfrac{AR}{RB} = \dfrac{m}{n}$, (1)

$\therefore \quad m(RB) = n(AR)$ and, $\overline{AE}$ and $\overline{RB}$ are in same direction.



$\therefore \quad m\left(\overline{RB}\right) = n\left(\overline{AR}\right)$

$\therefore \quad m\left(\overline{OB} - \overline{OR}\right) = n\left(\overline{OR} - \overline{OA}\right)$ (1)

$\therefore \quad m\left(\bar{b} - \bar{r}\right) = n\left(\bar{r} - \bar{a}\right)$

$\therefore \quad \bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$ (1)

Q.2 (B)

(i) Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are $-2, 1, -1$ and $-3, -4, 1$ respectively.

$\therefore \bar{a} = -2\hat{i} + \hat{j} - \hat{k}$ and $\bar{b} = -3\hat{i} - 4\hat{j} + \hat{k}$ (1)

A vector perpendicular to bth $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix} \qquad (1)$$

$$= (1-4)\,\hat{i} - (-2-3)\,\hat{j} + (8+3)\hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

$\therefore$ the direction ratios of the required line are –3, 5, 11 (1)

Now, $\sqrt{9+25+121}$

$$= \sqrt{155}$$

Direction cosine of the line are $\dfrac{-3}{\sqrt{155}}, \dfrac{5}{\sqrt{155}}, \dfrac{11}{\sqrt{155}}$ (1)

(ii) By the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$\therefore a = k \sin A, b = k \sin B, c = k \sin C$ (1)

Now, a^2, b^2, c^2 are in A.P.

$$\therefore b^2 - a^2 = c^2 - b^2$$

$$\therefore b^2 = a^2 + c^2 - b^2$$

$\therefore \dfrac{b^2}{2ac} = \dfrac{a^2+c^2-b^2}{2ac}$ (1)

$$\therefore \frac{k^2 \sin^2 B}{x(k\sin A)k(\sin C)} = \cos B$$

$$\therefore \frac{\sin^2 B}{2\sin A \sin C} = \cos B$$

$$\therefore \frac{\sin B}{\sin A \sin C} = \frac{2\cos B}{\sin B}$$

$\therefore \dfrac{\sin[\pi-(A+C)]}{\sin A \sin C} = 2\cot B$ .... $[\because A+B+C=\pi]$ (1)

$$\therefore \frac{\sin(A+C)}{\sin A \sin C} = 2\cot B$$

$$\therefore \frac{\sin A \cos C + \cos A \sin C}{\sin A \sin C} = 2\cot B$$

$$\therefore \frac{\sin A \cos C}{\sin A \sin C} + \frac{\cos A \sin C}{\sin A \sin C} = 2\cot B$$

$$\therefore \frac{\cos C}{\sin C} + \frac{\cos A}{\sin A} = 2\cot B$$

$\therefore \cot A + \cot C = 2\cot B$ (1)

Hence, $\cot A, \cot B, \cot C$ are in $A.P$

(iii) Let the three numbers be x, y, z

$x + y + z = 6$

$3(x + z) - y = 10$

$5(x + y) - yz = 3$

OR

$x + y + z = 6$

$3x - y + 3z = 10$

$5x + 5y - 4z = 3$ (1)

The equations can be written in matrix forms

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix}$$ (1)

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -27 \end{bmatrix}$$ (1)

$x + y + z = 6$

$-4y = -8 \Rightarrow y = 2$

$-9z = -27 \Rightarrow z = 3$

In equation $x + y + z = 6$

$x + 2 + 3 = 6$

$\therefore x = 1$

Three numbers are 1, 2, 3 (1)

[Note: If the student has started writing answer, then full marks will be given.]

Q.3 (A)

(i) Let m_1 and m_2 be the slopes of the lines represented by the equation

$ax^2 + 2hxy + by^2 = 0.$ ...(1)

Then their separate equations are $y = m_1 x$ and $y = m_2 x$

$\therefore$ their combined equation is $(m_1 x - y)(m_2 x - y) = 0$

i.e., $m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 = 0$ ...(2)

Since (1) and (2) represent the same two lines, comparing the coefficients, we get,

$$\frac{m_1 m_2}{a} = \frac{-(m_1 + m_2)}{2h} = \frac{1}{b}$$

$\therefore m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$ (1)

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2 = \frac{4h^2}{b^2} - \frac{4a}{b} = \frac{4(h^2 - ab)}{b^2}$$

$\therefore |m_1 - m_2| = \left|\frac{2\sqrt{h^2 - ab}}{b}\right|$ (1)

If θ is the acute angle between the lines, then

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, if $m_1 m_2 \neq -1$

$\tan\theta = \left|\frac{(2\sqrt{h^2 - ab})/b}{1 + (a/b)}\right|$, if $\frac{a}{b} \neq -1$

$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$, if $a + b \neq 0$. (1)

(ii) The lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ *and* $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ intersect,if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$ (1)

The equations of the given lines are

$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ *and* $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$

$\therefore x_1 = 1, y_1 = -1, z_1 = 1, x_2 = 3, y_2 = k, z_2 = 0$

$a_1 = 2, b_1 = 3, c_1 = 4, a_2 = 1, b_2 = 2, c_2 = 1$ (1)

Since these lines intersect, we get

$$\begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\therefore 2(3-8) - (k+1)(2-4) - 1(4-3) = 0$

$\therefore -10 + 2(k+1) - 1 = 0$

$\therefore 2(k+1) = 11$

$\therefore k + 1 = \frac{11}{2}$

$\therefore k = \frac{9}{2}$ (1)

(iii) Let p : switch s1 is closed

$\sim p$: switch s_1 is open

q : switch s_2 is closed

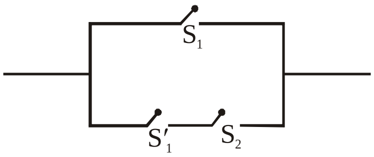
$\sim q$: switch s_2 is open

r : switch s_3 is closed

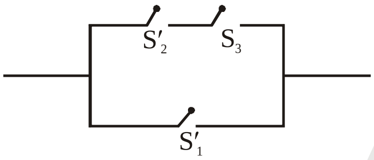
$\sim r$: switch s_3 is open (1)

Now,

$p \vee (\sim p \wedge q)$

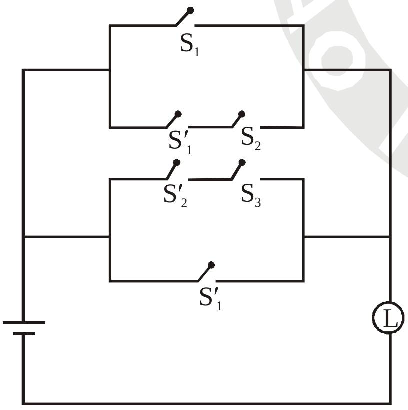


$(\sim q \wedge r) \vee \sim p$



(1)

$\therefore \left[p \vee (\sim p \wedge q)\right] \vee \left[(\sim q \wedge r) \vee \sim p\right]$



(1)

Q.3 (B)

(i) $\cos x - \sin x = 1$

Dividing by $\sqrt{1^2 + (-1)^2} = \sqrt{2}$

$$\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x = \frac{1}{\sqrt{2}}$$ (1)

$$\cos\frac{\pi}{4} \cos x - \sin\frac{\pi}{4} \sin x = \frac{1}{\sqrt{2}}$$

$$\cos\left(x + \frac{\pi}{4}\right) = \cos\frac{\pi}{4} \qquad .....(i)$$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \frac{\pi}{4}$; $n \in z$ (1)

$\therefore$ The general solution of equation (i) given by

$$x + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4} \ ; n \in z \qquad (1)$$

$$x = 2n\pi \ ; \quad x = 2n\pi - \frac{\pi}{2} \ ; n \in z \qquad (1)$$

(ii) The required equation of the plane parallel to the plane $x - 2y + 2z - 4 = 0$ is $x - 2y + 2z + \lambda = 0$

Now, distance of this plane from the point (1, 2, 3) $= \left|\frac{1(1) + (-2)(2) + 2(3) + \lambda}{\sqrt{1^2 + (-2)^2 + 2^2}}\right|$ (1)

$$1 = \left|\frac{1 - 4 + 6 + \lambda}{\sqrt{1 + 4 + 4}}\right| = \left|\frac{\lambda + 3}{3}\right|$$

$\therefore \quad \left|\frac{\lambda + 3}{3}\right| = 1$

$\therefore \quad \frac{\lambda + 3}{3} = \pm 1$ (1)

$\therefore \quad \frac{\lambda + 3}{3} = 1$ or $\frac{\lambda + 3}{3} = -1$

$\therefore \quad \lambda + 3 = 3$ or $\lambda + 3 = -3$

$\therefore \quad \lambda = 0$ or $\lambda = -6$ (1)

Hence, the equations of the required planes are

$x - 2y + 2z = 0$ and $x - 2y + 2z - 6 = 0.$ (1)

(iii) Let x units of food F_1 and y units of food F_2 be included in the diet of the sick person.

Then their total cost is $z =$ Rs. $(6x + 10y)$ (1)

This is the objective function which is to be minimized.

The constraints are as per the following table :

	Food F_1 (x)	Food F_2 (y)	Minimum requirement
Vitamin A	6	8	48
Vitamin B	7	12	64

From this table, the constraints are

$6x + 8y \geq 48,\ 7x + 12y \geq 64$

Also, the number of units of foods F_1 and F_2 cannot be negative.

$\therefore \quad x \geq 0,\ y \geq 0.$

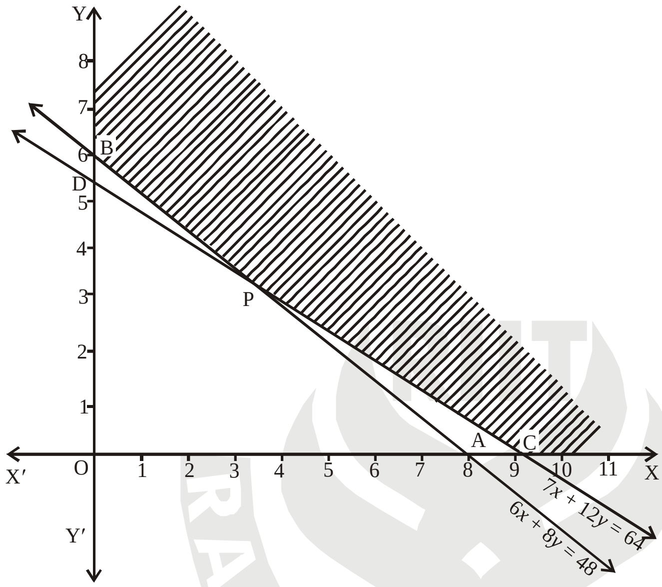
$\therefore$ the mathematical formulation of given LPP is

Minimize $z = 6x + 10y$, subjected to

$6x + 8y \geq 48,\ 7x + 12y \geq 64,\ x \geq 0,\ y \geq 0.$ (1)

First we draw the lines AB and CD whose equations are $6x + 8y \geq 48$ and $7x + 12y \geq 64$ respectively.

Line	Equation	Point on the X-axis	Point on the Y-axis
AB	$6x + 8y = 48$	$A\ (8, 0)$	$B\ (0, 6)$
CD	$7x + 12y = 64$	$C\left(\frac{64}{7}, 0\right)$	$D\left(0, \frac{16}{3}\right)$



(1)

The feasible region is shaded in the figure.

The vertices of the feasible region are C $\left(\frac{64}{7}, 0\right)$, P and B$(0, 6)$.

P is the point of intersection of the lines $6x + 8y = 48$ and $7x + 12y = 64$

Solving these eqautions we get

$\therefore \quad x = 4, \quad y = 3$

$\therefore \quad P = (4, 3)$

The values of the objective function $z = 6x + 10y$ at these vertices are

$$z(\text{C}) = 6\left(\frac{64}{7}\right) + 10(0) = \frac{384}{7} = 54.85$$

$$z(\text{P}) = 6(4) + 10(3) = 24 + 30 = 54$$

$$z(\text{B}) = 6(0) + 10(6) = 60$$

$\therefore$ the minimum value of z is 54 at the point (4, 3).

Hence, 4 units of food F_1 and 3 units of food F_2 should be included in the diet of the sick person to meet the minimal nutritional requirements, in order to have the minimum cost of Rs. 54. (1)

SECTION - II

Q.4 (A)

(i) (a) 0.8 (2)

$$E(x)=\sum x\,p(x)$$

$$=(-2)(0.1)+(-1)(0.1)+(0)(0.2)+(1)(0.2)+(2)(0.3)+(3)(0.1)$$

$$=0.8$$

(ii) (c) 2 (2)

$$I=\int_0^{\alpha}3x^2dx=8$$

$$8=\left[\frac{3x^3}{3}\right]_0^{\alpha}$$

$$\therefore \alpha^3=8$$

$$\therefore \alpha=2$$

(iii) (a) $x^4\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}=y$ (2)

$$y=\frac{c}{x}+c^2------(1)$$

diff.w.r.t.x

$$\frac{dy}{dx}=\frac{-c}{x^2}+0$$

$$c=-x^2\frac{dy}{dx}------(2)$$

Putting in equation (1)

$$y=\frac{-x^2\frac{dy}{dx}}{x}+\left(-x^2\frac{dy}{dx}\right)^2$$

$$y=-x\frac{dy}{dx}+x^4\left(\frac{dy}{dx}\right)^2$$

$$x^4\left(\frac{dy}{dx}\right)^2-x\frac{dy}{dx}=y$$

Q.4 (B)

(i) $I=\int e^{x}\left[\dfrac{\sqrt{1-x^{2}}\sin^{-1}x+1}{\sqrt{1-x^{2}}}\right]dx$

$=\int e^{x}\left[\dfrac{\sqrt{1-x^{2}}\sin^{-1}x}{\sqrt{1-x^{2}}}+\dfrac{1}{\sqrt{1-x^{2}}}\right]dx$

$=\int e^{x}\left[\sin^{-1}x+\dfrac{1}{\sqrt{1-x^{2}}}\right]dx$ (1)

$=\int e^{x}\left[\sin^{-1}x+\dfrac{d}{dx}\left(\sin^{-1}x\right)\right]dx$

$I=e^{x}\sin^{-1}x+c$ (1)

(ii) $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+......\infty}}}$

$y^{2}=\sin x+y$ (1)

$y^{2}-y=\sin x$

Diff. w.r.t. x

$2y\dfrac{dy}{dx}-\dfrac{dy}{dx}=\cos x$

$(2y-1)\dfrac{dy}{dx}=\cos x$

$\dfrac{dy}{dx}=\dfrac{\cos y}{2y-1}$ (1)

(iii) $I=\int_{0}^{\pi/2}\dfrac{1}{1+\cos x}dx$

$I=\int_{0}^{\pi/2}\dfrac{1}{2\cos^{2}x/2}dx$ (1)

$=\dfrac{1}{2}\int_{0}^{\pi/2}\sec^{2}x/2\,dx$

$=\dfrac{1}{2}\left[2\tan\dfrac{x}{2}\right]_{0}^{\pi/2}$

$=\tan\left(\dfrac{\pi}{4}\right)-\tan 0$

$=1$ (1)

(iv) $y = e^{ax}$

Taking log both side

$\log y = a\,x \log e = a\,x$

$$\frac{\log y}{x} = a \quad (1)$$

Diff. w.r.t. x

$$\frac{x\frac{1}{y}.\frac{dy}{dx} - \log y(1)}{x^2} = 0$$

$$x\frac{dy}{dx} = y\log y \quad (1)$$

(v) Let x = Number of heads

$n = 5$

$p = 1/2$

$q = 1/2$ (1)

$$P(x = 3 \text{ heads}) = {}^nC_x\, p^x q^{n-x}$$

$$= {}^5C_3 (1/2)^3 (1/2)^2$$

$$= \frac{5\times 4}{2\times 1}\times\frac{1}{2^3}\times\frac{1}{2^2}$$

$$= \frac{10}{2^5} = \frac{10}{32} = \frac{5}{16} = 0.3125 \quad (1)$$

Q.5 (A)

(i) $I = \int \sec^3 x\,dx$

$$= \int \sec x.\sec^2 x\,dx$$

$$= \sec x.\int \sec^2 x\,dx - \int\left[\frac{d}{dx}(\sec x).\int \sec^2 x\,dx\right]dx \quad (1)$$

$$= \sec x.\tan x - \int \sec x\cdot\tan x\cdot\tan x\,dx$$

$$= \sec x.\tan x - \int \sec x\left(\sec^2 x - 1\right)dx$$

$$= \sec x.\tan x - \int\left[\sec^3 x - \sec x\right]dx$$

$$I = \sec x.\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$= \sec x.\tan x - I + \log\left|\sec x + \tan x\right| + c \quad (1)$$

$$2I = \sec x.\tan x + \log\left|\sec x + \tan x\right| + c$$

$$\therefore I = \frac{1}{2}(\sec x.\tan x + \log\left|\sec x + \tan x\right|) + c \quad (1)$$

(ii) $y=\left(\tan^{-1}x\right)^2$

$\therefore \frac{dy}{dx}=2\left(\tan^{-1}x\right)\frac{d}{dx}\left(\tan^{-1}x\right) \quad =\left(2\tan^{-1}x\right)\frac{1}{1+x^2}$

$\therefore \left(1+x^2\right)\frac{dy}{dx}=2\tan^{-1}x$ (1)

Again differenitating w.r.t. x , we get

$\left(1+x^2\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(0+2x\right)=2+\frac{1}{1+x^2}$ (1)

$\therefore \left(1+x^2\right)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=2\frac{2}{1+x^2}$

$\therefore \left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}=2$ (1)

(iii) f is continous at $x=0$

$\therefore \lim_{x\to 0} f\left(x\right)=f\left(0\right)$ (1)

$R.H.S=f\left(0\right)=k$ ....$\left(given\right)$

$L.H.S=\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{1/x} \quad =\lim_{x\to 0}\left[\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right]^{1/x}$

$=\lim_{x\to 0}\left[\frac{1+\tan x}{1-\tan x}\right]^{1/x} \quad =\lim_{x\to 0}\left[\frac{\left(1+\tan x\right)^{1/\tan x}}{\left(1-\tan x\right)^{1/\tan x}}\right]^{\tan x/x}$ (1)

$=\frac{e^1}{e^{-1}}=e^2$

$from\left(1\right), k=e^2$ (1)

Q.5 (B)

(i) Equation of the curve : $y=x-\frac{4}{x}$............$\left(1\right)$

$\therefore \frac{dy}{dx}=1+\frac{4}{x^2}$ (1)

$\therefore$ Slope of the tangent at (x , y) $=1+\frac{4}{x^2}$

Equation of the line is y = 2x
Its slope is 2 (1)

$\therefore 1+\frac{4}{x^2}=2 \therefore \frac{4}{x^2}=1$

$\therefore x^2=4$

$\therefore x=\pm 2$ (1)

From (1), if $x = 2$, then $y = 2 - \frac{4}{2} = 0$

and if $x = -2$, then $y = -2 - \frac{4}{-2} = 0$

$\Rightarrow \therefore$ The required points are (2,0) and (–2,0) (1)

(ii) Let $I = \int \sqrt{x^2 - a^2}\, dx$

$$= \int \sqrt{x^2 - a^2} .1 dx$$

$$= \sqrt{x^2 - a^2} . \int 1 dx - \int \left[\frac{d}{dx} (\sqrt{x^2 - a^2}) . \int 1 dx \right] dx \quad (1)$$

$$= \sqrt{x^2 - a^2} . x - \int \frac{1}{2\sqrt{x^2 - a^2}} (2x - 0) . x\, dx \quad (1)$$

$$= \sqrt{x^2 - a^2} . x - \int \frac{x}{\sqrt{x^2 - a^2}} . x dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}} \quad (1)$$

$$= x\sqrt{x^2 - a^2} - I - a^2 \log \left| x + \sqrt{x^2 - a^2} \right| + c_1$$

$$\therefore 2I = x\sqrt{x^2 - a^2} - a^2 \log \left| x + \sqrt{x^2 - a^2} \right| + c_1$$

$$\therefore I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + \frac{c_1}{2}$$

$$\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + c, \text{ where } c = \frac{c_1}{2}. \quad (1)$$

(iii) Let $I=\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

$=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$ $\left[\text{By using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$ (1)

$=\int_0^{\pi}\frac{\pi\sin x-x\sin x}{1+\sin x}dx$

$=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx-\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

$2I=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}dx$ (1)

$=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}\times\frac{(1-\sin x)}{(1-\sin x)}dx$

$=\pi\int_0^{\pi}\frac{\sin x-\sin^2 x}{\cos^2 x}dx$

$=\pi\int_0^{\pi}\frac{\sin x}{\cos^2 x}-\frac{\sin^2 x}{\cos^2 x}dx$

$=\pi\int_0^{\pi}\left(\sec x\times\tan x-\tan^2 x\right)dx$

$=\pi\int_0^{\pi}\left(\sec x\times\tan x-\sec^2 x+1\right)dx$

$=\pi\left[\sec x-\tan x+x\right]_0^{\pi}$ (1)

$=\pi\left[(-1+\pi)-(1)\right]$

$=\pi(-2+\pi)$

$=\pi^2-2\pi$

$I=\frac{\pi^2}{2}-\pi$

$I=\pi\left(\frac{\pi}{2}-1\right)$ (1)

Q.6 (A)

(i) f is continous on $[-\pi,\pi]$

$\therefore$ f is continous at every point of $[-\pi,\pi]$

$(a)-\frac{\pi}{2}\in[-\pi,\pi]$ $\therefore$ f is continous at x = $-\frac{\pi}{2}$ (1)

$\therefore \lim_{x\to-\frac{\pi}{2}^-} f(x)=\lim_{x\to-\frac{\pi}{2}^+} f(x)=f\left(-\frac{\pi}{2}\right)$

$$\therefore \lim_{x\to -\frac{\pi}{2}} (-2\sin x) = \lim_{x\to -\frac{\pi}{2}} (\text{a+sinx+}\beta)$$

$$\therefore -2\sin\left(-\frac{\pi}{2}\right) = a\sin\left(-\frac{\pi}{2}\right) + b$$

$$\therefore (-2)(-1) = \text{a}(-1) + \text{b}$$

$$\therefore -\text{a+b} = 2$$

(b) $\frac{\pi}{2} \in [-\pi, \pi]$ $\therefore$ f is continous at x = $\frac{\pi}{2}$

$$\therefore \lim_{x\to \frac{\pi^-}{2}} f(x) = \lim_{x\to \frac{\pi^+}{2}} f(x) = f\left(\frac{\pi}{2}\right) \qquad (1)$$

$$\therefore \lim_{x\to \frac{\pi}{2}} (\text{a}\sin x + \text{b}) = \lim_{x\to \frac{\pi}{2}} (\cos x)$$

$$\therefore \text{a}\sin\frac{\pi}{2} + \text{b} = \cos\frac{\pi}{2}$$

$$\therefore \text{a+b} = 0 --------- (2)$$

solving (1) and (2), a= −1, b=1 (1)

(ii) Given:

$$\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 2$$

$$\therefore \frac{x^3 - y^3}{x^3 + y^3} = 10^2 = 100 \qquad (1)$$

$$\therefore x^3 - y^3 = 100x^3 + 100y^3$$

$$\therefore -99x^3 - 101y^3 = 0$$

Differentiating w.r.t. x, we get

$$-99(3x^2) - 101\left(3y^2\frac{dy}{dx}\right) = 0 \qquad (1)$$

$$\therefore 99x^2 + 101y^2\frac{dy}{dx} = 0$$

$$\therefore 101y^2\frac{dy}{dx} = -99x^2$$

$$\Rightarrow \therefore \frac{dy}{dx} = \frac{-99x^2}{101y^2} \qquad (1)$$

(iii) $p(x)=\binom{4}{x}\left(\frac{5}{9}\right)^{x}\left(\frac{4}{9}\right)^{4-x}$, $x=0,1,2,....4$

Comparing with $p(x)=\binom{n}{x}(p)^{x}q^{n-x}$

$\therefore n=4,\ p=\frac{5}{9},q=\frac{4}{9}$ (1)

$E(x)=np=4\times\frac{5}{9}=\frac{20}{9}$ (1)

$V(x)=npq=4\times\frac{5}{9}\times\frac{4}{9}=\frac{80}{81}$ (1)

[Note: If the student attempts this question or even writes the question number they will get full credit]

Q.6 (B)

(i) $f(x)=2x^{3}-21x^{2}+36x-20$

Diff. w.r.t.x

$f'(x)=6x^{2}-42x+36$ (1)

for maxima or minima $f'(x)=0$

$\therefore 6x^{2}-42x+36=0$

$6(x-1)(x-6)=0$

$\therefore x=1 or\ x=6$ (1)

$Now\ f''(x)=12x-42$

$f''(1)=12(1)-42=-30<0$

Hence f has a maximum at x = 1,

$\therefore$ Maximum value of $f(x)$ = $f(1)=2(1^{3})-21(1^{2})+36(1)-20$ $=-3$ (1)

Again $f''(6)=12(6)-42=30>0$

$\therefore f(x)$ has minimum at x=6

$\therefore$ Minimum value at x = 6 = $f(6)=2(6^{3})-21(6^{2})+36(6)-20$ $=-128$ (1)

(ii) The given equation can be written as

$(x^{2}+y^{2})\,dx=2xydy$

$i.e.\ \frac{dy}{dx}=\frac{x^{2}+y^{2}}{2xy}$

Above equation is homogeneous differential equation.

To solve it, we substitute $y=vx$ (1)

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ Equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x \cdot vx} = \frac{x^2\left(1+v^2\right)}{2x^2v}$$

$$= \frac{1+v^2}{2v}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}$$

$$\therefore \quad \frac{2v}{1-v^2}\,dv = \frac{dx}{x}$$ (1)

Which is in variables separatable form

$\therefore$ Integrating both sides, we get

$$\int \frac{2v}{1-v^2}\,dv = \int \frac{dx}{x} + c_1$$

$$\therefore -\log\left|1-v^2\right| = \log|x| + \log c$$ (1)

$$\therefore \quad \log\left|x\left(1-v^2\right)\right| = \log|c|$$

$$\therefore \quad x\left(1-v^2\right) = c$$

Resubstitution $v = \frac{y}{x}$, we get

$$x\left(1 - \frac{y^2}{x^2}\right) = c$$

$$\therefore \quad x\left(\frac{x^2 - y^2}{x^2}\right) = c$$

i.e. $\quad x^2 - y^2 = cx$, which is the requried general solution. (1)

(iii) c.d.f of X is given by

$$F(x) = \int_{-1}^{x} f(y)\,dy$$

$$= \int_{-1}^{x} \frac{y^2}{3}\,dy = \left[\frac{y^3}{9}\right]_{-1}^{x}$$

$$= \frac{x^3}{9} + \frac{1}{9}$$ (1)

Thus, $F(x) = \frac{x^3}{9} + \frac{1}{9}, \quad \forall x \in R$

Consider $P(X < 1) = F(1) = \frac{(1)^3}{9} + \frac{1}{9} = \frac{2}{9}$ (1)

$P(X > 0) = 0$ $\quad [\text{Range of } X \text{ is } (-1, 2)]$

$P(X > 0) = 1 - P(x \le 0)$

$= 1 - F(0)$

$= 1 - \left(\frac{0}{9} + \frac{1}{9}\right)$

$= \frac{8}{9}$ (1)

$P(1 < X < 2) = F(2) - F(1)$

$= \left[\frac{8}{9} + \frac{1}{9}\right] - \left[\frac{1}{9} + \frac{1}{9}\right]$

$= \frac{7}{9}$ (1)