Rao IIT Academy
Symbol of Excellence and Perfection

XII - CBSE BOARD
CODE (65/2/MT) SET - 2
Date: 18.03.2015 **MATHEMATICS - SOLUTIONS**

1. Let $\vec{a} = 2\hat{i} + 3\hat{i} - \hat{k}$

$\vec{b} = \hat{i} + \hat{j}$

and $\vec{a}\cdot\vec{b} = 2 + 3 + 0 = 5, \left|\vec{b}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|}$

$$= \frac{\left(2\hat{i} + 3\hat{j} - \hat{k}\right)\cdot\left(\hat{i} + \hat{j}\right)}{\sqrt{2}}$$

$$= \frac{2+3}{\sqrt{2}}$$

$$= \frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{5\sqrt{2}}{2}$$

2. $\hat{i}\cdot\left(\hat{j}\times\hat{k}\right) + \hat{j}\,.\left(\hat{k}\times\hat{i}\right)\cdot\left(\hat{i}\times j\right)$

Now $\hat{j}\times\hat{k} = i, \hat{k}\times\hat{i} = \hat{j}, \hat{i}\times\hat{j} = \hat{k}$

$\therefore\ \hat{i}\cdot\left(\hat{i}\right) + \hat{j}\cdot\left(\hat{j}\right) + \hat{k}\cdot\left(\hat{k}\right)$

$= 1 + 1 + 1$

$= 3$

3. $3x + 4y + 12z = 52$

$$\frac{3x}{\sqrt{3^2+4^2+12^2}} + \frac{4y}{\sqrt{3^2+4^2+12^2}} + \frac{12z}{\sqrt{3^2+4^2+12^2}} = \frac{52}{\sqrt{3^2+4^2+12^2}}$$

$$\frac{3}{13}x + \frac{4}{13}y + \frac{12}{13}z = 4$$

$\therefore$ direction cosines are $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$

4. $$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

5. $$x\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 + y^2 = 0$$

order $= 2$
order $= 2$
$\therefore$ order $\times$ degree $= 2 \times 2 = 4$

6. $y = A\cos\alpha x + B\sin\alpha x$

$$\frac{dy}{dx} = -A\alpha\sin\alpha x + B\alpha\cos\alpha x$$

$$\frac{d^2y}{dx^2} = -A\alpha^2\cos\alpha x - B\alpha^2\sin\alpha x$$

$$\frac{d^2y}{dx^2} = -\alpha^2\left[A\cos\alpha x + B\sin\alpha x\right]$$

$$\frac{d^2y}{dx^2} = -\alpha^2 y$$

$$\boxed{\frac{d^2y}{dx^2} + \alpha^2 y = 0}$$

7. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$$

$$\begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$\therefore\ a + 4b = -7,\ 2a + 5b = -8,\ 3a + 6b = -9$

or $a + 4b = -7$ ....(1) $2a + 5b = -8$ ....(2) $a + 2b = -3$ ....(3)

and $c + 4d = 2$, $2c + 5d = 4,\ 3c + 6d = 6$

$c + 4d = 2$ ....(4) $2c + 5d = 4$ ....(5) $c + 2d = 2$ ....(6)

Solving equation (1) and (2)

$b = -2 \quad a = 1$

Solving equation (4) and (6)

$d = 0 \qquad c = 2$

$$\therefore\ x = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$$

OR

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$|A| = 3[12-10] + 1[-30+25] + 1[30-30]$$

$$= 3[2] + [-5] + 1[0]$$

$$= 6 - 5$$

$$= 1$$

Minors

$M_{11} = 12 - 10 = 2,\quad M_{21} = -2 + 2 = 0,\quad M_{31} = 5 - 6 = -1,$

$M_{12} = -30 + 25 = -5,\quad M_{22} = 6 - 5 = 1,\quad M_{32} = -15 + 15 = 0,$

$M_{13} = 30 - 30 = 0,\quad M_{23} = -6 + 5 = -1,\quad M_{33} = 18 - 15 = 3,$

Cofactor

$C_{11} = 2,\ C_{21} = 0,\ C_{31} = -1,$

$C_{12} = 5,\ C_{22} = 1,\ C_{32} = 0,$

$C_{13} = 0,\ C_{23} = 1,\ C_{33} = 3,$

$$\text{Cofactor matrix of } A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

$$\text{Adjoint } A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\ A}{|A|} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Now, $A^{-1}\ A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\therefore\ A^{-1}\ A = I$

Hence proved

8. $f(x) = |x-3| + |x-4|$

$f(x) = -(x-3) - (x-4) \qquad x < 3$

$= (x-3) - (x-4) \qquad 3 \le x < 4$

$= (x-3) + (x-4) \qquad x \ge 4$

$\therefore\ f(x) = -2x + 7\ ,\ x < -3$

$= 1 \qquad ,\ 3 \le x < 4$

$= 2x - 7\ ,\ x \ge 4$

Differentiability of $x = 3$

L.H.D.

$$\lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \to 3^-} \frac{(-2x + 7) - 1}{x - 3}$$

$$= \lim_{x \to 3^-} \frac{-2x + 6}{x - 3}$$

$$= \lim_{x \to 3^-} \frac{-2(x-3)}{(x - 3)} = -2$$

RHD

$$\lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{1 - 1}{x - 3} = 0$$

$L.H.D \neq R.H.D$

$\therefore\ f(x)$ is not differentiable at $x = 3$

Differentiability at $x = 4$

L.H.D.

$$\lim_{x\to 4^-} \frac{f(x) - f(4)}{x - 4}$$

$$= \lim_{x\to 4^-} \frac{1 - 1}{x - 4} = 0$$

R.H.D.

$$\lim_{x\to 4^+} \frac{f(x) - f(4)}{x - 4}$$

$$= \lim_{x\to 4^+} \frac{2x - 7 - 1}{x - 4}$$

$$= \lim_{x\to 4^+} \frac{2(x - 4)}{(x - 4)}$$

$= 2$

$\therefore$ L.H.D $\neq$ R.H.D

$\therefore$ function is not differentiable at $x = 4$

9. $y = x^{e^{-x^2}}$

Taking logarithms on both sides

$$\log y = e^{-x^2} \log x$$

ifferentiating both sides w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{e^{-x^2}}{x} + \log x\, e^{-x^2} \cdot (-2x)$$

$$\frac{dy}{dx} = \frac{y}{x}\left[e^{-x^2} - 2x^2 \cdot \log x \cdot e^{-x^2}\right]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[e^{-x^2} - 2x^2 \log x .\, e^{-x^2}\right]$$

$$\frac{dy}{dx} = x^{e^{-x^2}} \frac{\left(e^{-x^2}\right)}{x}\left[1 - 2x^2 \log x\right]$$

OR

$$\log \sqrt{x^2 + y^2} = \tan^{-1} \frac{x}{y}$$

$$\frac{1}{2} \log \left(x^2 + y^2\right) = \tan^{-1}\left(\frac{x}{y}\right)$$

differentiating both sides w.r.t. x

$$\frac{1}{2\left(x^2 + y^2\right)} \times \left(2x + 2y \frac{dy}{dx}\right) = \frac{1}{1 + \frac{x^2}{y^2}} \left[\frac{y - x \frac{dy}{dx}}{y^2}\right]$$

$$\frac{x + y \frac{dy}{dx}}{x^2 + y^2} = \frac{y^2}{x^2 + y^2} \left[\frac{y - x \frac{dy}{dx}}{y^2}\right]$$

$$x + y \frac{dy}{dx} = y - x \frac{dy}{dx}$$

$$y \frac{dy}{dx} + x \frac{dy}{dx} = y - x$$

$$\frac{dy}{dx}(x + y) = y - x$$

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

10. If $y = \sqrt{x+1} - \sqrt{x-1}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x-1}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{\sqrt{x-1} - \sqrt{x+1}}{\sqrt{x^2 - 1}}\right]$$

$$\frac{dy}{dx} = \frac{-1}{2} \frac{\left[\sqrt{x+1} - \sqrt{x-1}\right]}{\sqrt{x^2 - 1}}$$

$$2\sqrt{x^2 - 1} \frac{dy}{dx} = -y \qquad .....(1)$$

$$\left(x^2 - 1\right)\left(\frac{dy}{dx}\right)^2 = \frac{y^2}{4}$$

Diff. w.r.t. x

$$\left(x^2-1\right)\cdot 2\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(2x)=\frac{2}{4}\ y\ \frac{dy}{dx}$$

Dividing by $2\frac{dy}{dx}$

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-\frac{y}{4}=0$$ Hence proved.

11. $$\int\frac{1-\cos x}{\cos x\left(1+\cos x\right)}\,dx$$

$$=\int\frac{1+\cos x-2\cos x}{\cos x\left(1+\cos x\right)}\,dx$$

$$=\int\frac{\left(1+\cos x\right)}{\cos x\left(1+\cos x\right)}\,dx-\int\frac{2\cos x}{\cos x\left(1+\cos x\right)}\,dx$$

$$=\int\frac{dx}{\cos x}-\int\frac{2}{1+\cos x}\,dx$$

$$=\int\sec x-2\int\frac{1}{2\cos^2\frac{x}{2}}\,dx$$

$$=\int\sec x-\int\sec^2\frac{x}{2}\,dx$$

$$=\log\left(\sec x+\tan x\right)-\frac{\tan\frac{x}{2}}{\frac{1}{2}}+c$$

$$=\log\left(\sec x+\tan x\right)-2\tan\frac{x}{2}+c$$

12. According to the given condition

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 3 \\ 4 & 2 & 6 \end{bmatrix}_{3\times 3} \begin{bmatrix} 200 \\ 150 \\ 200 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} 400+450+200 \\ 400+150+600 \\ 800+300+1200 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} 1050 \\ 1150 \\ 2300 \end{bmatrix}$$

Daily expense of family A = Rs. 1050
Daily expense of family B = Rs. 1150
Daily expense of family C = Rs. 2300
More children in family hamper the growth of the society.

13. $\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) = \dfrac{\pi}{2} - \tan^{-1} z$

$$\frac{x+y}{1-xy} = \tan\left(\frac{\pi}{2} - \tan^{-1} z\right)$$

$$\frac{x+y}{1-xy} = \cot\left(\tan^{-1} z\right)$$

$$\frac{x+y}{1-xy} = \cot\left(\cot^{-1}\frac{1}{z}\right)$$

$$\frac{x+y}{1-xy} = \frac{1}{z}$$

$$xz + yz = 1 - xy$$

$$xy + yz + zx = 1$$

14. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$C_1 \rightarrow C_1 + C_2 + C_3$

$$\begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = 0$$

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 - R_2 \;\&\; R_2 \rightarrow R_2 - R_3$

$$(a+b+c)\begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 1 & a & b \end{vmatrix} = 0$$

$$(a+b+c)\left[(b-c)(a-b)-(c-a)^2\right] = 0$$

$$(a+b+c)\left[ab - b^2 - ca + bc - c^2 - a^2 + 2\,ca\right] = 0$$

$$\frac{-1}{2}(a+b+c)\left[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca\right] = 0$$

$$\frac{-1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] = 0$$

$$(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] = 0$$

So, $\therefore a+b+c=0$ or $(a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

Here $(a-b)^2 + (b-c)^2 + (c-a)^2 \neq 0$, $a \neq b \neq c$

$\therefore \boxed{a+b+c=0}$

15. $\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0}=\lambda$ ....(1)

$x=3\lambda+1,\ y=-\lambda+1,\ z=-1$

Point $(3\lambda+1,-\lambda+1,-1)$

& $\frac{x-4}{2}=\frac{y}{0}=\frac{z+1}{3}=\mu$ ....(2)

$x=2\mu+4,\ \ y=0,\ z=3\mu-1$

If line (1) and line (2) are intersecting then

$3\lambda+1=2\mu+4$ ....(3) $-\lambda+1=0$ ......(4) $-1=3\mu-1$ ......(5)

from equation (4) $\lambda=1$, from equation (5) $\mu=0$

In equation (3)

L.H.S. = $3\lambda+1=4$

R.H.S. = $2\mu+4=4$

$\therefore$ L.H.S. = R.H.S.

$\therefore \lambda$ & μ exist

Thus line are intersecting

Point of Intersection = $(3\lambda+1,-\lambda+1,-1)$

$= (4, 0, -1)$

16. $r=(\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$

Let $\overline{r}=x\hat{i}+y+z\hat{k}$

$\therefore\ x\hat{i}+y\hat{j}+z\hat{k}=\hat{i}-\hat{j}+2\hat{k}+\mu(-3\hat{i}+\hat{j}+5\hat{k})$

$(x-1)\hat{i}+(y+1)\hat{j}+(z-2)\hat{k}=-3\mu\hat{i}+\mu\hat{j}+5\mu\hat{k}$

$x-1=-3\mu,\ y+1=\mu,\ z-2=5\mu$

$\frac{x-1}{-3}=\frac{y+1}{1}=\frac{z-2}{5}=\mu$

$x=-3\mu+1,\ \ y=\mu-1,\ \ z=5\mu+2$

The point represents Q on line is $-3\mu-2,\ \mu-1,\ 5\mu+2$

$P=(3, 2, 6)$

$\therefore$ direction ratios of $PQ=(-3\mu+1,\ \mu-3,\ 5\mu-4)$

Since the above line is parallel to the plane $x-4y+3z=1$

$\therefore$ drs of the normal to this plane are 1, –4, 3

$\therefore$ The normal of the plane is perpendicular to the given PQ

$\therefore \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$1(-3\mu - 2) + (-4)(\mu - 3) + 3(5\mu - 4) = 0$

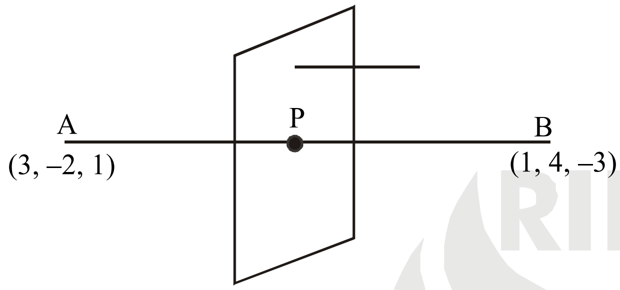
$-3\mu - 2 - 4\mu + 12 + 15\mu - 12 = 0$

$8\mu - 2 = 0$

$\mu = \frac{2}{8}$

$\boxed{\mu = \frac{1}{4}}$

OR



Point P is $\left(\frac{3+1}{2}, \frac{-2+4}{2}, \frac{1-3}{2}\right)$

$= [2, 1, -1]$

The normal of the plane is parallal to the line AB

$\therefore$ its direction ratio will be

$a = 1 - 3 = -2$
$b = 4 + 2 = 6$
$c = -3 - 1 = -4$

$\therefore$ equation of plane in cartesian form is

$-2(x-2) + 6(y-1) - 4(z+1) = 0$

$-2x + 4 + 6y - 6 - 4z - 4 = 0$

$-2x + 6y - 4z - 6 = 0$

$x - 3y + 2z + 3 = 0$

equation of plane in vector form is

$\vec{r} \cdot (\hat{i} - 3\hat{j} + 2\hat{k}) + 3 = 0$

17. $S = \{1, 2, 3, \ldots\ldots\ldots\ldots\ldots 100\}$

$n(S) = 100$

Let A : Event that the number on the card is divisible by 6.
B : Event that the number on the card is divisible by 8.
C : Event that the number on the card is divisible by 24.

$A = \{6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, 78, 84, 90, 96\}$

$n(A) = 16$

$B = \{8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96\}$

$n(B) = 12$

$C = \{24, 48, 72, 96\}$

$n(C) = 4$

$A \cup B \cap C' = \{6, 8, 12, 16, 18, 30, 32, 36, 40, 42, 54, 56, 60, 64, 66, 78, 80, 84, 88, 90\}$

$n(A \cup B \cap C') = 20$

$P(6 \text{ or } 8 \text{ but not } 24) = \dfrac{20}{100} = \dfrac{1}{5}$

18. Let $I = \int x \sin^{-1} x \, dx$

$= \int (\sin^{-1} x)(x) dx$

$= \sin^{-1} x \int x \, dx - \int \left[\dfrac{d}{dx} \sin^{-1} x \int x \, dx\right] dx$

$= (\sin^{-1} x) \dfrac{x^2}{2} - \int \dfrac{1}{\sqrt{1-x^2}} \times \dfrac{x^2}{2} dx$

$= \dfrac{x^2 \sin^{-1} x}{2} + \dfrac{1}{2} \int \dfrac{-x^2}{\sqrt{1-x^2}} dx$

$= \dfrac{x^2 \sin^{-1} x}{2} + \dfrac{1}{2} \int \dfrac{1-x^2-1}{\sqrt{1-x^2}} dx$

$= \dfrac{x^2 \sin^{-1} x}{2} + \dfrac{1}{2} \left[\int \left(\dfrac{1-x^2}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}}\right) dx\right]$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left[\left\{ \frac{1}{2} x\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right\} - \sin^{-1} x \right] + c$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left[\frac{1}{2} x\sqrt{1-x^2} - \frac{1}{2} \sin^{-1} x \right] + c$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} x\sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c$$

19. $\int_0^2 \left(x^2 + e^{2x+1} \right) dx$

We know that : as the limit of a sum

$$\int_0^a f(x)\, dx = \lim_{h\to 0} h\left[f(a) + f(a+2h) + f(a+3h) + \ldots\ldots\ldots + f(a+(n-1)h) \right] \quad \ldots(1)$$

We know that $h = \frac{b-a}{n}, nh = b - a$

Here $f(x) = x^2 + e^{2n+1}$

$\therefore \quad f(a) = a^2 + e^{2a+1} = a^2 + e^{2a}.e$

$f(a+h) = (a+h)^2 + e^{2(a+h)+1} = a^2 + h^2 + 2ah + e^{2a}.e^{2h}.e^1$

$f(a+2h) = (a+2h)^2 + e^{2(a+2h)+1} = a^2 + 4h^2 + 4ah + e^{2a}.e^{4h}.e$

$f(a+(n-1)^h) = (a+(n-1)h)^2 + e^{2(a+(n-1)h)+1}$

$f(a+(n-1)h) = a^2 + (n-1)^2 h^2 + 2a(n-1)h + e^{2a}.e^{2(n-1)h}.e^1$

put all the values of $f(a),\ f(a+h),\ f(a+2h)\ \ldots\ldots f(a+(n-1)h)$ in equation (1)

$$\int_0^a f(x)dx = \lim_{h\to 0} h\left[\left(a^2 + e^{2a}.e\right) + \left(a^2 + h^2 + 2ah + e^{2a}.e.e^{2h}\right) + \right.$$

$$\left. \left(a^2 + 4h^2 + 4ah + e^{2a}.e^{4h}.e\right) + \ldots\ldots\ldots \left(a^2 + (n-1)^2 h^2 + 2a(n-1)h + e^{2q}.e\,e^{2(n-1)h}\right) \right]$$

$$\int_0^a f(x)dx = \lim_{h\to 0} h\left[na^2 + h^2\left(1^2 + 2^2 + 3^2 + \ldots\ldots(n-1)^2\right) + 2ah\left(1 + 2 + 3 + \ldots\ldots\ldots + \overline{n-1}\right) \right.$$

$$\left. + e^{2a}.e\left(1 + e^{2h} + e^{4h} + \ldots\ldots e^{2(n-1)h}\right) \right]$$

$$\int_0^a f(x)dx = \lim_{h\to 0} h\left[na^2 + h^2\,\frac{n(n-1)\,(2n-1)}{6} + 2ah\,\frac{n(n-1)}{2} + e^{2a}.e\,\frac{1\left((e^{2h})^n - 1\right)}{e^{2h} - 1}\right]$$

$$\int_0^a f(x)dx = \lim_{h\to 0} h\left[na^2 + n(nh-h)\,(2nh-h) + 2ah\,n.(nh-h) + e^{2q}.e\,\frac{(e^{2nh}-1)}{e^{2h}-1}\right]$$

$$\int_0^a\left(x^2 + e^{2x+1}\right)dx = \lim_{h\to 0}\left[nha^2 + \frac{nh(nh-h)\,(2nh-h)}{6} + \frac{2a(nh)(nh-h)}{2} + \frac{e^{2a}.e}{2}\,\frac{2h(e^{2nh}-1)}{e^{2h}-1}\right]$$

We know that $nh = b - a$, and $h \to 0$, $n \to \infty$

$$\int_0^a\left(x^2 + e^{2x+1}\right)dx = \lim_{h\to 0}\left[(b-a)\,a^2 + \frac{(b-a)\,(b-a)\,(2(b-a))}{6} + \right.$$

$$\left.\frac{2a(b-a)(b-a)}{2} + \frac{e^{2a}.e}{2}(e^{2(b-a)} - 1)\right]$$

Here $a = 0$, $b = 2$.

$$\int_0^2\left(x^2 + e^{2x+1}\right) = \lim_{h\to 0}\left[2\times 0 + \frac{(2-0)\,(2-0).2(2-0)}{6} + \frac{2\times 0(2-0)(2-0)}{2} + \frac{e^{2\times 0}.e}{2}\left(e^4 - 1\right)\right]$$

$$\Rightarrow \int_0^2\left(x^2 + e^{2x+1}\right) = \left[\frac{16}{6} + \frac{e}{2}(e^4 - 1)\right] = \frac{8}{3} + \frac{e^5}{2} - \frac{e}{2}$$

$$\Rightarrow \int_0^2\left(x^2 + e^{2x+1}\right) = \frac{8}{3} + \frac{e^5}{2} - \frac{e}{2}$$

OR

$$I = \int_0^\pi \frac{x\tan x}{\sec x\cdot\operatorname{cosec} x}\,dx$$

$$= \int_0^\pi \frac{x\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}\times\dfrac{1}{\sin x}}\,dx$$

$$I = \int_0^\pi x\sin^2 x\,dx \qquad ....(1)$$

$$= \int_0^\pi (\pi - x)\sin^2(\pi - x)\,dx$$

$$I = \int_0^\pi (\pi - x)\sin^2 dx \qquad ....(2)$$

Adding equation (1) and (2)

$$2I = \int_0^{\pi} \pi \sin^2 x \, dx = 2\pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \quad [\because f(\pi - x) = f(x)]$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{1 - \cos^2 x}{2} \, dx$$

$$= \pi \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left[\left\{ \frac{\pi}{2} - \frac{1}{2} \sin \pi \right\} - \{0 - 0\} \right]$$

$$= \pi \left\{ \frac{\pi}{2} - 0 \right\}$$

$$2I = \frac{\pi^2}{2}$$

$$I = \frac{\pi^2}{4}$$

20. $y = \dfrac{x}{1 + x^2}$

$$\text{Slope} = \frac{dy}{dx}$$

$$= \frac{(1 + x^2).1 - x(2x)}{(1 + x^2)^2}$$

$$m = \frac{1 - x^2}{(1 + x^2)^2}$$

and $$\frac{dm}{dx} = \frac{(1 + x^2) . (-2x) - (1 - x^2) . 2(1 + x^2) . 2x}{(1 + x^2)^4}$$

$$= \frac{-2x (1 + x^2) \left[+ (1 + x^2) + 2 (1 - x^2) \right]}{(1 + x^2)^4}$$

$$= \frac{-2x (1 + x^2) [3 - x^2]}{(1 + x^2)^4}$$

$$= \frac{-2x (3 - x^2)}{(1 + x^2)^3}$$

Now consider

$$\frac{dm}{dx} = 0$$

$$\frac{-2x(3-x^2)}{(1+x^2)^3} = 0$$

$$x(3-x^2) = 0$$

$$x = 0;\ x = \sqrt{3};\ x = -\sqrt{3}$$

$$\frac{d^2m}{dx^2} = \frac{(1+x^2)^3\,(6x^2-6) - (2x^3-6x)\,3(1+x^2)^2\,2x}{(1+x^2)^6}$$

$$= \frac{6(1+x^2)\,(x^2-1) - 12x^2\,(x^2-3)}{(1+x^2)^4}$$

$$\left.\frac{d^2m}{dx^2}\right]_{\text{at } x=0} = \frac{6\,(1)\,(-1)}{1} = -6 < 0$$

$$\left.\frac{d^2m}{dx^2}\right]_{\text{at } x=\sqrt{3}} = \frac{6\,(1+3)\,(3-1)}{(1+3)^4} = \frac{6\times4\times2}{4^4} = \frac{12}{64} = \frac{3}{16} > 0$$

$$\left.\frac{d^2m}{dx^2}\right]_{\text{at } x=-\sqrt{3}} = \frac{6\,(1+3)\,(3-1)}{(1+3)^4} = \frac{3}{16} > 0$$

Slope is maximum when $x = 0$

Now $y = \frac{x}{1+x^2} = 0$

$\therefore$ Point $= (0,0)$

21. $\frac{dy}{dx} = \frac{y^2}{xy-x^2}$ ....(1)

This is a homogeneous equation

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{x\cdot vx - x^2}$$

$$v + x\frac{dv}{dx} = \frac{v^2}{v-1}$$

$$x\frac{dv}{dx} = \frac{v^2}{v-1} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$$

$$x\frac{dv}{dx} = \frac{v}{v-1}$$

$$\frac{v-1}{v}dv = \frac{dx}{x}$$

$$\left(1-\frac{1}{v}\right)dv = \frac{dx}{x}$$

Integrating on both sides

$$\int\left(1-\frac{1}{v}\right)dv = \int\frac{dx}{x}$$

$$v - \log v = \log x + c_1$$

$$\frac{y}{x} - \log\left(\frac{y}{x}\right) = \log x + \log c_2$$

$$\frac{y}{x} = \log\frac{y}{x} + \log x + \log c_2$$

$$\frac{y}{x} = \log_e(c_2 y)$$

$$c_2 y = e^{\frac{y}{x}}$$

$$y = ce^{\frac{y}{x}}$$

This is the general solution of the given equation.

OR

$$\sin 2x\frac{dy}{dx} - y = \tan x$$

$$\frac{dy}{dx} - \frac{y}{\sin 2x} = \frac{\tan x}{\sin 2x}$$

$$\frac{dy}{dx} + (-\operatorname{cosec} 2x)y = \frac{\sin x}{\cos x} \times \frac{1}{2\sin x\cos x}$$

$$\frac{dy}{dx} + (-\operatorname{cosec} 2x)y = \frac{1}{2\cos^2 x}$$

$$\frac{dy}{dx} + (-\operatorname{cosec} 2x)\, y = \frac{1}{2}\sec^2 x$$

This is the linear differential equation in the form of

$$\frac{dy}{dx} + py = Q$$

where $P = -\operatorname{cosec} 2x$

$$Q = \frac{1}{2}\sec x^2$$

$$I.F = e^{\int P dx} = e^{\int -\operatorname{cosec} 2x\, dx} = e^{-\frac{1}{2}\log|\tan x|} = (\tan x)^{-\frac{1}{2}}$$

$$I.F = \frac{1}{\sqrt{\tan x}}$$

Solution is

$$y \times I.F = \int Q \times I.F\, dx + c$$

$$y \times \frac{1}{\sqrt{\tan x}} = \int \frac{1}{2}\sec^2 x \frac{1}{\sqrt{\tan x}}\, dx + c$$

$$\frac{y}{\sqrt{\tan x}} = \frac{1}{2}\int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx + c$$

Put $\tan x = t$ in $R.H.S$

$$\sec^2 x\, dx = dt$$

$$\frac{y}{\sqrt{\tan x}} = \frac{1}{2}\int \frac{1}{\sqrt{t}} dt + c$$

$$\frac{y}{\sqrt{\tan x}} = \frac{1}{2}\int t^{\frac{-1}{2}}\, dt + c$$

$$\frac{y}{\sqrt{\tan x}} = \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$\frac{y}{\sqrt{\tan x}} = \sqrt{t} + c$$

$$\frac{y}{\sqrt{\tan x}} = \sqrt{\tan x} + c \qquad ....(1)$$

where $x = \frac{\pi}{4}$ then $y = 0$

in equation (1)

$$0 = \sqrt{\tan\frac{\pi}{4}} + c$$

$$c = -1$$

$\therefore$ solution is

$$\frac{y}{\sqrt{\tan x}} = \sqrt{\tan x} - 1$$

$$\therefore \boxed{y = \tan x - \sqrt{\tan x}}$$

22. Vector equation of plane is given as

$$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 6 \text{ and } \vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$$

The cartesian form of the above equations are

$$x + y + 3 - 6 = 0 \quad \text{and} \quad 2x + 3y + 4z + 5 = 0$$

Equation of plane passing through line of intersection of planes is $s_1 + \lambda s_2 = 0$

$$\therefore \quad (x + y + z - 6) + \lambda\,(2x + 3y + 4z + 5) = 0 \qquad ...(1)$$

Since the above plane passes through point (1, 1, 1)

$\therefore$ It should satisfy the above equation.

$$(1 + 1 + 1 - 6) + \lambda\,(2 + 3 + 4 + 5) = 0$$

$$(-3) + \lambda(14) = 0$$

$$\lambda = \frac{3}{14}$$

putting the value of $\lambda = \frac{3}{14}$ in equation (1) we get

$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$$

$$14x + 14y + 14z - 84 + 6x + 9y + 12z + 15 = 0$$

$$20x + 23y + 26z - 69 = 0$$

therefore the equation of plane in vector form is

$$\vec{r}.(20\hat{i} + 23\hat{j} + 26\hat{k}) = 69$$

23. $\because$ $|a - a| = 0$ is an even number

$\therefore$ $(a, a) \in R \ \forall \ a \in A$

$\Rightarrow$ R is reflexive

Again $(a, b) \in R$

$\Rightarrow$ $|a - b|$ is even

$\Rightarrow$ $|-(b - a)|$ is even

$\Rightarrow$ $|b - a|$ is even

$\Rightarrow$ $(b, a) \in R$

$\therefore$ R is symmetric.

Next, let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $|a - b|$ is even and $|b - c|$ is even

$\Rightarrow$ $a - b$ is even and $b - c$ is even

$\Rightarrow$ $(a - b) + (b - c)$ is even

$\Rightarrow$ $a - c$ is even

$\Rightarrow$ $|a - c|$ is even

$\Rightarrow$ $(a, c) \in R$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

$\because$ $|1 - 3| = 2, |3 - 5| = 2$ and $|1 - 5| = 4$ are all even, therefore, all the elements of $\{1, 3, 5\}$ are related to each other

Again $|2 - 4| = 2$ is even, therefore, all the elements of $\{2, 4\}$ are related to each other.

However, $|1 - 2| = 1, |1 - 4| = 3, |3 - 2| = 1, |3 - 4| = 1, |5 - 2| = 3$ and $|5 - 4| = 1$ are all odd, therefore, no element of the set $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

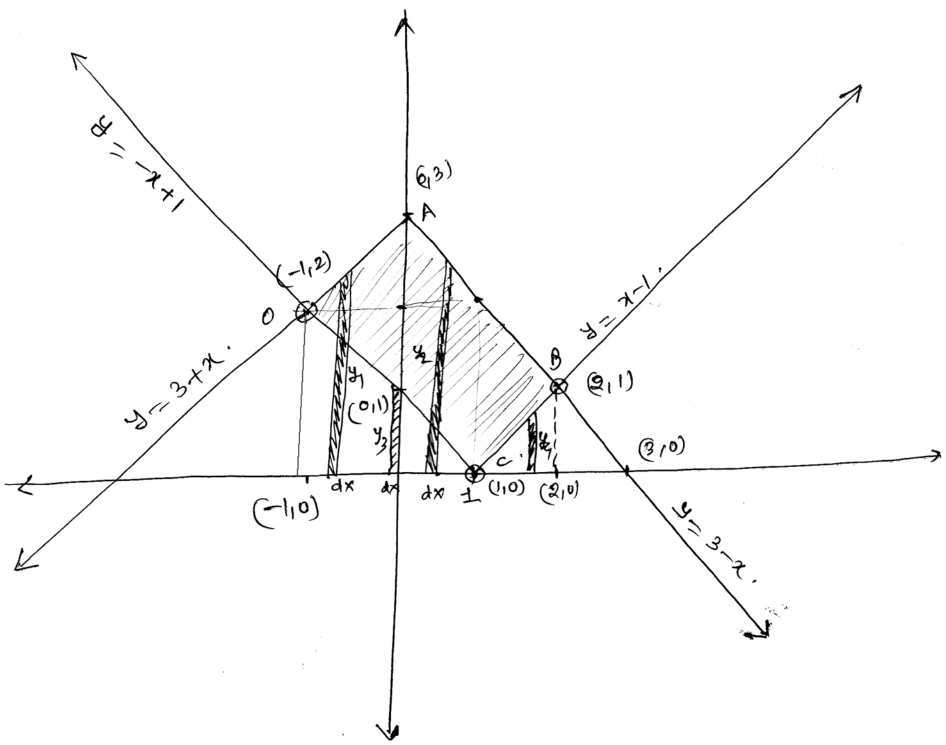
24. $y = |x - 1|$ $\qquad\qquad$ $y = 3 - |x|$

$y = (x-1)$ if $x \geq 1$ $\qquad\qquad$ $y = 3 - x \quad x \geq 0$

$\;\;= -(x-1)$ if $x < 1$ $\qquad\qquad$ $y = 3 + x \quad x < 0$

$y = -x+1$ if $x < 1$

Plotting $y = x - 1,\; y = -x + 1,\; y = 3 - x$ and $y = 3 + x$



Area of shaded region

$$= \left[\int_{-1}^{0} y_1 dx + \int_{0}^{2} y_2 dx\right] - \left[\int_{-1}^{1} y_3 dx + \int_{1}^{2} y_4 dx\right]$$

$$= \left[\int_{-1}^{0} (3+x)dx + \int_{0}^{2} (3-x)dx\right] - \left[\int_{-1}^{1} (-x+1)dx + \int_{1}^{2} (x-1)dx\right]$$

$$= \left[\int_{-1}^{0} 3dx + \int_{-1}^{0} xdx + \int_{0}^{2} 3dx - \int_{0}^{2} xdx\right] - \left[\int_{-1}^{1} -xdx + \int_{-1}^{1} dx + \int_{1}^{2} xdx - \int_{1}^{2} dx\right]$$

$$= \left[(3x)_{-1}^{0} + \left(\frac{x^2}{2}\right)_{-1}^{0} + (3x)_{0}^{2} - \left(\frac{x^2}{2}\right)_{0}^{2}\right] - \left[\left(\frac{-x^2}{2}\right)_{-1}^{1} + (x)_{-1}^{1} + \left(\frac{x^2}{2}\right)_{-1}^{2} - (x)_{1}^{2}\right]$$

$$= \left[(0 + 3) + \left(-\frac{1}{2}\right) + (6 - 0) - \left(\frac{4}{2} - 0\right)\right] - \left[-\left(\frac{1}{2} - \frac{1}{2}\right) + (1 + 1) + \left(\frac{4}{2} - \frac{1}{2}\right) - (2 - 1)\right]$$

$$= \left[3 - \frac{1}{2} + 6 - 2\right] - \left[0 + 2 + \frac{3}{2} - 1\right]$$

$$= 7 - \frac{1}{2} - 1 - \frac{3}{2}$$

$$= 6 - 2$$

$$= 4 \text{ sq. unit}$$

25. Let he purchases x items of A and y items of B.

$\therefore$ Acc. to the condition

$$2500x + 500y \le 50,000 \qquad ...(1)$$

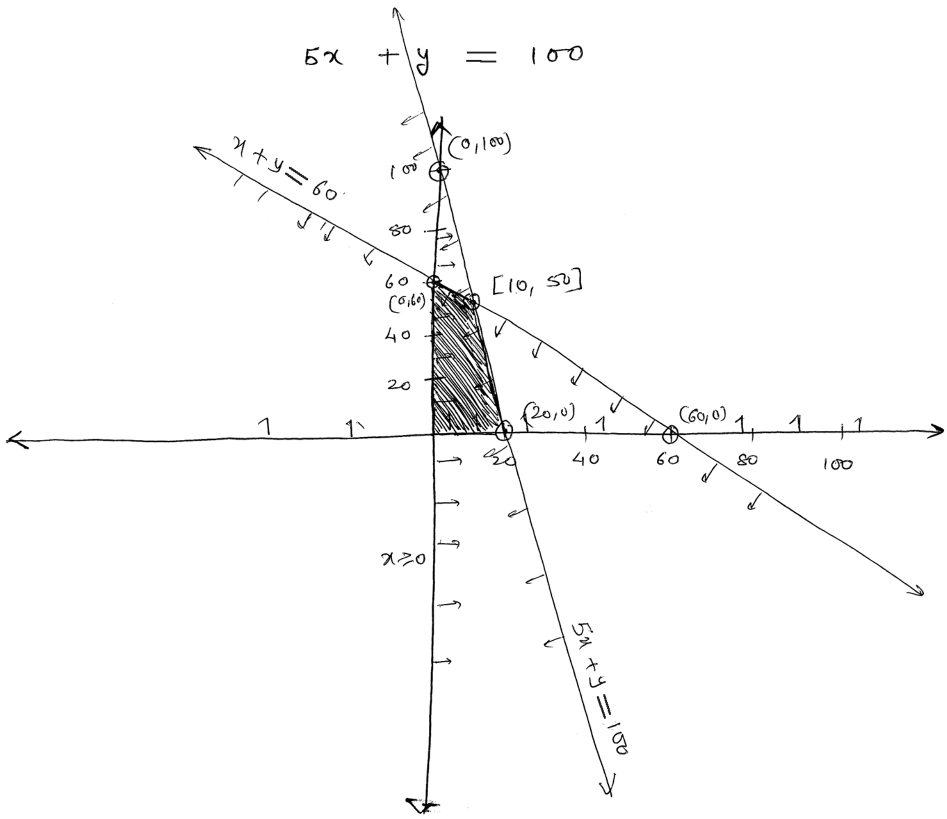
$$x + y \le 60 \qquad ...(2)$$

$$x \ge 0,\ y \ge 0 \qquad ...(3),(4)$$

$$z = 500x + 150y$$

Dividing equation (1) by 500 we get

$$5x + y \le 100$$



(x, y)	$z = 500x + 150y$	
(0, 0)	$z = 500(0) + 150(0) = 0$	
(0, 60)	$z = 500(0) + 150(60) = 9000$	
(10, 50)	$z = 500(10) + 150(50) = 12500$	$\longrightarrow$ Maximum
(20, 0)	$z = 500(20) + 150(0) = 10000$	

Thus he should purchase 10 items of x and 50 items of y.

OR

Let x and y be the number of packets of food P and Q respectively. Obviously $x \geq 0,\ y \geq 0$. Mathematical formulation of the given problem is as follows:

Minimise $Z = 6x + 3y$ (vitamin A)

subject to the constraints

$12x + 3y \geq 240$ (constraint on calcium), i.e. $\quad 4x + y \geq 80 \quad$ ... (1)

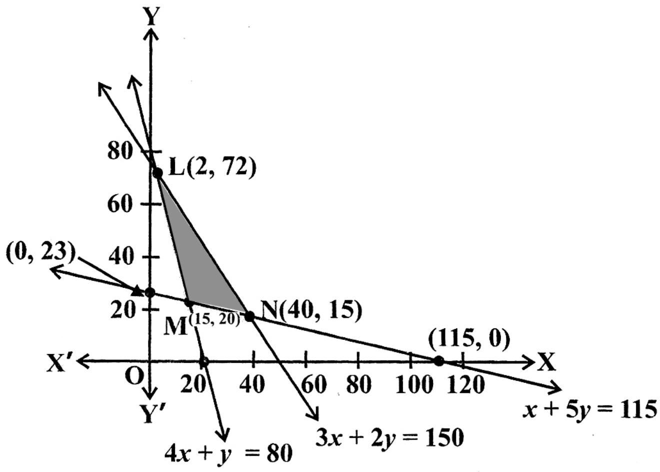
$4x + 20y \geq 460$ (constraint on iron), i.e. $\quad x + 5y \geq 115 \quad$ ... (2)

$6x + 4y \leq 300$ (constraint on cholesterol), i.e. $\quad 3x + 2y \leq 150 \quad$ ... (3)

$x \geq 0,\ y \geq 0 \quad$ ... (4)

Let us graph the inequalities (1) to (4).

The feasible region (shaded) determined by the constraints (1) to (4) is shown in figure and note that it is bounded.



The coordinates of the corner points L, M and N are (2, 72), (15, 20) and (40, 15) respectively. Let us evaluate Z at these points :

Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	150 $\leftarrow$	Minimum
(40, 15)	285	

From the table, we find that Z is minimum at the point (15, 20). Hence, the amount of vitamin A under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin A will be 150 units.

26. Let us define the following events:

E_1 = taking course of mediatation and yoga.

E_2 = taking a course of drugs

A = the patient gets a heart attack.

Here $P(E_1) = \frac{1}{2}; P(E_2) = \frac{1}{2}$

$$P(A/E_1) = \frac{40}{100}\left(1 - \frac{30}{100}\right) = \frac{28}{100}$$

$$P(A/E_2) = \frac{40}{100}\left(1 - \frac{25}{100}\right) = \frac{30}{100}$$

By Bayes' theorem

$P(E_1/A)$

$$= \frac{P(E_1)\cdot P(A/E_1)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{28}{100} \times \frac{1}{2}}{\frac{1}{2} \times \frac{28}{100} + \frac{1}{2} \times \frac{30}{100}}$$

$$= \frac{28}{58}$$

$$= \frac{14}{29}$$