EDUMATE

XII

MATHEMATICS

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CONTENTS

CHAPTER - 1

RELATIONS AND FUNCTIONS

Learning Outcomes

Recalls the idea of relation ; Recollects the concept of domain and range ; Illustrates different equivalence relations

Q.1

Let R be a relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ defined as $R = \{(x, y): y = 2x - 1\}$

i)   Write R in roster form and find it's domain and range   (2)

ii)   Is R an equivalence relation ? Justify   (2)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

i)   R ={(1,1), (2,3), (3,5)}   (1)

Domain = {1,2,3} ; Range = {1,3,5}   (1)

ii)   Since $(2, 2) \notin R$, R is not reflexive

$(2,3) \in R$ but $(3, 2) \notin R$; R is not symmetric   (1)

$(2,3) \in R$, $(3,5) \in R$ but $(2,5) \notin R$, R is not transitive

$\therefore R$ is not an equivalence relation   (1)

Learning Outcomes

Classifies and explains different equivalence relations

Q.2

The relation R defined on the $A = \{-1, 0, 1\}$ as $R = \{(a,b): a^2 = b\}$

i)   Check whether R is reflexive, symmetric and transitive   (2)

ii)   Is R an equivalence relation ?   (1)

(Scores: 3 ; Time: 5 mts)

Scoring Indicators

i) $(-1,-1) \notin R$, R is not reflexive

$(-1,1) \in R$ but $(1,-1) \notin R$,   R is not symmetric   (1)

$(-1,1) \in R$, $(1,1) \in R$ and $(-1,1) \in R$,   R is transitive   (1)

ii)   R is not reflexive, not symmetric and transitive.
So R is not an equivalence relation   (1)

Learning Outcomes

Classifies different types of relations ; Illustrates different types of relations

Q 3.

Let A = {1,2,3} . Give an example of a relation on A which is

i)   Symmetric but neither reflexive nor transitive   (2)
ii)   Transitive but neither reflexive nor symmetric   (2)

(Scores: 3 ; Time: 6 mts)

Scoring Indicators

i) R= $\{(1,2),(2,1)\}$ (1)

$(1,1) \notin R \Rightarrow$ R is not reflexive

$(1,2) \in R \Rightarrow (2,1) \in R$,   R is symmetric

$(1,2) \in R, (2,1) \in R$ but $(1,1) \notin R$, R is not transitive   (1)

ii) R= $\{(1,2),(1,3),(2,3)\}$

$(1,1) \notin R \Rightarrow$ R is not reflexive   (1)

$(1,2) \in R$ but $(2,1) \notin R$, R is not symmetric

$(1,2) \in R, (2,3) \in R \Rightarrow (1,3) \in R$, R is transitive   (1)

Learning Outcomes

Recalls the idea of a function ; Identifies the injectivity and surjectivity of a function

Q 4.

(i)Let f be a function defined by $f(x) = \sqrt{x}$ is a function if it defined from

($f:N \to N$, $f:R \to R$, $f:R \to R^+$, $f:R^+ \to R^+$)

(ii) Check the injectivity and surjectivity of the followingFunctions

a) $f:N \to N$ given by $f(x) = x^3$ (2)

b) $f:R \to R$ given by $f(x) = [x]$ (2)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

i) $f:R^+ \to R^+$ For $x, y \in N$,

ii) a) $f(x)=f(y)\Rightarrow x^3=y^3\Rightarrow x=y\Rightarrow$ f is injective (1)

For $2\in N$, there does not exist x in the domain N such that $f(x)=x^3=2$.
$\therefore$ f is not surjective (1)

b) $f:R\to R$ given by $f(x)=[x]$
It seen that $f(1.1)=1$ and $f(1.8)=1$; (1)
But $1.1\neq 1.8$; $\therefore$ f is not injective
There does not exist any element $x\in R$ such that $f(x)=0.7$
$\therefore$ f is not surjective (1)

Learning Outcomes

Illustrates composition of functions ; Constructs new functions using composition

Q 5 .

a) Find fog and gof if

i) f(x) = $|x|$ and g(x) = $|3x+4|$ (2)

ii) f(x) = $16x^4$ and g(x) = $x^{\frac{1}{4}}$ (2)

b) If $f(x)=\dfrac{4x+3}{6x-4}$, $x\neq\dfrac{2}{3}$, Prove that $fof(x)=x$ (2)

(Scores: 6 ; Time: 12 mts)

Scoring Indicators

a) i) f(x) = $|x|$ and g(x) = $|3x+4|$ (1)
$\Rightarrow fog(x)=f(g(x))$ = $f(|3x+4|)=\||3x+4|\|=|3x+4|$
$gof(x)=g(f(x))$ = $g(|x|)$ = $|3|x|+4|$ (1)

ii) $fog(x)=f(g(x))$ = $f(x^{\frac{1}{4}})=16(x^{\frac{1}{4}})^4$ = $16x$ (1)
$gof(x)=g(f(x))$ = $g(16x^4)=(16x^4)^{\frac{1}{4}}$ = $4x$ (1)

b) $fof(x)=f(f(x))$ (1)

$=\dfrac{4(\dfrac{4x+3}{6x-4})+3}{6(\dfrac{4x+3}{6x-4})-4}=\dfrac{16x+12+18x-12}{24x+18-24x+16}=\dfrac{34x}{34}=x$ (1)

Learning Outcomes

Recalls the idea of a function and its domain and range ; Identifies the inverse of a function

Q 6.

Let $S = \{(1,2),(2,3),(3,4)\}$

i)	Find the domain and range of S	(1)
ii)	Find S^{-1}	(1)
iii)	Find the domain and range of S^{-1}	(1)
iv)	Verify that S^{-1} is a function using the graph of S and S^{-1}	(2)

(Scores: 5 ; Time: 10 mts)

Scoring Indicators

i)	Domain = $\{1,2,3\}$; Range = $\{2,3,4\}$	(1)
ii)	$S^{-1} = \{(2,1),(3,2),(4,3)\}$	(1)
iii)	Domain = $\{2,3,4\}$; Range = $\{1,2,3\}$	(1)
iv)	Plotting S and S⁻¹ in graph paper	(1)
	Yes, S⁻¹ is a function because x coordinates do not intersect	(1)

Learning Outcomes

Identifies the condition for invertible function ; finds inverse

Q 7.

I) Consider $f : \{3,4,5,6\} \to \{8,10,12,13,14\}$ and $\qquad$ (3)

$\quad$ f = $\{(3,8),(4,10),(5,12),(6,14)\}$. State whether f has inverse ? Give reason

ii) Consider $f : R \to R$ given by $f(x) = 3x + 2$. Show that f is invertible. $\qquad$ (3)

$\quad$ Find the inverse of f

(Scores: 6 ; Time: 12 mts)

Scoring Indicators

i)	Distinct elements in set $\{3,4,5,6\}$ has distinct images.	(1)
	under f . $\therefore f$ is one- one	
	But 13 in the codomain has no pre image. $\therefore f$ is not onto.	(1)
	$\therefore f$ has no inverse	(1)

$\quad f(x) = 3x + 2$; then

ii) $\quad f(x_1) = f(x_2) \Rightarrow 3x_1 + 2 = 3x_2 + 2 \Rightarrow x_1 = x_2$ $\qquad$ (1)

$\quad$ Hence F is one - one

$\quad$ For $y \in R$, let $y = 3x + 2 \Rightarrow x = \dfrac{y-2}{3} \in R$ $\qquad$ (1)

$\quad f(x) = f(\dfrac{y-2}{3}) = 3(\dfrac{y-2}{3}) + 2 = y \Rightarrow$ f is onto

Define $g: R \to R$ such that $g(y) = \dfrac{y-2}{3}$

$gof(x) = g(f(x)) = g(3x+2) = \dfrac{3x+2-2}{3} = x$

$fog(y) = f(g(y)) = f(\dfrac{y-2}{3}) = 3(\dfrac{y-2}{3}) + 2 = y$ 　　　　(1)

Learning Outcomes

Identifies the composition of an operation ; Applies the concept of composition ;
Identifies and solves inverse of a function

Q 8.

(1)

Choose the correct answer from the bracket

If $x \neq 1$ and $f(x) = \dfrac{x+1}{x-1}$ is a real function, then $fof(2)$ = -----

(1, 2, 3, 4)

i)　　　What is the inverse of f 　　　　(2)

ii)　　$Find\ f(3) + f^{-1}(3)$ 　　　　(1)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

i)　　2 　　　　(1)

ii)　　Let g: $range\ of\ f \to R-\{1\}$ be the inverse of f

Let y be any arbitrary element in the range of f , then $y = f(x) = \dfrac{x+1}{x-1}$

$\Rightarrow xy - y = x+1 \Rightarrow x(y-1) = y+1 \Rightarrow x = \dfrac{y+1}{y-1}, y \neq 1$

Let us define

g: $range\ of\ f \to R-\{1\}$ as $g(y) = \dfrac{y+1}{y-1}$

$gof(x) = g(f(x)) = g(\dfrac{x+1}{x-1}) = \dfrac{\dfrac{x+1}{x-1}+1}{\dfrac{x+1}{x-1}-1} = x$ 　　　　(1)

$\therefore f^{-1} = g \Rightarrow f^{-1}(y) = \dfrac{y+1}{y-1}, y \neq 1$ 　　　　(1)

iii)　　$f(3) = 2$, $f^{-1}(3) = 2 \Rightarrow$

$f(3) + f^{-1}(3) = 2+2 = 4$ 　　　　(1)

Learning Outcomes

Identifies the binary operation as a function ; Illustrates various properties of a binary operation

Q 9.

i) Determine whether the following is a binary operaton or not?　　　　(2)
　　　Justify
　　　$a*b = 2^a b$　defined on Z

ii) Determine whether　* is commutative or associative if　　　　(2)

$$a*b = \frac{ab}{6}, a,b \in R$$

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

i)　　　$a*b = 2^a b$

　　　If a is negative, then 2^a becomes a fraction　　　　(1)

　　　Eg : $-1*3 = 2^{-1}.3 = \frac{3}{2} \notin Z$; $\therefore$ * is not a binary operation　　(1)

　　　$a*b = \frac{ab}{6} \implies b*a = \frac{ba}{6} = \frac{ab}{6} = a*b$

ii)　　　$\implies$ * is commutative　　　　(1)

$$(a*b)*c = \frac{\frac{ab}{6}.c}{6} = \frac{abc}{36} \implies a*(b*c) = \frac{a.\frac{bc}{6}}{6} = \frac{abc}{36}$$

　　　$\therefore a*(b*c) = a*(b*c) \implies$ * is associative　　　　(1)

Learning Outcomes

Identifies the identity element of a binary operation ; Identifies that the inverse of an element belongs to the set

Q10.

Consider the binary operation　$*:Q \to Q$　where Q is the set of rational numbers is defined as　$a*b = a+b-ab$

　i)　　Find　$2*3$　　　　(1)
　ii)　　Is identity for * exist? If yes, find the identity element　　(2)
　iii)　　Are elements of Q invertible? If yes, find the inverse of an element
　　　in Q　　　　(2)

(Scores: 5 ; Time: 10 mts)

Scoring Indicators

　i)　　$2*3 = 2+3-6 = -1$　　　　(1)

ii) $a*e = a+e-ae = a \implies e-ae = 0$ (1)

$\implies e(1-a) = 0 \implies e = 0$

$\implies$ e*a = a

$\implies$ e+a-ea = a

$\implies$ e-ea = 0

$\implies$ e=0 is the identity element (1)

iii) $a*a^{-1} = a+a^{-1}-aa^{-1} = 0$ (1)

iv) $\implies a^{-1}(1-a) = -a \implies a^{-1} = \dfrac{-a}{1-a} = \dfrac{a}{a-1}$ (1)

CHAPTER – 2

INVERSE TRIGONOMETRIC FUNCTIONS

Learning Outcomes

Restates the domain and range of inverse trigonometric function, Finds principal values, Applies the identifies

Q1.
  i) Choose the correct answer from the bracket
     If $cos^{-1}x = y$, then y is equal to
  $(a) - \pi \le y \le \pi$ $(b)\ 0 \le y \le \pi$ $(c)\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ $(d)\ \ 0 < y < \pi$ (1)

  (i)   Find the value of $cos^{-1}(cos\frac{7\pi}{3})$ (2)

  (ii)   Solve for x : $tan^{-1}\left(\frac{1+x}{1-x}\right) = 2tan^{-1}x,\ x > 0$ (2)

(Scores: 5 ; Time: 10 mts)

Scoring Indicators
  i)   Range of $cos^{-1}x$ is $[0,\pi] \Rightarrow 0 \le y \le \pi$ (1)
  ii) $cos^{-1}(cosx) = x$, if $x \in [0,\pi]$ which is the
         principal value branch of $cos^{-1}x$ (1)
  $\therefore cos^{-1}\left(cos\frac{7\pi}{3}\right) = cos^{-1}\left(cos\ 2\pi + \frac{\pi}{3}\right) = cos^{-1}\left(cos\frac{\pi}{3}\right) = \frac{\pi}{3}$, $\frac{\pi}{3} \in [0,\pi]$ (1)
  iii) $tan^{-1}(1) + tan^{-1}x = 2tan^{-1}x \Rightarrow \frac{\pi}{4} = 2tan^{-1}x - tan^{-1}x$ (1)
  $\Rightarrow tan^{-1}x = \frac{\pi}{4} \Rightarrow x = tan\frac{\pi}{4} \therefore x = 1$ (1)

Learning Outcomes

Formulates various identities, Applies the identities, Recalls trigonometric functions

Q 2
  i) Choose the correct answer from the bracket (1)
  $cos(tan^{-1}x)$, $|x| < 1$, is equal to
  $[\ \ (a)\frac{x}{\sqrt{1-x^2}} (b)\frac{1}{\sqrt{1-x^2}} (c)\frac{1}{\sqrt{1+x^2}} (d)\frac{x}{\sqrt{1+x^2}}\ ]$

  ii) Prove that $sin^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13} = sin^{-1}\frac{63}{65}$ (3)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators
  (i) $tan^{-1}\frac{x}{1} = cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \Rightarrow cos(tan^{-1}x)$ (1)

  $$= cos\left(cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right) = \frac{1}{\sqrt{1+x^2}}$$

(ii) LHS $tan^{-1}\frac{4}{\sqrt{5^2-4^2}} + tan^{-1}\frac{5}{\sqrt{13^2-5^2}} = tan^{-1}\frac{4}{3} + tan^{-1}\frac{5}{12}$ (1)

$= tan^{-1}\left(\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}}\right) = tan^{-1}\left(\frac{48+15}{36-20}\right) = tan^{-1}\left(\frac{63}{16}\right)$ (1)

$= sin^{-1}\left(\frac{63}{\sqrt{63^2-16^2}}\right) = = sin^{-1}\left(\frac{63}{\sqrt{4225}}\right) = = sin^{-1}\left(\frac{63}{65}\right)$ (1)

Learning Outcomes

Finds the principal values, Applies the identities
Q 3.

Match the following   (4)

A	B
(i) $sin^{-1}\left(\frac{1}{2}\right)$	a) $\frac{\pi}{3}$
(ii) $cos^{-1}\left(\frac{1}{2}\right) - sin^{-1}\left(-\frac{1}{2}\right)$	b) $\frac{\pi}{6}$
(iii) $sin^{-1}\left(sin\frac{3\pi}{4}\right)$	c) $\frac{\pi}{2}$
(iv) $tan^{-1}\left(2sin\left(cos^{-1}\left(\frac{1}{2}\right)\right.\right.$	d) $\frac{\pi}{4}$
	e) π

(Scores: 4 ; Time: 6 mts)

Scoring Indicators

(i) $sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (1)

(ii) $cos^{-1}\left(\frac{1}{2}\right) - sin^{-1}\left(-\frac{1}{2}\right) = cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$ (1)

$sin^{-1}\left(sin\frac{3\pi}{4}\right) = sin^{-1}\left(sin(\pi - \frac{\pi}{4})\right) = \frac{\pi}{4}$ (1)

iii) $tan^{-1}\left(2sin\left(cos^{-1}\left(\frac{1}{2}\right)\right)\right) = tan^{-1}\left(2sin(\frac{\pi}{3})\right) = tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right)$ (1)

$= tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Learning Outcomes

Formulates various identities, Applies the identities
Q 4.

i) Choose the correct answer from the bracket   (1)

$tan^{-1}\left(\frac{1-tanx}{1+tanx}\right)$ is equal to

[(a)$\frac{\pi}{4} + \frac{x}{2}$ (b)$\frac{\pi}{4} - x$ (c)$\frac{\pi}{4} - \frac{x}{2}$ (d) $x - \frac{\pi}{4}$]

ii)Express $tan^{-1}\left(\frac{1-sin\,x}{cos x}\right)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.   (3)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

(i) $\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) = \frac{\pi}{4} - x$ (1)

(ii) $\tan^{-1}\left(\frac{1-\sin x}{\cos x}\right) = \tan^{-1}\left(\frac{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}\right) =$ (1)

$\tan^{-1}\left(\frac{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)}\right) = \tan^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}\right)$ (1)

$= \tan^{-1}\left(\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \frac{\pi}{4} - \frac{x}{2}$ (1)

Learning Outcomes

Formulates various identities, Applies the identities

Q 5

Prove that $\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{2}{5}\right) = \tan^{-1}\left(\frac{1}{12}\right)$ (2)

(Scores: 2 ; Time: 4 mts)

Scoring Indicators

$\tan^{-1}\left(\frac{\frac{1}{2}-\frac{2}{5}}{1+\frac{1}{2}\times\frac{2}{5}}\right) = \tan^{-1}\left(\frac{\frac{5-4}{10}}{\frac{10+2}{10}}\right)$ (1)

$= \tan^{-1}\left(\frac{\frac{1}{10}}{\frac{12}{10}}\right) = \tan^{-1}\left(\frac{1}{12}\right)$ (1)

Learning Outcomes

Sketches the graphs of inverse T – functions, Formulates various identities, Applies the identities

Q6.

   (i)   In which quadrants are the graph of $\cos^{-1}x$ lies, $x \in [-1, 1]$ (1)

   (ii)   Choose the correct answer from the bracket

        If $\cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$, then $\sin^{-1}x + \sin^{-1}y = $ ......

         $[\ (a)\frac{2\pi}{3}\ (b)\frac{\pi}{3}\ (c)\frac{\pi}{6}\ d)\ \ \pi\ ]$ (1)

   (iii)   If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$ then prove that $x + y + xy = 1$ (2)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

(i)   First and Second quadrants   (1)

(ii) $\frac{\pi}{2} - sin^{-1}x + \frac{\pi}{2} - sin^{-1}y = \frac{\pi}{3}$ (1)

$\Rightarrow sin^{-1}x + sin^{-1}y = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(iii) $tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy} = tan\frac{\pi}{4}$ (1)

$\Rightarrow \frac{x+y}{1-xy} = 1 \Rightarrow x + y + xy = 1$ (1)

CHAPTER - 3

MATRICES

Learning Outcomes

Interprets and classifies the concept of a Matrix ; Identifies the properties of Matrices

Q .1

If $\begin{bmatrix} x & -y & -z \\ & -y & +z \\ & & z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$, then find the values of x, y and z $\hspace{2cm}$ (2)

$\hspace{8cm}$ **(Scores: 2 ; Time: 5 mts)**

Scoring Indicators

z= 3

$- y + z = 5 \quad \Rightarrow y = z\ -5 = 3\ -5 = -2$ $\hspace{4cm}$ (1)

$x - y + z = 0 \Rightarrow x = y\ - z = -2\ +3, z = 1$ $\hspace{3cm}$ (1)

Learning Outcomes

Identifies and compares the order of Matrices for finding the product

Q.2

i) If $B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ find B^2 $\hspace{6cm}$ (2)

ii) If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find x $\hspace{1.5cm}$ (2)

$\hspace{8cm}$ **(Scores: 4 ; Time: 8 mts)**

Scoring Indicators

i)$B^2 = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4+3 & 6+0 \\ 2+0 & 3+0 \end{bmatrix}$ $\hspace{3cm}$ (1)

$= \begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix}$ $\hspace{7cm}$ (1)

ii) $A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\hspace{2cm}$ (1)

$\therefore x = 0$ $\hspace{9cm}$ (1)

Learning Outcomes

Identifies and compares the order of matrices for finding their product

Q.3

Let the order of a matrix A is 2×3, order of a matrix B is 3 × 2 and order of a matrix C is 3 ×3 , then match the following

	Matrix	Order	
i)	CB	3×3	(1)
ii)	AC	3×2	(1)
iii)	BAC	3×2	(1)

(Scores: 3; Time: 5 mts)

Scoring Indicators

i) Order of matrix CB is   3×2   (1)

ii) Order of matrix AC is   2×3   (1)

iii)   Order of matrix BAC is 3×3   (1)

Learning Outcomes

Differentiates different types of Matrices ; Identifies the properties of Matrices
Q.4

i) If $U = [2 \quad -3 \quad 4]$, $V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$X = [0 \quad 2 \quad 3]$, $Y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, then find UV , XY , UV +XY   (3)

ii) What must be the matrix X if $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$,   (2)

iii) If $[1 \ x \ 1] \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$ then find the value of x   (2)

(Scores:7; Time: 10 mts)

Scoring Indicators

i) $UV = [2 \quad -3 \quad 4] \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = [6 - 6 + 4] = [4]$ **(1)**

$XY = [0 \quad 2 \quad 3] \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = [0 + 4 + 12] = [16]$ **(1)**

UV +XY =[**4**] + [**16**] = [**20**]   (1)

ii) 2 X $= \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$ (1)

$\Rightarrow X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (1)

iii) $[\,1 \; x \; 1\,] \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$

i)

$[\,1 \; x \; 1\,] \begin{bmatrix} x+2-6 \\ 0+5-2 \\ 0+3-4 \end{bmatrix} = 0$ (1)

$[\mathbf{1} \; \pmb{x} \; \mathbf{1}] \begin{bmatrix} x-4 \\ 3 \\ -1 \end{bmatrix} = 0$

$[x \; -4 \; +3x \; -1] = [\,0\,]$

$\Rightarrow x-4+3x-1=0 \quad \Rightarrow x = \dfrac{5}{4}$ (1)

Learning Outcomes

Interpret and classify the concept ; Identifies the properties of transpose of a Matrices

Q.5

Choose the correct answer from the bracket

i)If $A = \left[a_{ij}\right]_{2X2}$ where $a_{ij} = i + j$, then A is equal to

a) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ (1)

ii) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ is a

  a) symmetric matrix   b) skew- symmetric matrix   (1)
  c) null matrix   d) identity matrix

iii) If $A = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 8 \\ -6 & 3 \end{bmatrix}$ then A + B is equal to

a) $\begin{bmatrix} 21 & -16 \\ -12 & 15 \end{bmatrix}$ b) $\begin{bmatrix} 10 & 6 \\ -4 & 8 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (1)

iv) If A $= \begin{bmatrix} 10 & 6 \\ -4 & 8 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then AI_2 is equal to

a)$\begin{bmatrix} 10 & 6 \\ -4 & 8 \end{bmatrix}$ b) $\begin{bmatrix} 21 & -16 \\ -12 & 15 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (1)

v) . If A is square matrix, A′ , is its transpose, then $\frac{1}{2}(A - A')$ is

  a) a symmetric matrix   b) a skew-symmetric matrix
  c) a unit matrix   d) a null matrix   (1)

(Scores:5; Time: 10 mts)

Scoring Indicators

i)   A $=a_{11}$ = 1+1 =2 , a_{12} = 1+2 =3, a_{21} = 2+1 = 3

a_{22} = 2+2 = 4 , A = $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ (1)

ii) A^T = $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ = A is symmetric matrix   (1)

iii) $A + B = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ -6 & 3 \end{bmatrix}$ = $\begin{bmatrix} 10 & 6 \\ -4 & 8 \end{bmatrix}$ (1)

iv)   If A I_2 = $\begin{bmatrix} \mathbf{10} & \mathbf{6} \\ \mathbf{-4} & \mathbf{8} \end{bmatrix}\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$ = $\begin{bmatrix} \mathbf{10} & \mathbf{6} \\ \mathbf{-4} & \mathbf{8} \end{bmatrix}$ (1)

v)   a skew-symmetric matrix   (1)

Learning Outcomes

Identifies matrix multiplication is not commutative ;Identifies the properties of transpose of a matrix

Q.6

Let A and B be two symmetric matrices of same order. Then show that AB – BA is a skew- symmetric matrix   (2)

(Scores: 2; Time: 5 mts)

Scoring Indicators

Given A = A^T , B = B^T
$(AB –BA)^T$ = $(AB)^T – (BA)^T$ (1)
        = $B^T A^T$ - $A^T B^T$ = BA – AB = - (AB - BA)
       Therefore AB – BA is a skew- symmetric matrix   (1)

Learning Outcomes

Identifies and compares the order of a matrix for finding their product

Q.7

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \text{then}$$

   i) Find 4A and A^2 (3)

   ii) **Show that A^2- 4A = $5I_3$**   (2)

(Scores: 5; Time: 10 mts)

Scoring Indicators

i) $4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$ (1)

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+4 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$
(1)

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$ (1)

ii) $A^2 - 4A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$ (1)

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5I_3$$ (1)

Learning Outcomes

Identifies that   matrix multiplication is not commutative

Q.8

If $A = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$ and $B = \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$

   i) Show that $AB = \begin{bmatrix} cos\alpha cos\beta cos(\alpha - \beta) & sin\beta cos\alpha cos(\alpha - \beta) \\ sin\alpha cos\beta cos(\alpha - \beta) & sin\alpha sin\beta cos(\alpha - \beta) \end{bmatrix}$

        (3)

   ii) If the product AB is null matrix, then show that α -β is a multiple of $\pi/2$ (2)

(Scores:5; Time: 10 mts)

Scoring Indicators

i) $\quad$ AB $= \begin{bmatrix} cos^2\alpha & cos\alpha sin\alpha \\ cos\alpha sin\alpha & sin^2\alpha \end{bmatrix} \begin{bmatrix} cos^2\beta & cos\beta sin\beta \\ cos\beta sin\beta & sin^2\beta \end{bmatrix}$ (1)

$= \begin{bmatrix} cos^2\alpha cos^2\beta + cos\alpha sin\alpha\, cos\beta sin\beta & cos^2\alpha cos\beta sin\beta + cos\alpha sin\alpha sin^2\beta \\ cos\alpha sin\alpha cos^2\beta + sin^2\alpha cos\beta sin\beta & cos\alpha sin\alpha + sin^2\alpha sin^2\beta \end{bmatrix}$ (1)

$= \begin{bmatrix} cos\alpha cos\beta(cos\alpha cos\beta + sin\alpha\, sin\beta) & sin\beta cos\alpha(cos\alpha cos\beta + sin\alpha\, sin\beta) \\ sin\alpha cos\beta(cos\alpha cos\beta + sin\alpha\, sin\beta) & sin\alpha sin\beta(cos\alpha cos\beta + sin\alpha\, sin\beta) \end{bmatrix}$
(1)

$= \begin{bmatrix} cos\alpha cos\beta cos(\alpha - \beta) & sin\beta cos\alpha cos(\alpha - \beta) \\ sin\alpha cos\beta cos(\alpha - \beta) & sin\alpha sin\beta cos(\alpha - \beta) \end{bmatrix}$

ii) $\quad$ Given $\begin{bmatrix} cos\alpha cos\beta cos(\alpha - \beta) & sin\beta cos\alpha cos(\alpha - \beta) \\ sin\alpha cos\beta cos(\alpha - \beta) & sin\alpha sin\beta cos(\alpha - \beta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (1)

$$\Rightarrow cos(\alpha - \beta) = 0$$

$\Rightarrow (\alpha - \beta)$ is a multiple of $\pi/2$ (1)

Learning Outcomes

Identifies and compares the order of Matrices for finding their product
Q.9

Let $\quad$ A $= \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$

i) $\quad$ Find A^2 (1)
ii) If $\quad A^4 = I_2$, then show that ab = 1 (2)

(Scores:3; Time: 6 mts)

Scoring Indicators

$A^2 = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}$ (1)

i) $A^4 = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} = \begin{bmatrix} a^2 b^2 & 0 \\ 0 & a^2 b^2 \end{bmatrix}$ (1)

$\begin{bmatrix} a^2 b^2 & 0 \\ 0 & a^2 b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow ab = 1$ (1)

Learning Outcomes

Identifies the properties of transpose of a Matrix

Q.10

If the matrices A = $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ and B $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$, then

i) Find AB , B^T, $(AB)^T$ (3)
ii) Show that $(AB)^T = B^T A^T$ (2)

(Scores: 5; Time: 8 mts)

Scoring Indicators

$AB = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$ (1)

$B^T = \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix}$ (1)

$(AB)^T = \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$ (1)

ii) $B^T A^T = \begin{bmatrix} 1 & 0 & 5 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$ (1)

$= \begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix} = (AB)^T$ (1)

CHAPTER-4

DETERMINANTS

Learning Outcomes

Identifies the properties of determinants and verifies them through examples.
Identifies the methods of evaluating a determinant.

Q.1

(i) Choose the correct answer from the bracket. If $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix} = 40$, then $\begin{vmatrix} 1 & 4 & 3 \\ -3 & -1 & 5 \\ 2 & 2 & 2 \end{vmatrix} = ?$

$\{$ (a) 0 , (b) -40 , (c) 40 , (d) 2 $\}$ (1)

(ii) Show that $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) (c) 40 (1)

(ii) $\Delta = abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$ take a,b,c from C_1, C_2, C_3 (1)

$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ take a,b,c from R_1, R_2, R_3

$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} \begin{matrix} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 + R_1 \end{matrix}$ (1)

$= a^2b^2c^2(-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = -a^2b^2c^2(0-4) = 4a^2b^2c^2$ (1)

Learning Outcomes

Identifies the properties of determinants and verifies them through examples.

Identifies the methods of evaluating a determinant

Q.2

(i) Choose the correct answer from the bracket. Let the value of a determinant is Δ.
Then the value of a determinant obtained by interchanging two rows is
{ (a) Δ , (b) $-\Delta$, (c) 0 , (d) 1 }   (1)

(ii) Show that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ (3)

(Scores:4; Time: 8 mts)

Scoring Indicators

(i)   (b) $-\Delta$ (1)

(ii) Operating $C_1 \rightarrow C_1 + C_2 + C_3$, we have

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$ (1)

$$= 2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix}$$

$C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$

$$= 2\begin{vmatrix} a+b+c & -a & -b \\ a+b+c & -b & -c \\ a+b+c & -c & -a \end{vmatrix} = 2\begin{vmatrix} c & -a & -b \\ a & -b & -c \\ b & -c & -a \end{vmatrix} C_1 \rightarrow C_1 + C_2 + C_3$$ (1)

$$= 2(-1)(-1)\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= 2(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ (1)

Learning Outcomes

Identifies the properties of determinants and verifies them through examples.
Identifies the methods of evaluating a determinant.

Q.3

The value of the determinant $\begin{vmatrix} \sin 10 & -\cos 10 \\ \sin 50 & \cos 50 \end{vmatrix}$ is (a) – 1 (b) $\dfrac{\sqrt{3}}{2}$ (c) $\dfrac{1}{2}$ (d) – 2   (1)

(ii) Using properties of determinants, show that

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$$ (3)

(Scores:4; Time: 8 mts)

Scoring Indicators

(i) (b) Since, $\sin 10 \cos 50 + \cos 10 \sin 50 = \sin 60 = \dfrac{\sqrt{3}}{2}$ (1)

$$\Delta = \begin{vmatrix} a & a^2 & a+b+c \\ b & b^2 & a+b+c \\ c & c^2 & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

ii) Let (1)

$$= (a+b+c)\begin{vmatrix} a & a^2 & 1 \\ b-a & b^2-a^2 & 0 \\ c-a & c^2-a^2 & 0 \end{vmatrix} \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$= (a+b+c)\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$ (1)

$$\begin{aligned} &= (a+b+c)(b-a)(c-a)(c+a-b-a) \\ &= (a+b+c)(b-a)(c-a)(c-b) \\ &= (b-c)(c-a)(a-b)(a+b+c) \end{aligned}$$ (1)

Learning Outcomes

Identifies the properties of determinants and verifies them through examples.

Identifies the methods of evaluating a determinant. Identifies the peculiarities of odd order skew symmetric matrix.

Q.4

(i) Choose the correct answer from the bracket. The value of the determinant

$$\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} \text{ is …}$$

{(a) $p+q+r$, (b)1 , (c)0 , (d) $3pqr$ } (1)

(ii) Evaluate $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$ (3)

(Scores:4; Time: 8 mts)

SCORING INDICATORS

(i) (c) 0(since the given determinant is the determinant of a
third order skew symmetric matrix) (1)

(ii) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$

$R_2 \to R_2 - 2R_1, \ R_3 \to R_3 - 3R_1$ (1)

$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$ = a [7a²+3ab-6a²-3ab] (1)

$= a(a^2) = a^3$ (1)

Learning Outcomes

Recognizing the concept of minors and cofactors, Identifies the adjoint and its
properties, Identifies the methods of evaluating a determinant.

Q.5

(i) Choose the correct answer from the bracket. Consider a square matrix of order 3.
Let C_{11}, C_{12}, C_{13} are cofactors of the elements a_{11}, a_{12}, a_{13} respectively, then
$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ is
{ (a) 0 , (b) $|A|$, (c) 1 , (d) none of these }

(1)

(ii) Verify for the matrix $A = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix}$ that $A(adjA) = (adjA)A = |A|I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) (b) $|A|$ (1)

(ii) $|A| = \begin{vmatrix} 5 & -2 \\ 3 & -2 \end{vmatrix} = -4$

$C_{11} = -2, C_{12} = -3, C_{21} = 2, C_{22} = 5$

$adjA = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix}$ (1)

Now; $A(adjA) = \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} \times \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$

$$= -4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I \tag{1}$$

$$(adjA)A = \begin{bmatrix} -2 & 2 \\ -3 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & -2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$= -4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A|I$$

Hence $A(adjA) = (adjA)A = |A|I$ (1)

Learning Outcomes

Recognising the concept of minors and cofactors, Identifies the adjoint and its properties, Identifies the methods of evaluating a determinant, Computing the inverse of a matrix.

Q.6

(i) Choose the correct answer from the bracket. If $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ and

$A(adjA) = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then the value of k is …

{ (a) 0, (b) 3, (c) 1, (d) 2 } (1)

(ii) Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

(3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) (c) 1 (1)

$$|A| = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \text{ and } A(adjA) = |A|I$$

(ii) Let $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$

$= 1(8-6) + 1(0+9) + 2(0-6) = -1$ (1)

$C_{11} = 2, C_{12} = -9, C_{13} = -6, C_{21} = 0, C_{22} = -2$

$C_{23} = -1, C_{31} = -1, C_{32} = 3, C_{33} = 2$

$$adj(A) = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} \tag{1}$$

$$A^{-1} = \frac{adj(A)}{|A|} = (-1)\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$
(1)

Learning Outcomes

Recognizing the concept of minors and cofactors, Identifies the adjoint and its properties, Identifies the methods of evaluating a determinant, Computing the inverse of a matrix, Identifies the singular and non singular matrices.

Q.7

(i) Choose the correct answer from the bracket. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$ and $A^{-1} = kA$, then the value of 'k'is

{(a)7 , (b) -7 , (c) $\frac{1}{7}$,(d) $-\frac{1}{7}$ }
(1)

(ii)If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, (a) find A^2 (b) Show that $A^2 = A^{-1}$

(4)
(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) (c) $\frac{1}{7}$
(1)

$$|A| = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7 \; ; A^{-1} = \frac{adjA}{|A|} = \frac{-1}{7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}$$

(ii) $A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$
(1)

Now; $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1 \neq 0$
(1)

$C_{11} = 0, C_{12} = 0, C_{13} = 1, C_{21} = 0, C_{22} = -1, C_{23} = -1, C_{31} = 1, C_{32} = 2, C_{33} = 1$

(1)

$adjA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} ; A^{-1} = \frac{adjA}{|A|} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = A^2$
(1)

Learning Outcomes

Identifies the properties of determinants and verifies through examples, Computing the inverse of a matrix, Verifying the properties of inverse of matrices.

Q.8

    (i)   Choose the correct answer from the bracket. If each element of a third order square matrix 'A' is multiplied by 3, then the determinant of the newly formed matrix is

        { (a) $9|A|$, (b) $3|A|$, (c) $27|A|$, (d) $(|A|)^3$ }   (1)

    (ii)   Consider the matrix $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$

        (a) Show that 'A' satisfies the equation $x^2 + 4x - 42 = 0$ (2)

        (b) Hence find A^{-1} (2)

        (c)

                    (Scores:5; Time: 10 mts)

Scoring Indicators

    (i) (c) $27|A|$ (1)

    (ii) (a) $A^2 = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} \times \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix}$ (1)

Now; $A^2 + 4A - 42I = \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix} + 4\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} - 42\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$ (1)

    (b) We have; $A^2 + 4A - 42I = 0$

        $\Rightarrow A \times A + 4A - 42I = 0$

Multiply both sides by A^{-1}

$\Rightarrow A^{-1}A \times A + 4A^{-1}A - 42A^{-1}I = 0$ (1)

$\Rightarrow A + 4I = 42A^{-1} \Rightarrow \dfrac{1}{42}(A + 4I) = A^{-1}$

$\Rightarrow A^{-1} = \dfrac{1}{42}\left\{\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$\Rightarrow A^{-1} = \dfrac{1}{42}\left\{\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}\right\}$ (1)

Learning Outcomes

Identifies the properties of determinants and verifies through examples, Computing the inverse of a matrix, Verifying the properties of inverse of matrices.

Q.9

(i) If A and B are matrices of order 3 such that $|A| = -1; |B| = 3$, then $|3AB|$ is

    (a)-9 (b)-27 (c)-81   (d)9   (1)

(ii) If $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, Show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

           (4)

(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) (c)-81 (since $|3AB| = 27|A||B|$)   (1)

(ii) $|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = \sec^2 x \neq 0$, therefore A is invertible.   (1)

$adjA = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\sec^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$ (1)

$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \times \frac{1}{\sec^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$= \cos^2 x \begin{bmatrix} 1 - \tan^2 x & -2\tan x \\ \tan x + \tan x & -\tan^2 x + 1 \end{bmatrix}$ (1)

$= \cos^2 x \begin{bmatrix} 1 - \dfrac{\sin^2 x}{\cos^2 x} & -2\dfrac{\sin x}{\cos x} \\ 2\dfrac{\sin x}{\cos x} & -\dfrac{\sin^2 x}{\cos^2 x} + 1 \end{bmatrix}$

$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & -\sin^2 x + \cos^2 x \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$ (1)

Learning Outcomes

Identifies the properties of determinants and verifies through examples, Redefine the area of triangle using determinants, Solving problems related to co linearity of three points.

Q.10

(i) If $A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$ What is the value of $|3A|$?   (1)

(ii) Find the equation of the line joining the points (1,2)and (-3, -2) using determinants. (2)

(Scores: 3; Time: 6 mts)

Scoring Indicators

(i) $|3A| = 3^3|A| = 27\begin{vmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & -1 & -2 \end{vmatrix} = 27(-2+3) = 27$ (1)

(ii) Let (x,y) be the coordinate of any point on the line, then (1,2), (-3, -2) and (x, y) are collinear.
   Hence the area formed will be zero.

$\begin{vmatrix} 1 & 2 & 1 \\ -3 & -2 & 1 \\ x & y & 1 \end{vmatrix} = 0$ (1)

$\Rightarrow (-2-y) - 2(-3-x) + 1(-3y+2x) = 0$

$\Rightarrow 4x - 4y + 4 = 0 \Rightarrow x - y + 1 = 0$ (1)

Learning Outcomes

Identifies the properties of determinants and verifies through examples, Identifies the methods of evaluating determinants, Solving the system of equations

Q.11

(i) Find the values of x in which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ (1)

(ii) Using the property of determinants, show that the points
   $A(a, b+c)$, $B(b, c+a)$, $C(c, a+b)$ are collinear. (2)

(iii) Examine the consistency of system of following equations:
   $5x - 6y + 4z = 15$, $7x + y - 3z = 19$, $2x + y + 6z = 46$ (2)

(Scores:5; Time: 10 mts)

SCORING INDICATORS

(i) $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \Rightarrow 3 - x^2 = 3 - 8 \Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8}$ (1)

$$\text{(ii) } \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \overset{C_2 \to C_2 + C_1}{=} \begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0$$ (1)

since area is zero, points are collinear (1)

iii) The coefficient $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$ (1)

$|A| = 5(24+3) + 6(428+6) + 4(7-8) = 419 \neq 0$

Since, the system is consistent and has unique solution. (1)

Learning Outcomes

Identifies the methods of evaluating determinants, Computing the inverse of a matrix, Solving the system of equations

Q.12

If A = $\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$

(i) Find $|A|$ (1)
(ii) Find A^{-1} (2)
(iii) Solve the linear equations   3x- 2y + 3z = 8
$\qquad\qquad\qquad\qquad\qquad$ 2x + y – z = 1
$\qquad\qquad\qquad\qquad\qquad$ 4x – 3y + 2z= 4 (2)

(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) $|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = 3(2 - 3) + 2 (4 + 4) + 3 (-6 - 4) = - 17$ (1)

(ii) $|A| \neq 0$, hence its inverse exists.

$A^{-1} = \dfrac{1}{|A|} adj\ A$ (1)

$C_{11} = -1, C_{12} = -8, C_{13} = -10 \quad C_{21} = -5, \ C_{22} = -6, C_{23} = 1, C_{31} = -1, C_{32} = 9 \ C_{33} = 7$

$adj\ A = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}, A^{-1} = -\dfrac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$ (1)

(iii)   The given system of linear equations is of the form

$AX = B,$ where $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$X = A^{-1}.B = -\dfrac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

(1)

$= \dfrac{-1}{17} \begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix} = \dfrac{-1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\therefore$ We have, $x = 1$, $y = 2$, $z = 3$

(1)

Learning Outcomes

Identifies the methods of evaluating determinants, Computing the inverse of a matrix, Solving the system of equations

Q.13

'Arjun' purchased 3 pens, 2 purses and 1 instrument box and pays Rs. 410. From the same Shop 'Deeraj' purchases 2 pens, 1 purse and 2 instrument boxes and pays Rs.290, while 'Sindhu' purchases 2pens, 2 purses, 2 instrument boxes and pays Rs. 440.

(i) Translate the equation into system of linear equations (2)
(ii) The cost of one pen, one purse and one instrument box using matrix method. (4)

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i) Let the price of one pen is Rs.x, one purse is Rs.y and one instrument box be Rs.z
$3x + 2y + z = 410$; $2x + y + 2z = 290$; $2x = 2y + 2z = 440$ (1)

(ii)The system can be represented by the matrix equation $AX = B$

Where; $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 410 \\ 290 \\ 440 \end{bmatrix}$ (1)

$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{vmatrix} = -4 \neq 0$ (1)

$C_{11} = -2, C_{12} = 0, C_{13} = 2, C_{21} = -2, C_{22} = 4, C_{23} = -2, C_{31} = 3, C_{32} = -4, C_{33} = -1$

$adjA = \begin{bmatrix} -2 & -2 & 3 \\ 0 & 4 & -4 \\ 2 & -2 & -1 \end{bmatrix}$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-4} \begin{bmatrix} -2 & -2 & 3 \\ 0 & 4 & -4 \\ 2 & -2 & -1 \end{bmatrix} \qquad (1)$$

Solution is given by

$$X = A^{-1}B = \frac{1}{-4} \begin{bmatrix} -2 & -2 & 3 \\ 0 & 4 & -4 \\ 2 & -2 & -1 \end{bmatrix} \times \begin{bmatrix} 410 \\ 290 \\ 440 \end{bmatrix} \qquad (1)$$

$$= \frac{1}{-4} \begin{bmatrix} -80 \\ -600 \\ -200 \end{bmatrix} = \begin{bmatrix} 20 \\ 150 \\ 50 \end{bmatrix} \qquad (1)$$

Hence the cost one pen is Rs.20, one purse is Rs. 150 and one instrument box is Rs. 50.

CHAPTER - 5

CONTINUITY AND DIFFERENTIABILITY

Learning Outcomes

Solves problems related to continuity of functions, solves problems using logarithmic differentiation and finds the derivatives using parametric form

Q.1

(i) Choose the correct answer from the bracket. If $x = \sin t$ and $y = \cos t$, $then$ $\frac{dy}{dx}$ $is\ equal\ to\ ...$

{ *(a)* $\tan t$, *(b)* $\cot t$, *(c)* $-\tan t$, *(d)* $-\cot t$ } *(1)*

(ii) Find the value of k such that the function $f(x)$ defined by

$$f(x) = \begin{cases} kx - 1 \text{ , if } x \leq \frac{\pi}{2} \\ \sin x \text{ , if } x > \frac{\pi}{2} \end{cases} \text{ is a continuous function } at\ x = \frac{\pi}{2}$$ *(2)*

(iii) *If* $y^x = e^{y-x}$, *Prove that* $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$ *(3)*

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i) *(c)* $-\tan t$ *(1)*

(ii) $k \frac{\pi}{2} - 1 = 1$ *(1)*

$k = \frac{4}{\pi}$ *(1)*

iii) $x \log y = y - x$ *(1)*

$\frac{dy}{dx}\left(\frac{x}{y} - 1\right) = -1 - \log y$ *(1)*

$\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$ *(1)*

Learning Outcomes

Solves problems related to continuity of functions, solves problems using logarithmic differentiation and finds the derivatives using parametric form

Q.2

(i) Choose the correct answer from the bracket. *If* $y = \log \cos x$, *then the value of* $\frac{dy}{dx}$ $at\ x = \frac{\pi}{4}$ $is\ ...$

{ *(a)* ∞ , *(b)* 1 , *(c)* 0 , *(d)* -1 } *(1)*

(ii)   Find the value of $'a'$ such that the function $f(x)$ defined by $f(x) =$
$$\begin{cases} ax + 3, & \text{if } x \le 2 \\ a^2x - 1, & \text{if } x > 2 \end{cases}$$
is continuous function at $x = 2$ *(2)*

(iii)   *If* $x = a\cos^3\theta$ *and* $y = a\sin^3\theta$ *Prove that* $1 + \left(\frac{dy}{dx}\right)^2 = \sec^2\theta$ *(3)*

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i)   *(d)* -1 *(1)*

(ii) $2a + 3 = 2a^2 - 1$ *(1)*
       $a = 2, \ a = -1$ *(1)*

(iii) $\frac{dx}{d\theta} = 3a\cos^2\theta \, (-\sin\theta)$ $\frac{dy}{d\theta} = 3a\sin^2\theta \, (\cos\theta)$ *(1)*
$\frac{dy}{dx} = -\tan\theta$ *(1)*
$1 + \left(\frac{dy}{dx}\right)^2 = \sec^2\theta$ *(1)*

Learning Outcomes

Solves problems related to continuity of functions, solves problems using logarithmic differentiation and finds the derivatives using parametric form

Q.3

(i)   Choose the correct answer from the bracket. *If* $x = at^2$, $y = 2at$, *then the value of* $\frac{dy}{dx}$ *at* $t = 1$ *is*
        $\{ (a) \ \mathbf{0}, \ (b) \ \mathbf{1}, \ (c) \ \mathbf{2}, \ (d) \ \infty \}$ *(1)*

(ii)   *Is the function defined by* $f(x) = \begin{cases} x + 3, & \text{if } x \le 1 \\ x - 3, & \text{if } x > 1 \end{cases}$ *a continuous function?*
Justify   *(2)*

(iii) Find $\frac{dy}{dx}$, *if* $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ *(3)*

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i)   *(a)* **1**   *(1)*
        Left limit $\ne$ *Right limit*   *(1)*
    It is not continuous   *(1)*
   (iii). $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ *(1)*
$y = \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) = \tan^{-1}\tan\left(\frac{\pi}{2} - x\right) = \quad \frac{\pi}{2} - x$ *(1)*

$$\frac{dy}{dx} = -1 \qquad\qquad (1)$$

Learning Outcomes

Identifies the method to find derivatives, when a function is defined implicitly, Solves problems related to continuity of functions, solves problems using logarithmic differentiation

Q.4

(i) Choose the correct answer from the bracket. *If* $x^2 + 2xy + 2y^2 = 1$, *then* $\frac{dy}{dx}$ *at the point where* $y = 1$ *is equal to*

{ *(a)* **0** , *(b)* **1** , *(c)* **2** , *(d)* **−1** } (1)

(ii) *Find the relationship between* **a** *and* **b** *so that the function* **f** *defined by*

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 2 \\ bx - 3, & \text{if } x > 2 \end{cases} \text{ is continuous at } x = 2$$

(2)

(iii) *If* $x^y y^x = 16$, *then find* $\frac{dy}{dx}$ *at (2, 2)* (3)

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i)   (a) 0   (1)
(ii) $2a + 1 = 2b - 3$ (1)
        $a - b = -2$ (1)
(iii) $y \log x + x \log y = \log 16$ (1)

$$y\frac{1}{x} + \log x \frac{dy}{dx} + \frac{x}{y}\frac{dy}{dx} + \log y = 0 \qquad (1)$$

$$\frac{dy}{dx}(2,2) = -1 \qquad\qquad (1)$$

Learning Outcomes

Identifies the method to find derivatives, when a function is defined implicitly, Solves problems related to continuity of functions, solves problems using logarithmic differentiation

Q.5

(i) *Find the values of* **a** *and* **b** *such that the function* **f** *defined by* $f(x) =$

$$\begin{cases} 2, & \text{if } x \le 1 \\ ax + b, & \text{if } 1 < x < 5 \\ 14, & \text{if } x \ge 5 \end{cases}$$

is a continuous function (3)
   (ii) *Verify Lagrange's Mean Value theorem for the function* $f(x) = 2x^2 - 10x + 29$
     in [2, 9] (3)

(Scores:6; Time: 12 mts)

Scoring Indicators

(i) $a + b = 2$ *(1)*
        $5a + b = 14$ *(1)*
        $a = 3, \ b = -1$ *(1)*
        $f(2) = 17 \ \ f(9) = 101$ *(1)*
ii) $f^1(c) = 4c - 10 = 12$ *(1)*

        $c = \frac{22}{4} \in [2,9]$ *(1)*

Learning Outcomes

Identifies the method to find derivatives, when a function is defined implicitly, Solves problems related to continuity of functions, solves problems using logarithmic differentiation

Q.6

   (i) *If* $f(x) = \log\left[e^x \left(\frac{3-x}{3+x}\right)^{\frac{1}{3}}\right]$ *, then find* $f^1(1)$ (3)

   (ii) *Verify Rolle's theorem for the function* $f(x) = \sin 3x$ *on* $\left[0, \frac{\pi}{3}\right]$ (3)

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i) $f(x) = \log e^x + \frac{1}{3}[\log(3-x) - \log(3+x)\,]$ *(1)*
        $f^1(x) = 1 + \frac{1}{3}\left(\frac{-1}{3-x} - \frac{1}{3+x}\right)$ *(1)*
        $f^1(1) = \frac{3}{4}$ *(1)*
(ii) $f(0) = 0 \ \ f\left(\frac{\pi}{3}\right) = 0$ *(1)*

        $f^1(c) = 3\cos 3c = 0$ *(1)*
        $c = \frac{\pi}{6} \in \left[0, \frac{\pi}{3}\right]$ *(1)*

Learning Outcomes

Identifies the method to find derivatives and finds a relation satisfying the function

Q.7

Let $y = \left(x + \sqrt{1+x^2}\right)^m$

i)Find $\dfrac{dy}{dx}$ (2)

ii)Show that $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - m^2 y = 0$ (3)

(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) $\dfrac{dy}{dx} = m\left(x + \sqrt{1+x^2}\right)^{m-1}\left[1 + \dfrac{1}{2\sqrt{1+x^2}}2x\right]$

$= m\left(x + \sqrt{1+x^2}\right)^{m-1}\left[\dfrac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}\right]$ (1)

$= \dfrac{m\left(x + \sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}} = \dfrac{my}{\sqrt{1+x^2}}$ (1)

ii)We have $\sqrt{1+x^2}\,\dfrac{dy}{dx} = my$ Diff:w r t x

$\sqrt{1+x^2}\,\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\dfrac{1}{2\sqrt{1+x^2}}(2x) = m\dfrac{dy}{dx}$ (1)

multiplying by $\sqrt{1+x^2}$

$(1+x^2)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}x = m\dfrac{dy}{dx}\sqrt{1+x^2}$ (1)

$(1+x^2)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}x = m(my)$

$(1+x^2)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}x - m^2 y = 0$ (1)

Learning Outcomes

Identifies the method to find derivatives when a function is defined implicitly, solves problems using logarithmic differentiation

Q.8

i)Find $\dfrac{dy}{dx}$ if sinx+cosy=xy (2)

ii)Find $\dfrac{dy}{dx}$ if $y = (\sin x)^{\log x}$ (3)

37

(Scores: 5; Time: 10 mts)

Scoring Indicators

i)Cosx-siny $\dfrac{dy}{dx}$ =x$\dfrac{dy}{dx}$ +y   (1)

$\dfrac{dy}{dx}$ (x+siny)=cosx-y   (1)

$\dfrac{dy}{dx} = \dfrac{\cos x - y}{\sin y + x}$

ii)Let y=$(\sin x)^{\log x}$

logy=log(sinx)^{logx}

logy=logx.log(sinx)   (1)

Diff:wrt x

$\dfrac{1}{y}\dfrac{dy}{dx} = \log x\left(\dfrac{1}{\sin x}\right)\cos x + \log(\sin x)\dfrac{1}{x}$ (1)

$\dfrac{dy}{dx} = y\left(\dfrac{\log \sin x}{x} + \log x \cot x\right)$

$\dfrac{dy}{dx} = (\sin x)^{\log x}\left(\dfrac{\log \sin x}{x} + \log x \cot x\right)$ (1)

Learning Outcomes

Identifies the method to find derivatives, when a function is defined implicitly, Solves problems related to continuity of functions, solves problems using logarithmic differentiation .

Q.9

Differentiate the following

i) y=$\sin\left[\sqrt{\sin \sqrt{x}}\right]$ (3)

ii) y=$\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$ (2)

(Scores : 5; Time: 10 mts)

Scoring Indicators

i) $\dfrac{dy}{dx} = \cos\sqrt{\sin \sqrt{x}}\; D\left(\sqrt{\sin \sqrt{x}}\right)$

$= \cos\sqrt{\sin \sqrt{x}}\left[\dfrac{1}{2\sqrt{\sin \sqrt{x}}}\right]D\left(\sin \sqrt{x}\right)$ (1)

$$= \cos \sqrt{\sin \sqrt{x}} \left[\frac{1}{2\sqrt{\sin \sqrt{x}}} \right] \cos \sqrt{x} \left(\frac{1}{2\sqrt{x}} \right)$$

$$= \frac{\cos \sqrt{\sin \sqrt{x}} \cos \sqrt{x}}{4\sqrt{x}\sqrt{\sin \sqrt{x}}}$$

(1)

(ii)   Put x=tant

$$y = \tan^{-1} \left(\frac{\sqrt{1 + \tan^2 t} + 1}{\tan t} \right)$$

$$= \tan^{-1} \left(\frac{\sec t + 1}{\tan t} \right) = \tan^{-1} \left(\frac{\cos t + 1}{\sin t} \right)$$

(1)

$$= \tan^{-1} \left(\frac{2\cos^2 \frac{t}{2}}{2\sin \frac{t}{2} \cos \frac{t}{2}} \right) = \tan^{-1} \left(\cot \frac{t}{2} \right)$$

$$= \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \frac{t}{2} \right) \right) = \frac{\pi}{2} - \frac{t}{2}$$

(1)

$$= \frac{\pi}{2} - \frac{\tan^{-1} x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2} \left(\frac{1}{1 + x^2} \right)$$

(1)

Learning Outcomes

Identifies the method to find derivatives when a function is defined implicitly .

Q.10

Let siny=xsin(a+y)

i)Express x as a function of y   (1)

ii)Show that $\dfrac{dy}{dx} = \dfrac{\sin^2 (a + y)}{\sin a}$

(2)

(Scores:3; Time: 6 mts)

Scoring Indicators

Sin y=xsin(a+y)   (1)

$$x = \frac{\sin y}{\sin(a + y)}$$

Diff:wrt y

$$\frac{dx}{dy} = \frac{\sin(a + y)\cos y - \sin y\cos(a + y)}{\sin^2(a + y)}$$

(1)

$$= \frac{\sin(a + y - y)}{\sin^2(a + y)} = \frac{\sin(a)}{\sin^2(a + y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$

(1)

Learning Outcomes

Understands the concept of continuity and differentiability and Solves problems related to continuity of functions

Q.11

   i)   Let f:R→R given by f(x)= $[x]$,where $[x]$ denotes greatest
           integer less than or equal to x. Is f(x) continuous at x=0?   (1)
   ii)   Let f:R→R given by f(x)= $|x|$. Is f(x) differentiable at x=0?   (2)
   iii)   Let f:R→R given by f(x)= $|x|$.Is f(x) differentiable at x≠0?   (2)

(Scores: 5; Time: 8 mts)

Scoring Indicators

   i)   f(x) is not continuous at x=0 ,since it breaks at integral values

   ii) $f^1(0+) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$

                                                     (1)

        $f^1(0-) = \lim\limits_{h \to 0} \dfrac{f(0 - h) - f(0)}{-h}$

                                                     (1)

        $f^1(0+) \neq f^1(0-)$ so not differentiable at x = 0
        At x>0,f(x)=x ,a straight line graph   (1)
        At x<0,f(x)=-x, a straight line graph. So differentiable at x≠0   (1)

CHAPTER – 6

APPLICATION OF DERIVATIVES

Learning Outcomes

Interprets rate of change of quantities as derivatives.
Applies the concept in various situations, finds the intervals in which a function is increasing or decreasing, identifies the local maximum and local minimum of function and Derives the conditions for local maxima and local minima using first and second derivatives.

Q.1

   (i)   Choose the correct answer from the bracket. The rate of change of the area of a circle with respect to its radius r at r = 10cm is
      { (a) 10π (b) 20π (c) 30π (d) 40π}   (1)

   (ii)   Find the intervals in which the function f given by $f(x) = x^2 - 6x + 5$ is
     (a) Strictly increasing   (b) Strictly decreasing   (2)

   (iii) Find the local minimum and local maximum value, if any,
     of the function $f(x) = x^3 - 6x^2 + 9x + 8$ (2)

(Scores: 5; Time: 10 mts)

Scoring Indicators

   (i)   (b) $A = \pi r^2$
$$\frac{dA}{dr} = 2\pi r$$
$$\frac{dA}{dr} = 2\pi \times 10 = 20\pi \qquad\qquad\qquad\qquad (1)$$

   (ii) $f^1(x) = 2x - 6$
        $2x - 6 = 0$
        $x = 3$
        $(-\infty,\ 3\ )$ is strictly decreasing   (1)
        $(3,\ \infty\ )$ is strictly increasing   (1)

   (iii) $f^1(x) = 3x^2 - 12x + 9$
        $f^{11}(x) = 6x - 12$
        $For\ maxima, minima\ f^1(x) = 0 \ \rightarrow 3x^2 - 12x + 9 = 0$
        $3(x - 3)(x - 1) = 0$ (1)
                          $x = 3,\ \ x = 1$
        At $x = 3\ f^{11}(x) = 6 \times 3 - 12 = 18 - 12 = 6 > 0$,
        $f$is minimum, the local minimum value of $f = 8$

        At $x = 1\ f^{11}(x) = 6 \times 1 - 12 = -6 < 0$,
        $f$is maximum, the local maximum value of $f = 12$ (1)

Learning Outcomes

Recalls the equation of straight lines and their properties,
Finds the equation of tangent and normal to a curve at a point, Applies the concept to find approximate changes in one variable with respect to another.

Q.2

(i)   Choose the correct answer from the bracket. The slope of the tangent to the curve $y = x^3 - 2x + 3$ at $x = 1$ is …

{ (a) 0   (b) 1   (c) 2   (d) 3 }   (1)

(ii)   Find points on the curve $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at which the tangents are

(a) Parallel to x-axis   (b) parallel to y – axis.   (2)

(iii)   Use differential to approximate $\sqrt{25.6}$.   (3)

                                                                          (Scores: 6; Time: 12 mts)

Scoring Indicators

(i) (b) 1 , since $\frac{dy}{dx} = 3x^2 - 2$

   $\frac{dy}{dx}(x = 1) = 3 - 2 = 1$ (1)

(ii) $\frac{2x}{25} + \frac{2y}{9} \cdot \frac{dy}{dx} = 0,$ $\frac{dy}{dx} = \frac{-9x}{25y}$

   $\frac{dy}{dx} = 0$, since tangents are parallel to x- axis

   $\frac{-9x}{25} = 0,$ $x = 0$

   $\therefore y = \pm 3$ The points are (0, 3) and (0, -3)   (1)

   $\frac{-25y}{9x} = 0$, since tangents are parallel to y- axis, y = 0

   $\therefore x = \pm 5$ The points are (5, 0) and (-5, 0)   (1)

iii)   Let $y = \sqrt{x}$, $x = 25, \Delta x = 0.6$

   $\Delta y = \sqrt{x + \Delta x} - x = \sqrt{25.6} - \sqrt{25}$

   $= \sqrt{25.6} - 5$ (1)

$\sqrt{25.6} = \Delta y + 5$ $dy = \frac{dy}{dx} \Delta x$

   $dy = \frac{1}{2\sqrt{x}} 0.6$ (1)

$$dy = \frac{1}{2 \times 5} 0.6$$

   $dy = \frac{1}{10} 0.6 = 0.06$

   $\sqrt{25.6} = 0.06 + 5$ $= 5.06$ (1)

Learning Outcomes

Finds the intervals in which a function is increasing or decreasing,
Identifies the equation of tangents, Derives absolute maxima and minima using first and second derivatives

Q.3

(i)   Choose the correct answer from the bracket. The function $f(x) = cos\ x$ is strictly decreasing in the interval is …

   { (a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,   (b) $(0,\ 2\pi)$, (c) $(0,\ \pi)$, (d) $(-\pi,\ \pi)$ }   (1)

(ii)   Find the equation of the tangent to the curve $y = x^2 - 4x + 5$ which is parallel to the line $2x + y + 5 = 0$ (2)

(iii)   Find the absolute maximum and minimum values of a function f given by $f(x) = x^3 + 3x^2 - 9x + 8$ on [-4,2]   (3)

(Scores:6; Time: 12 mts)

Scoring Indicators

(i)   (c) $(0,\ \pi)$ (1)

(ii) $\frac{dy}{dx} = 2x - 4$

   $2x - 4 = -2$ (1)

   $x = 1, \quad y = 2$

   Equation of tangent is $y - 2 = -2(x - 1)$ (1)

(iii) $f^1(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3)$

   $f^1(x) = 3(x + 3)(x - 1)$ (1)

   $f^{11}(x) = 6x + 6$

   For maxima, minima $f^1(x) = 0 \rightarrow 3(x + 3)(x - 1)\ = 0$
   $3(x + 3)(x - 1) = 0$
   $x = -3, \quad x = 1$ (1)

   $$f(1) = 21$$
   $$f(-3) = 35$$
   $$f(-4) = 28$$
   $$f(2) = 20$$

   The absolute maximum value of f [-4,2] is 35 , at x= -3   (1)

Learning Outcomes

Finds the intervals in which a function is increasing or decreasing, Recalls the equation of straight lines and their properties, Finds the equation of tangent and normal to a curve at a point .

Q.4

i)Prove that f(x)=logsinx is strictly increasing in $\left[0,\frac{\pi}{2}\right]$ and strictly decreasing in $\left[\frac{\pi}{2},\pi\right]$

(2)

ii)Consider the parametric forms $x=t+\dfrac{1}{t}$ and $y=t-\dfrac{1}{t}$ of the curve then

a)Find $\dfrac{dy}{dx}$

(2)

b)Find the equation of tangent at t=2   (1)

c)Find the equation of normal at t=2   (1)

(Scores: 6 ; Time:12 mts)

Scoring Indicators

i) $f^{1}(x)=\dfrac{1}{\sin x}(\cos x)$ =cotx   (1)

In $\left[0,\frac{\pi}{2}\right]$,cotx>0 So f(x) is increasing

In$\left[\frac{\pi}{2},\pi\right]$,cotx <0 So f(x) is decreasing   (1)

ii)a) $\dfrac{dx}{dt}=1-\dfrac{1}{t^{2}}=\dfrac{t^{2}-1}{t^{2}}$

$\dfrac{dy}{dt}=1+\dfrac{1}{t^{2}}=\dfrac{t^{2}+1}{t^{2}}$

So $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{t^{2}+1}{t^{2}-1}$ (2)

b) At t=2 , x= $\dfrac{5}{2}$ and y=$\dfrac{3}{2}$

At t=2,slope of the tangent $\dfrac{dy}{dx}=\dfrac{5}{3}$

Equation of tangent is $y-\dfrac{3}{2}=\dfrac{5}{3}\left(x-\dfrac{5}{2}\right)$ (1)

ie , 10x-6y-16=0 , 5x-3y-8 = 0

c) At t=2,slope of the normal=$\dfrac{-1}{slope\ of\ \tan gent}=\dfrac{-3}{5}$

Equation of normal is $y-\dfrac{3}{2}=\dfrac{-3}{5}\left(x-\dfrac{5}{2}\right)$ (1)

ie, 3x+5y=15

Learning Outcomes

Applies the concept to find approximate changes in one variable with respect to
another, Apply the concept of derivatives in life situation and finds the solution

using the conditions for local maxima and local minima

Q5

i) Find the approximate change in volume of a cube of side x metre caused by an increase in the side by 3% (3)

ii) A wire of length 28m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum using differentiation? (4)

(Scores: 7; Time:12 mts)

Scoring Indicators

i)Let x be the side and V be the volume of the cube

Change in side ,Δx=3% of x

$$= \frac{3x}{100}$$ (1)

Change in volume , $\Delta V = \frac{dV}{dx} \Delta x$

$$= \frac{d(x^3)}{dx} \cdot \frac{3x}{100}$$

$$= 0.09x^3$$ (2)

iii) Let x be the length of the square piece and 28-x be the length of the circular piece

Perimetre of square=x

ie , 4a=x , a=x/4

$$\text{Area of square} = a^2 = \frac{x^2}{16}$$ (1)

Circumference of circle =28-x

ie , $2\pi r$= 28-x $\qquad \therefore r = \frac{28-x}{2\pi}$

$$\text{Area of Circle} = \pi r^2 = \pi\left(\frac{28-x}{2\pi}\right)^2$$ (1)

$$= \frac{(28-x)^2}{4\pi}$$

$Total\ area\ A = \frac{x^2}{16} + \frac{(28-x)^2}{4\pi}$

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(28-x)}{4\pi}(-1) = \frac{x}{8} - \frac{28-x}{2\pi}$$

$$\frac{d^2y}{dx^2} = \frac{1}{8} - \frac{1}{2\pi}(-1) \quad , which\ is\ +ve$$ (1)

So Area is minimum

$For\ a\ \min imum\ , \frac{dA}{dx} = \frac{x}{8} + \frac{28-x}{2\pi} = 0$

$$\therefore \frac{\pi x - 4(28 - x)}{8\pi} = 0$$

$$\therefore x = \frac{112}{\pi + 4}$$

So length of square piece $= \dfrac{112}{\pi + 4}$

(1)

Length of other piece $= 28 - \dfrac{112}{\pi + 4} = \dfrac{28\pi}{\pi + 4}$

Learning Outcomes

Identifies the extreme values of f(x) as local maximum and local minimum of the function and Derives the conditions for local maxima and local minima using first and second derivatives.

Q.6

Consider the function $y = \dfrac{\log x}{x}, x > 0$

i)Find the extreme points of f(x)   (2)
ii)find the maximum or minimum value if any   (2)

(Scores: 4; Time:8 mts)

Scoring Indicators

i) $\dfrac{dy}{dx} = \dfrac{x\left(\frac{1}{x}\right) - \log x (1)}{x^2} = \dfrac{1 - \log x}{x^2}$

(1)

At extreme points

$\dfrac{dy}{dx} = 0 \Rightarrow 1 - \log x = 0$

$\Rightarrow \log x = 1$
$\Rightarrow x = e, x > 0$

When x=e, $y = \dfrac{\log e}{e} = \dfrac{1}{e}$

(1)

So the extreme point is (e,1/e)

ii) $\dfrac{d^2 y}{dx^2} = \dfrac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)2x}{x^4} = \dfrac{-3x + 2x\log x}{x^4}$

$= \dfrac{2\log x - 3}{x^3}$

(1)

When x=e, $\dfrac{d^2 y}{dx^2} < 0$

$\therefore$ y is maximum at x=e
Maximum value is 1/e

(1)

Learning Outcomes

Applies the concept of derivatives in life situation and finds the solution using the conditions for local maxima and local minima .

Q.7

A rectangle sheet of tin with adjascent sides 45cm and 24cm is to be made into a box without top,by cutting off equal squares from the corners and folding up the flaps

   i)   Taking the side of the square cut off as x, express the volume of the box as the function of x   (2)

   ii)   For what value of x,the volume of the box will be maximum   (3)

                                         (Scores: 5; Time: 10 mts)

Scoring Indicators

Length of the box	=45-2x	
Breadth of the box	=24-2x	
Height of the box	=x	(1)
VolumeV	=(45-2x)(24-2x)x	
	=(1080-138x+4x²)x	
	=4x³-138x²+1080x	(1)

$$\frac{dy}{dx}=12x^2-276x+1080$$

$$\frac{d^2y}{dx^2}=24x-276$$

                                             (1)

For maxima, $\frac{dy}{dx}=0$

                   12x²-276x+1080=0

                    x²-23x+90=0

                    x = 18, 5

x=18 is impossible ∴ x=5   (1)

when x=5, $\frac{d^2y}{dx^2}<0$

The volume of the box is maximum at x=5   (1)

Learning Outcomes

Recalls the equation of straight lines and their properties,
Finds the equation of tangent and normal to a curve at a point. Applies the concept to find approximate changes in one variable with respect to another.

Q.8
i)   The slope of the tangent to the curve y=x³ inclined at an angle 60º to x-axis is
…….........   (1)

ii) Consider the function $y^2 =4x+5$
   a) Find a point on the curve at which the tangent is parallel to the line y=2x+7 (1)
   b) Find the equation of the tangent at the point of tangency   (2)
   c) Find the approximate value of $\sqrt{0.037}$ (2)

(Scores: 6; Time: 12 mts)

Scoring Indicators

   i)   Slope=tan60=$\sqrt{3}$ (1)

   ii)   a) $y^2 =4x+5$

   $2y\dfrac{dy}{dx} = 4 \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{4}{2y} = \dfrac{2}{y}$ (1)

Given tangent is parallel to y=2x+7

Slope of the tangent =slope of the line

$\dfrac{2}{y} = 2 \Rightarrow$ y=1

When y=1, 1=4x+5 $\Rightarrow$ x=-1   (1)

So the point of contact is (-1,1)

   b) Equation of tangent at (-1,1) is

y-1=2(x+1)   (1)

y=2x+3

   c) Let f(x)= $\sqrt{x}$.

Let x=0.04 and Δx= - 0.003   (1)

 We have

   f(x+Δx) - f(x) = f¹(x) Δx

  f(0.037) - f(0.04)= $\dfrac{1}{2\sqrt{0.04}}(-0.003)$

   $\therefore \sqrt{0.037} = \sqrt{0.04} - \dfrac{0.003}{2(0.2)}$

             =0.1925   (1)

Learning Outcomes

Identifies the local maximum and local minimum of function and estimate the
Absolute maxima and Absolute minima.

Q.9

Consider the function f(x)=x² in[-2,1]
 i)Find the local maximum or minimum if any   (2)
ii)Find the absolute maximum and minimum   (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

    i) $f^1(x)=2x$ $f^{11}(x)=2>0$ So f(x) has local minimum   (1)

            For maximum or minimum $f^1(x)=0$

            2x=0 so x=0

            Local minimum value of f at x=0 is f(0)=0   (1)

            At the end points we have

            x=-2,   f(-2)=4 and   (1)

            x=1 ,   f(1)=1

    ii)   Absolute maximum value of f

              =max{4,1,0}=4

            Absolute minimum value of f

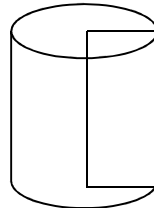   =min{4,1,0}=0   (1)

Learning Outcomes

Applies the concept of derivatives in life situation and finds the solution using the conditions for local maxima and local minima.

Q10.

Of all the Cylinders with given surface area, show that the volume is maximum when height is equal to the diameter of the base (4)

                                          (Scores: 4; Time: 8 mts)

Scoring Indicators

Let r be the radius,h be the height ,
V be the volume and S be the surface area

S= $2\pi r^2 + 2\pi rh$

$\therefore$ h= $\dfrac{S-2\pi r^2}{2\pi r}$ (1)

Now V= $\pi r^2 h$

        = $\pi r^2\left(\dfrac{S-2\pi r^2}{2\pi r}\right) = \dfrac{rS-2\pi r^3}{2}$

$\dfrac{dV}{dr}=\dfrac{S-6\pi r^2}{2}$ and $\dfrac{d^2V}{dr^2}=-6\pi r<0$

                                                (2)

For a maximum, $\dfrac{dV}{dx}=0$

$$\frac{S - 6\pi r^2}{2} = 0$$

$$S - 6\pi r^2 = 0$$

$$2\pi r^2 + 2\pi rh - 6\pi r^2 = 0$$

So   h=2r

So volume is maximum when h=2r   (1)

INTEGRATION

Learning Outcomes

Recalls the differentiation and standard results of Integration

Q.1

Match the following (4)

Integrals	Functions
i) $\int \sin x\, dx$	$\log|\sin x| + c$
ii) ii) $\int \cos x\, dx$	$-\log|\cos x| + c$
iii) $\int \tan x dx$	$-\cos x + c$
iii) $\int \cot x\, dx$	$\sin x + c$

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $\int \sin x\, dx = -\cos x + c$ (1)

ii) $\int \cos x\, dx = \sin x + c$ (1)

iii) $\int \tan x dx = -\log|\cos x| + c$ (1)

iv) $\int \cot x\, dx = \log|\sin x| + c$ (1)

Learning Outcomes

Familarise the concept of definite integral; Applies in various problems

Q.2

Choose the correct answer from the bracket. Verify your answer

i) $\int_{-1}^{1} x dx$ is equal to (1)
   (0, 1 , 2 ,3)

ii) $\int_{0}^{3} \frac{3}{x} dx$ is equal to (1)
   (3log3 , log3 , 9log3 3log9)

iii) $\int_{0}^{\frac{\pi}{2}} \sin x\, dx$ (1)
   (0 ,−1 , 1 , 2)

iv) $\int_{-2}^{4} |x + 1|\, dx$ is equal to (3)
   (12 , 13 , 14 , 15)

(Scores: 6 ; Time: 12 mts)

Scoring Indicators

i) $\int_{-1}^{1} x\,dx = \left(\frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{2}(1\text{-}1) = 0$ (1)

ii) $\int_{1}^{3} \frac{3}{x}\,dx = 3\,(logx)_{1}^{3} = 3[\,log3 - log1\,] = 3log3$ (1)

iii) $\int_{0}^{\frac{\pi}{2}} sinx\,dx = -(cosx)_{0}^{\frac{\pi}{2}} = -(cos\frac{\pi}{2} - cos0) = -(0 - 1) = 1$ (1)

iv) $\int_{-2}^{4} |x+1|\,dx = \int_{-2}^{-1} -(x+1)\,dx + \int_{-1}^{4} (x+1)\,dx$ (1)

$= -\left(\frac{x^2}{2} + x\right)_{-2}^{-1} + \left(\frac{x^2}{2} + x\right)_{-1}^{4}$ (1)

$= -\left(\frac{1}{2} - 1 - 2 + 2\right) + \left(8 + 4 - \frac{1}{2} + 1\right) = 13$ (1)

Learning Outcomes

Applies the method of substitution in various problems.

Q.3

Evaluate

i) $\int (x^6 - 2x^4 + 2x^2)\,dx$ (1)

ii) $\int (x^2 + 1)\sqrt{x+1}\,dx$ (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $\int (x^6 - 2x^4 + 2x^2)\,dx = \frac{x^7}{7} - 2\frac{x^5}{5} + 2\frac{x^3}{3} + c$ (1)

ii) put $x + 1 = t^2$ $dx = 2t\,dt$ and $x^2 + 1 = t^4 - 2t^2 + 2$ (1)

$\int (x^2 + 1)\sqrt{x+1}\,dx = \int (t^4 - 2t^2 + 2)\,2t^2\,dt$

    (1)

$= \int (2t^6 - 4t^4 + 4t^2)\,dt = 2\frac{t^7}{7} - 4\frac{t^5}{5} + 4\frac{t^3}{3}$

$= 2\frac{(x+1)^{\frac{7}{2}}}{7} - 4\frac{(x+1)^{\frac{5}{2}}}{5} + 4\frac{(x+1)^{\frac{3}{2}}}{3}$ (1)

Learning Outcomes

Applies trigonometric identities in integration.

Q.4

Evaluate

i) $\int \frac{1}{t^2} dt$ (1)

ii) $\int \frac{cosx-sinx}{1+2sinxcosx} dx$ (3)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

i) $\int \frac{1}{t^2} dt = \frac{-1}{t}$ +c (1)

ii) $\int \frac{cosx-sinx}{1+2sinxcosx} dx = \int \frac{cosx-sinx}{sin^2x+cos^2x+2sinxcosx} dx$ (1)

$= \int \frac{cosx-sinx}{(sinx+cosx)^2} dx$ put $sinx + cosx = $ t $\Rightarrow (cosx - sinx)dx$ =dt (1)

$= \int \frac{1}{t^2} dt = \frac{-1}{t}$ +c $= \frac{-1}{sinx+cosx} + c$ (1)

Learning Outcomes

Illustrates the properties of definite integrals

Q.5

Evaluate

i) $\int_0^1 e^{-5x} dx$ (2)

ii) $\int_0^1 xe^{-5x} dx$ (3)

(Scores: 5; Time: 10 mts)

Scoring Indicators

i) $\int_0^1 e^{-5x} dx = \left(\frac{e^{-5x}}{-5}\right)_0^1$ (1)

$= \frac{-1}{5} (e^{-5} - e^0) = \frac{-1}{5} (e^{-5} - 1)$ (1)

ii) $\int_0^1 xe^{-5x} dx = \left(-x\frac{e^{-5x}}{5} - \frac{e^{-5x}}{25}\right)_0^1$

$= \frac{1}{25} (-5e^{-5} - e^{-5} + 0 + 1)$ (1)

$= \frac{1}{25} (1 - 6e^{-5})$ (1)

Learning Outcomes

Derives the definite integral as the limit of a sum.

Q.6

Evaluate the following definite integral as the limit of a sum $\int_0^4 (x+1)dx$ (4)

(Scores: 4; Time: 10 mts)

Scoring Indicators

$f(x) = x + 1$, $a = 0$, $b = 5$, $nh = b - a = 4 - 0 = 4$ (1)

By definition

$\int_a^b f(x)dx = \lim\limits_{h\to0} h\{f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)\}$ (1)

$\int_0^4 (x+1)dx = \lim\limits_{h\to0} h\{1 + (h+1) + (2h+1) + (3h+1) \ldots + ((n-1)h+1)\}$

$= \lim\limits_{h\to0} h\{n + h + 2h + 3h \ldots + (n-1)h\}$

$= \lim\limits_{h\to0} h\{n + h[1 + 2 + 3 \ldots + (n-1)]\}$

$= \lim\limits_{h\to0} h\{n + h[1 + 2 + 3 \ldots + (n-1)]\}$ (1)

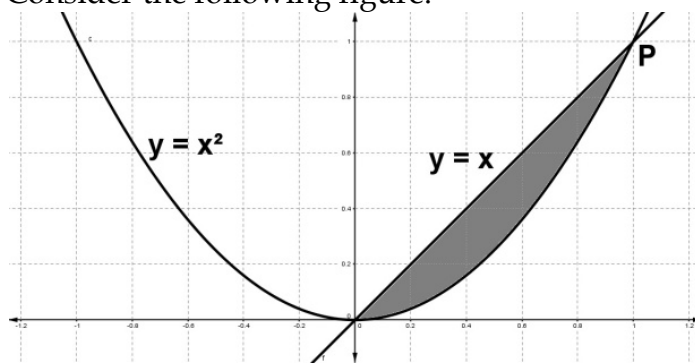$= \lim\limits_{h\to0} h\{n + h \frac{n(n-1)}{2}\} = 12$ (1)

CHAPTER – 8

APPLICATION OF INTEGRALS

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration.

Q.1

Consider the following figure.

    (i) Find the point of intersection (P) of the parabola and the line.   (2)
    (ii) Find the area of the shaded region.   (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) We have, $y = x^2$ and $y = x \Rightarrow x = x^2 \Rightarrow$

    $\Rightarrow x^2 - x = 0 \Rightarrow \Rightarrow x(x-1) = 0 \Rightarrow x = 0,1$ (1)

    When $x = 0$, $y = 0$ and $x = 1$, $y = 1$.

    Therefore the points of intersections are (0,0) and (1,1).   (1)

(ii) Required area $= \int_0^1 x\,dx - \int_0^1 x^2\,dx = \left(\frac{x^2}{2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ (2)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration.

Q.2

Consider the following figure:
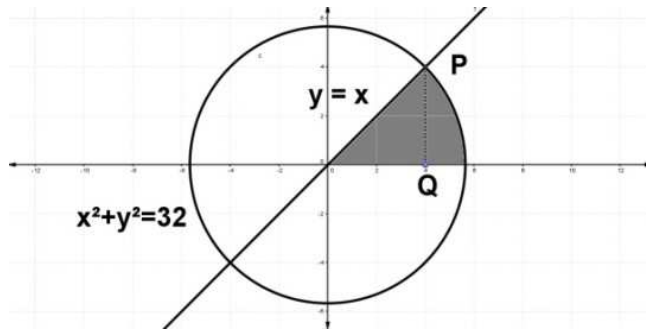
i)   Find the point of Intersection P of the circle $x^2 + y^2 = 32$ and the line y = x. (1)

(ii) Find the area of the shaded region.   (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

$$x^2 + x^2 = 32 \Rightarrow 2x^2 = 32 \Rightarrow x^2 = 16,$$
$$x=4$$

(i)

Therefore the point of intersection P is (4,4).   (1)

(ii) We have $x^2 + y^2 = 32 \Rightarrow y = \sqrt{32 - x^2}$.

The required area $= \int\limits_{0}^{4} x\,dx + \int\limits_{4}^{\sqrt{32}} \sqrt{32 - x^2}\,dx = \left(\dfrac{x^2}{2}\right)_{0}^{4} + \left[\dfrac{x}{2}\sqrt{32 - x^2} + \dfrac{32}{2}\sin^{-1}\dfrac{x}{\sqrt{32}}\right]_{4}^{\sqrt{32}}$ (1)

$= \left(\dfrac{16}{2}\right) + \left[16\sin^{-1}1 - \dfrac{4}{2}\sqrt{32 - 16} - 16\sin^{-1}\dfrac{4}{\sqrt{32}}\right]$ (1)

$= 8 + \left[16\dfrac{\pi}{2} - 2\times 4 - 16\sin^{-1}\dfrac{1}{\sqrt{2}}\right]$

$= 8 + [8\pi - 8 - 4\pi] = 4\pi$ (1)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration, Identifies the area bounded by the curve $y = f(x); y = g(x)$ as

$$\left|\int\limits_{a}^{b} [f(x) - g(x)]dx\right|$$

Q.3

(i) Draw the graph of the function $y = x^2$ and $x = y^2$ in a coordinate axes.   (2)

(ii) Find the point of intersection of the above graphs.   (1)

(iii)   Find the area of the region bounded by the above two curves.   (2)

(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) The two functions are parabolas as shown in the figure.
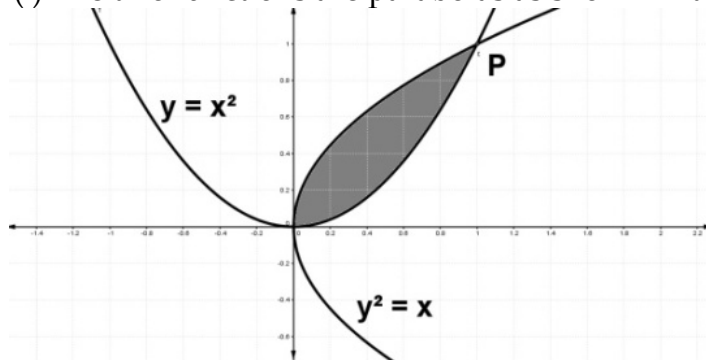
(2)

(ii) We have, $y = x^2$ and $x = y^2$

$x = (x^2)^2 \Rightarrow x - x^4 = 0 \Rightarrow x(1 - x^3) = 0 \Rightarrow x = 0,1$

When $x = 1$, $y = 1$ and $x = 0$, $y = 0$.

Therefore the point is (0,0) and (1,1). (1)

(iii) The required area $= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$ (1)

$$= \left(\frac{x^{3/2}}{3/2}\right)_0^1 - \left(\frac{x^3}{3}\right)_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$ (1)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration, Identifies the area bounded by the curve $y = f(x); y = g(x)$ as

$$\left|\int_a^b [f(x) - g(x)]dx\right|$$

Q.4

(i) Consider the figure below.
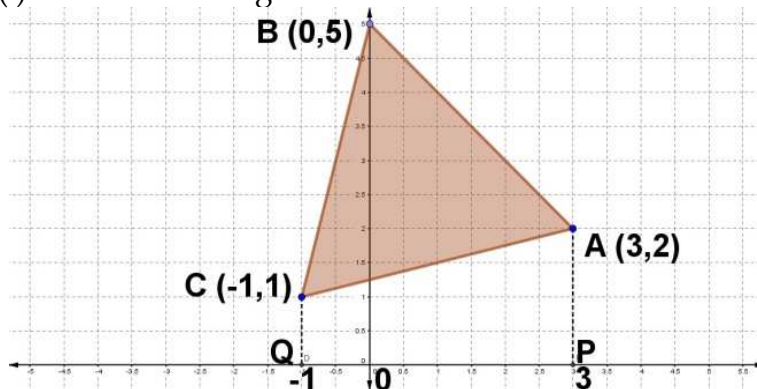
Find the area of the triangle using

    (i) Integration (4)

    (ii) Determinant method, compare the results. (2)

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(Scores: 6; Time: 12 mts)

Scoring Indicators

Integration method.

Equation of BC is $\dfrac{y-5}{1-5}=\dfrac{x-0}{-1-0}$

$\Rightarrow y-5=4x \Rightarrow 4x-y+5=0 \Rightarrow y=4x+5$

Equation of AB is $x+y-5=0 \Rightarrow y=5-x$

Equation of AC is $x-4y+5=0 \Rightarrow y=\dfrac{x}{4}+\dfrac{5}{4}$ (1)

The required area = Area of the region PABCQP-Area of the region PACQP

$$=\int_{-1}^{0}(4x+5)dx+\int_{0}^{3}(5-x)dx-\int_{-1}^{3}\left(\dfrac{x}{4}+\dfrac{5}{4}\right)dx$$ (1)

$$=\left(2x^2+5x\right)_{-1}^{0}+\left(5x-\dfrac{x^2}{2}\right)_{0}^{3}-\left(2x^2+\dfrac{5}{4}x\right)_{-1}^{3}$$ (1)

$$=-2+5+15-\dfrac{9}{2}-\left(\dfrac{9}{8}+\dfrac{15}{4}-\dfrac{1}{8}+\dfrac{5}{4}\right)=\dfrac{27}{2}-(6)=\dfrac{15}{2}$$ (1)

(ii) Determinant method.

Area of the triangle $=\dfrac{1}{2}\begin{vmatrix}1 & 3 & 2\\1 & 0 & 5\\1 & -1 & 1\end{vmatrix}$ (1)

$$=\dfrac{1}{2}\{1(0+5)-1(3+2)+1(15-0)\}$$

$$=\dfrac{1}{2}\{5-5+15\}=\dfrac{15}{2}$$ (1)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration, Identifies the area bounded by the curve $y=f(x); y=g(x)$ as $\left|\int_{a}^{b}[f(x)-g(x)]dx\right|$

Q.5

Consider the functions $f(x)=\sin x$ and $g(x)=\cos x$ in the interval$[0,2\pi]$

(i) Find the x coordinates of the meeting points of the functions. (1)
(ii) Draw the rough sketch of the above functions. (2)
(iii) Find the area enclosed by these curves in the given interval (3)

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i) $f(x) = \sin x$ and $g(x) = \cos x$ meet at multiples of $\dfrac{\pi}{4}$.

$x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ (1)

(ii)
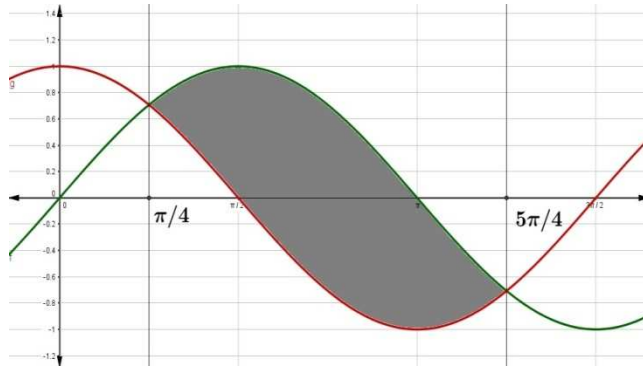
(2)

(iii) Area $= 2\{$ Area under $f(x) = \sin x$ from $\dfrac{\pi}{4}$ to π - Area under $g(x) = \cos x$ from $\dfrac{\pi}{4}$

to $\dfrac{\pi}{2}$ $\}$ (1)

$= 2\left\{ \displaystyle\int_{\frac{\pi}{4}}^{\pi} \sin x\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx \right\} = 2\left\{ \left[-\cos x \right]_{\frac{\pi}{4}}^{\pi} - \left[\sin x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right\} = 2\left\{ -\left[-1 - \dfrac{1}{\sqrt{2}} \right] - \left[1 - \dfrac{1}{\sqrt{2}} \right] \right\}$ (1)

$= 2\sqrt{2}$ (1)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration, Identifies the area bounded by the curve $y = f(x); y = g(x)$ as

$\left| \displaystyle\int_{a}^{b} [f(x) - g(x)]\,dx \right|$

Q.6

Using the figure answer the following questions
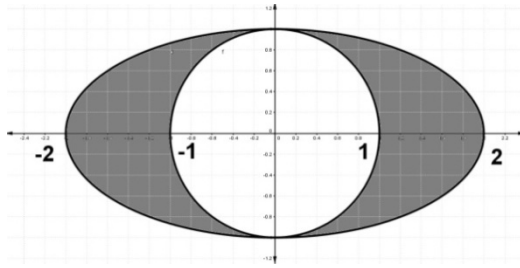
(i) Define the equation of the ellipse and circle in the given figure.   (1)
(ii) Find the area of the ellipse using integration.   (3)
(iii) Find the area of the shaded region (Area of the circle can be found using direct formula).   (2)

(Scores: 6; Time: 12 mts)

Scoring Indicators

(i) Equation of the ellipse is $\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$ and circle is $x^2+y^2=1$ (1)

(ii) We have , $\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$

$\Rightarrow y^2 =1-\dfrac{x^2}{4} \Rightarrow y=\dfrac{1}{2}\sqrt{4-x^2}$ (1)

Area of the ellipse $= 4\displaystyle\int_0^2 y\,dx$

$= 4\displaystyle\int_0^2 \dfrac{1}{2}\sqrt{4-x^2}\,dx = 2\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$

$= 2\left[\dfrac{x}{2}\sqrt{4-x^2}+\dfrac{4}{2}\sin^{-1}\dfrac{x}{2}\right]_0^2$ (1)

$= 2\left[\dfrac{2}{2}\sqrt{4-4}+2\sin^{-1}1-0-2\sin^{-1}0\right]$

$= 2\left[0+2\dfrac{\pi}{2}-0\right]=2\pi$ (1)

(iii)   Area of the circle $= \pi r^2 = \pi\times 1 = \pi$ (1)
Required area = Area of ellipse – area of the circle $= 2\pi-\pi = \pi$ (1)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and transforms the concept of area in different cases, Evaluating area under a curve using integration, Identifies the area bounded by the curve $y = f(x); y = g(x)$ as

$\left|\displaystyle\int_a^b [f(x)-g(x)]dx\right|$

Q.7

(i) Area of the shaded portion in the figure is equal to

(a) $\int_{d}^{c} f(x)dx$

(b) $\int_{c}^{d} f(x)dx$

(c) $\int_{d}^{c} f(y)dy$

(d) $\int_{c}^{d} f(y)dy$
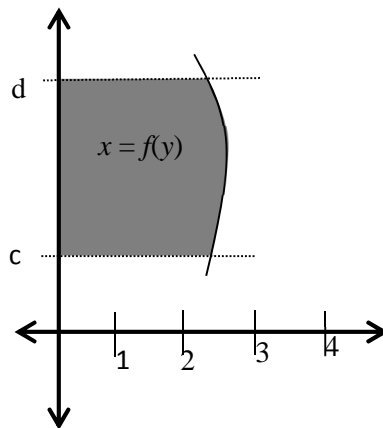
(1)

(ii)   Shade the area enclosed by
$x^2 = 4y, y=2, y=4$ and the y- axis in the first quadrant.   (2)

(iii)   Find the area of the enclosed area in the first quadrant.   (2)

(Scores: 5; Time: 12 mts)

Scoring Indicators

(i) (d) $\int_{c}^{d} f(y)dy$ (1)

(ii)
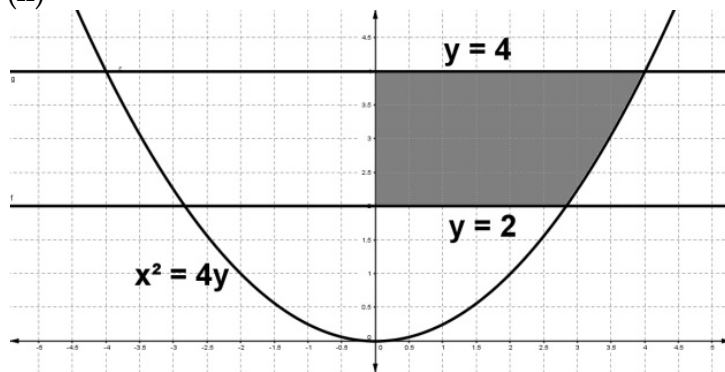
(2)

(iii)   Area

$$= \int_{2}^{4} \sqrt{4y}\,dy = 2\left(\frac{y^{3/2}}{3/2}\right)_{2}^{4} = \frac{4}{3}\left(4^{3/2} - 2^{3/2}\right)$$ (1)

$$= \frac{4}{3}\left(8 - 2\sqrt{2}\right) = \frac{8}{3}(4 - \sqrt{2})$$ (1)

Learning Outcomes

Recalls the concept of definite integral as the area under the curve, Interprets and

transforms the concept of area in different cases, Evaluating area under a curve using integration, Identifies the area bounded by the curve $y = f(x); y = g(x)$ as

$$\left| \int_a^b [f(x) - g(x)]dx \right|$$

Q.8

Find the area of the circle, $x^2 + y^2 = 16$ which is exterior to parabola $y^2 = 6x$. **(6)**

(Scores: 6; Time: 12 mts)

Scoring Indicators

Given, $x^2 + y^2 = 16$ and $y^2 = 6x$

$\Rightarrow x^2 + 6x = 16 \Rightarrow x^2 + 6x - 16 = 0 \Rightarrow (x+8)(x-2) = 0 \Rightarrow x = -8, 2$ (1)
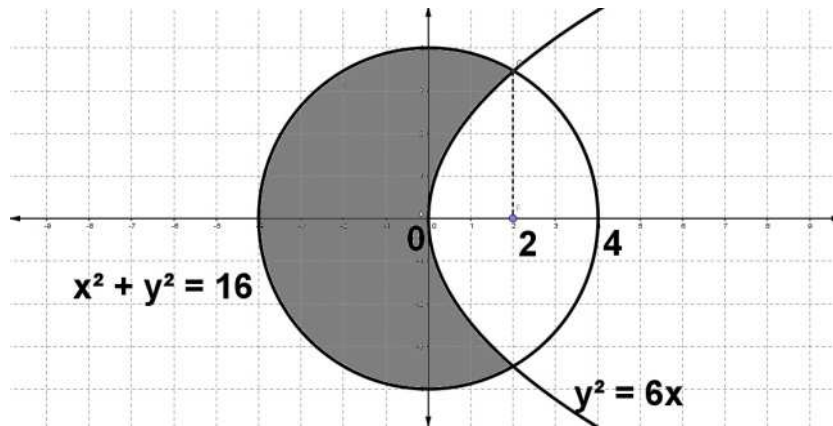
(1)

Area = Area of the circle – Interior area of the parabola.

$$= 16\pi - 2\left\{ \int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx \right\}$$

(1)

$$= 16\pi - 2\left\{ \sqrt{6}\left(\frac{2x^{\frac{3}{2}}}{3} \right)_0^2 + \left(\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4} \right)_2^4 \right\}$$

(1)

$$= 16\pi - 2\left\{ \sqrt{6}\frac{2 \times 2^{\frac{3}{2}}}{3} + \left(0 + 8\sin^{-1}(1) - \frac{2}{2}\sqrt{12} - 8\sin^{-1}\frac{2}{4} \right) \right\}$$

(1)

$$= 16\pi - 2\left\{ \frac{8}{\sqrt{3}} + \left(8\frac{\pi}{2} - 2\sqrt{3} - 8\sin^{-1}\frac{1}{2} \right) \right\}$$

$$= 16\pi - 2\left\{ \frac{8}{\sqrt{3}} + \left(4\pi - 2\sqrt{3} - 8\frac{\pi}{6} \right) \right\}$$

$$= 16\pi - 2\left\{ \frac{2\sqrt{3} + 8\pi}{3} \right\} = 16\pi - \frac{4\sqrt{3} + 16\pi}{3}$$

$$= \frac{48\pi - 4\sqrt{3} - 16\pi}{3} = \frac{32\pi - 4\sqrt{3}}{3}$$

(1)

CHAPTER – 9

DIFFERENTIAL EQUATIONS

Learning Outcomes

Identifies the order and degree of a Differential Equation. Identifies the arbitrary constants. Form Differential equation.

Q.1

(i) Consider the differential equation given below

$$\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

Write the order and degree of the DE (if defined)   (1)
(ii) Form the differential equation of the family of curves represented by the equation $y^2 = a(b^2 - x^2)$, a and b are arbitrary constants.   (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i)   4 ; degree is not defined   (1)

(ii) $2y\dfrac{dy}{dx} = -a2x \Rightarrow y\dfrac{dy}{dx} = -ax$ (1)

        Differentiating (2) with respect to x (1)

$$\Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx}\frac{dy}{dx} = -a \Rightarrow y\frac{dy}{dx} = \left[y\frac{d^2y}{dx^2} + \left[\frac{dy}{dx}\right]^2\right]x \qquad (1)$$

Which is the differential equation.

Learning Outcomes

Identifies the order and degree of a Differential Equation. Identifies the arbitrary constants. Form Differential equation.

Q.2
   i)   Find the Differential equation satisfying the family of curves,
           $y = ae^{3x} + be^{-2x}$, a and b are arbitrary constants.   (3)
   ii)   Hence write the degree and order of the DE   (1)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) $y = ae^{3x} + be^{-2x}$ ……… (1)   (1)

Differentiating with respect to x (1)

$\dfrac{dy}{dx} = ae^{3x} \times 3 + be^{-2x} \times -2$ ……… (2)

Differentiating (2) with respect to x,

$\Rightarrow \dfrac{d^2 y}{dx^2} = 9ae^{3x} + 4be^{-2x}$ …………(3)   (1)

Now, (3) + 2×(2) $\Rightarrow \dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx} = 15ae^{3x}$

$\Rightarrow \dfrac{1}{15}\left[\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx}\right] = ae^{3x}$ ….(4)

(3) - 3×(2) $\Rightarrow \dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx} = 10be^{-2x}$

$\Rightarrow \dfrac{1}{10}\left[\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx}\right] = be^{-2x}$ ….(5)

Using (4),(5) in (1), we have,

$y = \dfrac{1}{15}\left[\dfrac{d^2 y}{dx^2} + 2\dfrac{dy}{dx}\right] + \dfrac{1}{10}\left[\dfrac{d^2 y}{dx^2} - 3\dfrac{dy}{dx}\right]$

$= \dfrac{1}{30}\left[5\dfrac{d^2 y}{dx^2} - 5\dfrac{dy}{dx}\right] = \dfrac{1}{6}\left[\dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx}\right]$

$\Rightarrow \dfrac{d^2 y}{dx^2} - \dfrac{dy}{dx} - 6y = 0$ (1)

(ii) Order:2 ; degree: 1   (1)

Learning Outcomes

Identifies the arbitrary constants. Form Differential equation.

Q.3
  i) Choose the correct answer from the bracket

The solution of the differential equation $xdy + ydx = 0$ represents

[(a) a rectangular hyperbola (b) a parabola whose centre is origin
(c) a straight line whose centre is origin(d) a circle whose centre is origin.]   (1)
(ii) From the DE of the family of circles touching the x-axis at origin   (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators
  i) (c) a straight line whose centre is origin

ii)   Let (0,a) be the centre of the circle. Therefore the equation of the circle is $x^2 + (y-a)^2 = a^2 \Rightarrow x^2 + y^2 = 2ay$ (1)

$$\Rightarrow \frac{x^2 + y^2}{y} = 2a \dots\dots$$ (1)

Differentiating with respect to x,
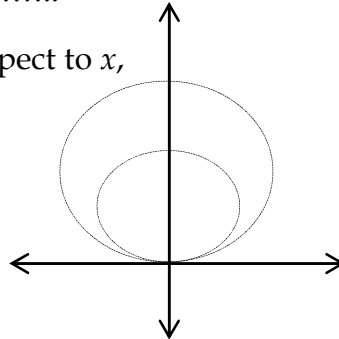
$$\frac{y(2x + 2y\frac{dy}{dx}) - (x^2 + y^2)\frac{dy}{dx}}{y^2} = 0$$

$$\Rightarrow 2xy + 2y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - y^2\frac{dy}{dx} = 0 \Rightarrow 2xy + y^2\frac{dy}{dx} - x^2\frac{dy}{dx} = 0$$ (1)

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$ (1)

Learning Outcomes

Solves the DE of the form variable separable. Distinguishes the general and particular solution of a DE.

Q.4

Find a particular solution satisfying the given condition
$(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x;$ y = 1, when x = 0   (3)

(Scores: 3; Time: 8 mts)

Scoring Indicators

$(x^2(x+1) + (x+1))\frac{dy}{dx} = x(2x+1)$

$(x^2 + 1)(x+1)\frac{dy}{dx} = x(2x+1) \Rightarrow dy = \frac{2x^2 + x}{(x^2 + 1)(x+1)}dx$ (1)

Integrating on both sides,

$$\int dy = \int \frac{2x^2 + x}{(x^2 + 1)(x + 1)} dx$$

Splitting into partial fractions we have,

$$\int dy = \int \frac{1}{2(x+1)} + \frac{1}{2}\frac{3x-1}{(x^2+1)} dx \qquad (1)$$

$$y = \frac{1}{2}\log(x+1) + \frac{3}{4}\int \frac{2x}{(x^2+1)} dx - \frac{1}{2}\int \frac{1}{(x^2+1)} dx$$

$$y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1} x + c$$

Given y = 1, when x = 0.

$$1 = \frac{1}{2}\log(1) + \frac{3}{4}\log(1) - \frac{1}{2}\tan^{-1} 0 + c \Rightarrow 1 = c$$

Hence,

$$y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1} x + 1 \qquad (1)$$

Learning Outcomes

Solving the DE of the form variable separable. Distinguishing the general and particular solution of a DE.

Q.5

Consider the DE $xy\frac{dy}{dx} = (x+2)(y+2)$,

(i) Find the equation of the family of curves (3)
(ii) Find equation of the curve passing through the point(1,-1) (2)

(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) $xy\frac{dy}{dx} = (x+2)(y+2) \Rightarrow \frac{y}{y+2} dx = \frac{x+2}{x} dx$ (1)

Integrating on both sides,

$$\Rightarrow \int \frac{y}{y+2} dy = \int \frac{x+2}{x} dx \Rightarrow \int 1 - \frac{2}{y+2} dy = \int 1 + \frac{2}{x} dx \qquad (1)$$

$$\Rightarrow y - 2\log|y+2| = x + 2\log|x| + c \,\dots(1) \qquad (1)$$

(ii) Since (1) passes through (1,-1), we have;

$$\Rightarrow -1 - 2\log|-1+2| = 1 + 2\log|1| + c \Rightarrow -2 = c \qquad (1)$$

$$(1) \Rightarrow y - 2\log|y+2| = x + 2\log|x| - 2 \qquad (1)$$

Learning Outcomes

Distinguishing the general and particular solution of a DE. Identifies the homogeneous DE. Solve the homogeneous DE.

Q.6

Consider the differential equation $xdy - ydx = \sqrt{x^2 + y^2}\, dx$

  (i) Find $\dfrac{dy}{dx}$ (1)

  (ii) Solve the above differential equation   (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) $xdy - ydx = \sqrt{x^2 + y^2}\, dx$

$xdy = (y + \sqrt{x^2 + y^2})dx \Rightarrow \dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$ (1)

(ii)This is Homogeneous DE,

Hence put, $y = v\,x$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ (1)

$\Rightarrow v + x\dfrac{dv}{dx} = \dfrac{vx + \sqrt{x^2 + v^2 x^2}}{x} \Rightarrow x\dfrac{dv}{dx} = v + \sqrt{1 + v^2} - v \Rightarrow x\dfrac{dv}{dx} = \sqrt{1 + v^2}$

$\Rightarrow \dfrac{dv}{\sqrt{1 + v^2}} = \dfrac{dx}{x}$

Integrating on both sides,

$\Rightarrow \displaystyle\int \dfrac{dv}{\sqrt{1 + v^2}} = \int \dfrac{dx}{x} \Rightarrow \log\left|v + \sqrt{1 + v^2}\right| = \log|x| + \log|c|$ (1)

$\Rightarrow v + \sqrt{1 + v^2} = cx \Rightarrow \dfrac{y}{x} + \sqrt{1 + \dfrac{y^2}{x^2}} = cx$

$\Rightarrow y + \sqrt{x^2 + y^2} = cx^2$ (1)

Learning Outcomes

Identifies linear DE. Solves linear DE.

Q.7

i.Choose the correct answer from the bracket

The general solution of the DE $\dfrac{dy}{dx} = e^{x-y}$ is   (1)

[(a) $e^y + e^x = c$ (b) $e^y - e^x = c$ (c) $e^{-y} + e^{-x} = c$ (d) $e^{-y} - e^{-x} = c$]

(ii) Solve the DE $\dfrac{dy}{dx} - \dfrac{2xy}{1+x^2} = x^2 + 2$ (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) $e^y - e^x = c$ (1)

(ii) $\dfrac{dy}{dx} - \dfrac{2xy}{1+x^2} = x^2 + 2$

$\Rightarrow P = -\dfrac{2x}{1+x^2}, Q = x^2 + 2$ (1)

$IF = e^{\int Pdx} = e^{-\int \frac{2x}{1+x^2}dx} = e^{-\int \frac{2x}{1+x^2}dx} = e^{-\log(1+x^2)} = \dfrac{1}{1+x^2}$

Solution is; $y(IF) = \int Q(IF)dx$

$\Rightarrow y(\dfrac{1}{1+x^2}) = \int (x^2+2)(\dfrac{1}{1+x^2})dx$ (1)

$\Rightarrow \dfrac{y}{1+x^2} = \int \dfrac{x^2+2}{1+x^2}dx \Rightarrow \dfrac{y}{1+x^2} = \int 1 + \dfrac{1}{1+x^2}dx$

$\Rightarrow \dfrac{y}{1+x^2} = x + \tan^{-1} x + c$

$\Rightarrow y = (1+x^2)(x + \tan^{-1} x + c)$ (1)

Learning Outcomes

Solving the DE of the form variable separable. Distinguishing the general and particular solution of a DE.

Q.8

(i)   Choose the correct answer from the bracket
Determine the order and degree of the differential equation,

$2x\dfrac{d^4y}{dv^4} + 5x^2 \left(\dfrac{dy}{dx}\right)^3 - xy = 0.$

(1)

[(a). Fourth order, first degree(b) Third order, first degree
(c) First order, fourth degree(d)First order, third degree]

(ii) The population of a country doubles in 50 years. How many years will   (3)
it be five times as much? Assume that the rate of increase is proportional to
the number inhabitants. (hint: $\log 2 = 0.6931; \log 5 = 1.6094$)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i)   (a) Fourth order, first degree   (1)

ii) $\dfrac{dP}{dt} \alpha \;\; P$, where P-population of the country

$\dfrac{dP}{dt} = kP \Rightarrow \dfrac{dP}{P} = kdt \Rightarrow \int \dfrac{dP}{P} = k\int dt$ (1)

$\Rightarrow \log P = kt + c$

When $t = 0$; $P = P_o$

$\Rightarrow \log P_o = k(0) + c \Rightarrow \log P_o = c$

When $t = 50$; $P = 2P_o$

$\Rightarrow \log 2P_o = k(50) + \log P_o \Rightarrow \log 2 + \log P_o = k(50) + \log P_o$

$\Rightarrow 0.6931 = k(50) \Rightarrow k = \dfrac{0.6931}{50} = 0.01386$ (1)

When population is $P = 5P_o$

$\Rightarrow \log 5P_o = 0.01386t + \log P_o \Rightarrow \log 5 + \log P_o = 0.01386t + \log P_o$

$\Rightarrow 1.6094 = 0.01386t \Rightarrow t = \dfrac{0.01386}{1.6094} = 116.12$ years   (1)

LEARNING OUTCOMES

Solving the DE of the form variable separable. Distinguishing the general and particular solution of a DE

Q.9

The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds   (3)

(Scores: 3; Time: 6 mts)

Scoring Indicators

Let the rate of change of the volume of the balloon be k (where k is a constant).

$\Rightarrow \dfrac{dv}{dt} = k \Rightarrow \dfrac{d\left(\dfrac{4}{3}\pi r^3\right)}{dt} = k \Rightarrow \dfrac{4\pi}{3} 3r^2 \dfrac{dr}{dt} = k$ (1)

$\Rightarrow 4\pi r^2 dr = kdt$

Integrating both sides, we get:

$$4\pi \int r^2 dr = k \int dt$$

$$\Rightarrow 4\pi \cdot \frac{r^3}{3} = kt + C$$

$$\Rightarrow 4\pi r^3 = 3(kt + C) \qquad \ldots(1)$$

Now, at $t = 0$, $r = 3$:

(1)

$\Rightarrow 4\pi \times 3^3 = 3 (k \times 0 + C)$

$\Rightarrow 108\pi = 3C$

$\Rightarrow C = 36\pi$

At $t = 3$, $r = 6$:

$\Rightarrow 4\pi \times 6^3 = 3 (k \times 3 + C)$

$\Rightarrow 864\pi = 3 (3k + 36\pi)$

$\Rightarrow 3k = 252\pi$

$\Rightarrow k = 84\pi$

Substituting the values of k and C in equation (1), we get:

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$

(1)

CHAPTER - 10

VECTORS

Learning Outcomes

Finds the unit vector, Finds difference of two vectors and magnitude of a vector, Identifies the vector product of two vectors.

Q.1

(i) Choose the correct answer from the bracket. The unit vector in the direction of the vector
$\vec{a} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$ is
$\{(a)\hat{\imath} + \hat{\jmath} + \hat{k}$, $(b) \frac{1}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath} - \frac{1}{2}\widehat{k}$, (c) $\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$, (d) $\frac{1}{\sqrt{5}}\hat{\imath} + \frac{1}{\sqrt{5}}\hat{\jmath} - \frac{2}{\sqrt{5}}\hat{k}\}$ (1)
(ii) Find $|\ \vec{a} - \vec{b}\ |$, if two vector $\vec{a}$ and $\vec{b}$ are such that $|\ \vec{a}\ | = 2$, $|\ \vec{b}\ | = 4$ and
$\vec{a} . \vec{b} = \frac{11}{2}$ (2)
(iii) Find $|\ \vec{a} \times \vec{b}\ |$, if$\vec{a} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$ (2)

(Scores: 5; Time: 10 mts)

Scoring Indicators

(i) (c) $\frac{2}{3}\hat{\imath} + \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$ (1)
(ii)$|\ \vec{a} - \vec{b}\ |^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - 2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2$ (1)
$= 4 + 16 - 11 = 9$
$|\ \vec{a} - \vec{b}\ | = \sqrt{9} = 3$ (1)
(iii) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = -5\hat{\imath} + 5\hat{\jmath} + 5\hat{k}$ (1)
$|\ \vec{a} \times \vec{b}\ | = \sqrt{75} = 5\sqrt{3}$ (1)

Learning Outcomes

Solves the problem using scalar product, Applies the condition of coplanarity in related situations

Q.2

(i) Choose the correct answer from the bracket. The angle between the vectors $\vec{a}$ and $\vec{b}$ with magnitude 1 and 2 respectively having $\vec{a} \cdot \vec{b} = \sqrt{3}$

$$\{(a)\frac{\pi}{3}\ ,\ (b)\frac{\pi}{4},\ (c)\frac{\pi}{6},(d)\ \frac{\pi}{2}\} \tag{1}$$

(ii) If the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{c} = k\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ are coplanar then find the value of k   (2)

(Scores: 3; Time: 6 mts)

Scoring Indicators

(i)   (c) , $cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\cdot\vec{b}|} = \dfrac{\sqrt{3}}{2},\qquad \theta = \dfrac{\pi}{6}$ (1)

(ii) $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar , then $\overrightarrow{[a\ \ \vec{b}\ \ \vec{c}\ ]} = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ k & -2 & 3 \end{vmatrix} = 0 \tag{1}$$

$$1(-3+2) - 1(6-k) + 1(-4+k) = 0 \tag{1}$$
$$2k - 11 = 0, k = 11/2$$

Learning Outcomes

Illustrate different types of vectors and magnitude, Differentiates the direction cosines, derives the section formula in vector form.

Q.3

   (i)   Write two different vectors having same magnitude   (1)
   (ii)   Find the direction cosines of the vector $2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ (2)
   (iii) Consider two points A and B with position vectors $\overrightarrow{OA} = \vec{a} - 4\vec{b}$ and
          $\overrightarrow{OB} = \vec{a} - \vec{b}$. Find the position vector of a point R which divides the line
          joining A and B in the ratio 2 : 1 internally   (2)

(Scores: 3; Time: 6 mts)

Scoring Indicators

$(i)\ \vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \vec{b} = 2\hat{\imath} + 3\hat{\jmath}\text{-}\hat{k}$ (1)
  Any two different vectors with same magnitude
(ii) Magnitude $= \sqrt{4+1+9} = \sqrt{14}$ (1)
          $\therefore$ Direction cosines are $\dfrac{2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$ (1)
(iii) $\overrightarrow{OR} = \dfrac{m\vec{b} + n\vec{a}}{m+n} = \dfrac{2(\vec{a}-\vec{b}) + 1(\vec{a}-4\vec{b})}{3} = \overrightarrow{a} - 2\vec{b}$ (2)

Learning Outcomes

Identifies the scalar projection and vector projection, finds the sum and differences of two vectors, Solves different problems using vector product of two vectors.

Q.4

(i)   Choose the correct answer from the bracket. A vector $\vec{a}$ makes an angle $\frac{3\pi}{2}$ with a given directed line l, in the anticlockwise direction, then the projection vector of $\vec{a}$ on line l

$(a) zero$ $(b)\ zero\ vector$ $(c)\ unit\ vector$ $(d)\ \frac{3\pi}{2}$ (1)

(ii)   Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$ (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i) (b) , since $\vec{a}$ is perpendicular to l, then the projection of $\vec{a}$ is zero.
Projection vector is a zero vector   (1)

(ii) $\vec{a} + \vec{b} = 3\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$
$\vec{a} - \vec{b} = \hat{\imath} + 2\hat{\jmath} + 0\hat{k}$ (1)
$Vector\ perpendicular\ to\ \vec{a} + \vec{b}\ and\ \vec{a} - \vec{b}$

$(\vec{a} + \vec{b})\ x(\vec{a} - \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$ (1)

$|\ (\vec{a} + \vec{b})\ x(\vec{a} - \vec{b})\ | = \sqrt{16 + 4 + 36} = \sqrt{56}$
Unit vector perpendicular to $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$
$= -\frac{4}{\sqrt{56}}\hat{\imath} + \frac{2}{\sqrt{56}}\hat{\jmath} + \frac{6}{\sqrt{56}}\hat{k}$ (1)

Learning Outcomes

Writes the direction cosines, Finds the unit vector, identifies the direction of angles.

Q.5

(i) Choose the correct answer from the bracket. If a unit vector $\hat{a}$ makes angles $\frac{\pi}{4}$ with $\hat{\imath}$ and $\frac{\pi}{3}$ with $\hat{\jmath}$ and an acute angle θ with $\hat{k}$, then θ is
$\{ (a)\ \frac{\pi}{6}$, $(b)\ \frac{\pi}{4}$, $(c)\ \frac{\pi}{3}$,$(d)\ \frac{\pi}{2} \}$ (1)
(ii) Find a unit vector $\hat{a}$ (1)
(iii) Write down a unit vector in XY plane, making an angle 60^0 with the positive direction of x – axis   (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

(i)   (c), Since $l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $m = \cos\frac{\pi}{3} = \frac{1}{2}$ $n = \cos\theta$

$l^2 + m^2 + n^2 = 1$

$n^2 = 1 - (\frac{1}{2})^2 - (\frac{1}{\sqrt{2}})^2 = \frac{1}{4}$

$n = \frac{1}{2}$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$ (1)

(ii)   Unit vector $\hat{a} = l\,\hat{\imath} + m\,\hat{\jmath} + n\,\hat{k}$

$= \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{1}{2}\hat{k}$ (1)

Let $\vec{r} = x\hat{\imath} + y\hat{\jmath}$

$|\vec{r}| = \sqrt{x^2 + y^2}$

$x = \cos 60 = \frac{1}{2}$, $y = \sin 60 = \sqrt{3}/2$ (1)

$\vec{r} = \frac{1}{2}\,\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$ (1)

Learning Outcomes

Interprets scalar triple product geometrically, Examines different properties of vector product, Computes volume of parallelepiped

Q.6

(i)   If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar , then $[\vec{a}\ \vec{b}\ \vec{c}]$ is …   (1)

(ii)   Find p, if $\vec{a} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{b} = \hat{\imath} + p\,\hat{\jmath} + \hat{k}$ and

$\vec{a} \times \vec{b} = -3\hat{\imath} + 4\,\hat{\jmath} + 5\,\hat{k}$ (2)

(Scores: 3; Time: 6 mts)

Scoring Indicators

(i)   0   (1)

(ii) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 2 \\ 1 & p & -1 \end{vmatrix} = -3\hat{\imath} + 4\,\hat{\jmath} + 5\,\hat{k}$ (1)

$= \hat{\imath}\,(1 - 2p) - \hat{\jmath}(-2-2) + \hat{k}\,(2p + 1) = -3\hat{\imath} + 4\,\hat{\jmath} + 5\,\hat{k}$

$1 - 2p = -3$

$2p = 4$

$p = 2$ (1)

Learning Outcomes

Examines different properties of vector product, Interprets scalar triple product,

Computes the area of the triangle

Q.7

(i) $\vec{a} \times \vec{a}$ is equal to …..   (1)

(ii)   Find the value of $[\ \vec{a} + \vec{c},\ \vec{a} + \vec{b},\ \vec{a} + \vec{b} + \vec{c}\ ]$ (3)

(iii)   Find the area of a triangle having the points A (2, 3, 1), B (1, 1, 2) and   C (1, 2, 1)   (3)

(Scores: 7; Time: 12 mts)

Scoring Indicators

(i)   0   (1)

(ii) $(\vec{a} + \vec{c}).\left[\left(\vec{a} + \vec{b} \right) \times \left((\vec{a} + \vec{b} + \vec{c})\right)\right]$

$=(\vec{a} + \vec{c}).\left[\left(\vec{a} + \vec{b} \right) \times \vec{a} + \left(\vec{a} + \vec{b} \right) \times \vec{b} + \left(\vec{a} + \vec{b} \right) \times \vec{c}\right]$

$= (\vec{a} + \vec{c}).\left[\vec{a} \times \vec{a} + \vec{b} \times \vec{a} + \vec{a} \times \vec{b} + \vec{b} \times \vec{b} + \vec{a} \times \vec{c} + + \vec{b} \times \vec{c}\right]$ (1)

$= (\vec{a} + \vec{c}).\left[\vec{a} \times \vec{c} + \vec{b} \times \vec{c}\right]$

$= (\vec{a} + \vec{c}).(\vec{a} \times \vec{c}) + (\vec{a} + \vec{c}).(\vec{b} \times \vec{c})$ (1)

$= [\vec{a}\ \vec{a}\ \vec{c}] + [\vec{c}\ \vec{a}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{c}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]$ (1)

iii) $\overrightarrow{AB} = -\hat{\imath} - 2\,\hat{\jmath} + \hat{k}$

       $\overrightarrow{AC} = -\hat{\imath} - \hat{\jmath} + 0\,\hat{k}$ (1)

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -2 & 1 \\ -1 & -1 & 0 \end{vmatrix}$

             $= \hat{\imath} - \hat{\jmath} - \hat{k}$ (1)

     Area of the triangle $= \frac{1}{2}\ |\overrightarrow{AB} \times \overrightarrow{AC}|$

     $= \frac{1}{2}\sqrt{1 + 1 + 1}$

     $= \frac{\sqrt{3}}{2}$ square units   (1)

Learning Outcomes

Framing a vector when two points are given, understanding the concepts of unit vector, angle between vectors .

Q.8

Consider the points A(0,-2,1),B(1,-1,-2) and C(-1,1,0) lying in the plane

i)Compute $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (1)

ii)Find $\overrightarrow{AB} \times \overrightarrow{AC}$ (1)

iii)Find a unit vector perpendicular to the plane   (1)

iv)Find CosA

(1)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $\overline{AB}$ =(i-j-2k)-(0i-2j+1k)=i+j-3k (1)

$\overline{AC}$ =-i+3j-k

ii) $\overline{AB} \times \overline{AC}$ = $\begin{vmatrix} i & j & k \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$ =8i+4j+4k (1)

iii)Unit vector perpendicular to the plane= $\dfrac{\overline{AB} \times \overline{AC}}{\left|\overline{AB} \times \overline{AC}\right|}$

$$= \frac{8i + 4j + 4k}{\sqrt{8^2 + 4^2 + 4^2}}$$

$$= \frac{1}{\sqrt{6}}(2i + j + k)$$ (1)

iv)cosA = $\dfrac{\overline{AB}.\overline{AC}}{\left|\overline{AB}\right|.\left|\overline{AC}\right|}$ = $\dfrac{1(-1) + 1(3) + (-3)(-1)}{\sqrt{1+1+9}\sqrt{1+9+1}}$ = $\dfrac{5}{\sqrt{11}\sqrt{11}}$

$$= \frac{5}{11}$$ (1)

Learning Outcomes

Concepts of dot, cross and triple products of two vectors

Q.9

Let $\overline{a}$ =7i-2j+k, $\overline{b}$ =i-2j+2k and $\overline{c}$ =3i-8j
i)Compute $\overline{a} \times \overline{b}$ and $\overline{a} \times \overline{c}$ (2)
ii)Are the products $\overline{a}.(\overline{b} \times \overline{c})$ and $(\overline{a} \times \overline{b}).\overline{c}$ obtained are same? (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i)

$\overline{axb}$ = $\begin{vmatrix} i & j & k \\ 7 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix}$ =-2i-13j-12k (1)

$\overline{bxc}$ = $\begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 3 & -8 & 0 \end{vmatrix}$ =16i+6j-2k (1)

ii) $\bar{a}.(\bar{b} \times \bar{c})$=(7i-2j+k).(16i+6j-2k)

$\qquad$ =7.16+-2.6+1.-2=98 $\qquad$ (1)

$(\bar{a} \times \bar{b}).\bar{c}$ =(-2i-13j-12k).(3i-8j)

$\qquad$ =-2x3+13x8+0=98 $\qquad$ (1)

Learning Outcomes

Identifies the scalar projection and vector projection, finds the sum and differences of two vectors, Solves different problems using vector product of two vectors

Q.10

Let $\bar{a}$ =i+j+k, $\bar{b}$ =2i+3j , $\bar{c}$ =3i+5j-2k and $\bar{d}$ =-j+k

i)Find $\bar{b}-\bar{a}$ $\qquad$ (1)

ii)Find the unit vector along $\bar{b}-\bar{a}$ $\qquad$ (1)

iii)Prove that $\bar{b}-\bar{a}$ and $\bar{d}-\bar{c}$ are parallel vectors $\qquad$ (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $\bar{b}-\bar{a}$ =(2i+3j)-(i+j+k)=i+2j-k $\qquad$ (1)

ii)Unit vector along $\bar{b}-\bar{a}=\dfrac{\bar{b}-\bar{a}}{\left|\bar{b}-\bar{a}\right|}$ $\qquad$ (1)

$$=\frac{i+2j-k}{\sqrt{1+4+1}}=\frac{i+2j-k}{\sqrt{6}}$$

iii) $\bar{d}-\bar{c}$ =(-j+k)-(3i+5j-2k)

$\qquad$ =-3i-6j+3k $\qquad$ (1)

$\qquad$ =-3(i+2j-k)=-3($\bar{b}-\bar{a}$)

So $\bar{b}-\bar{a}$ and $\bar{d}-\bar{c}$ are parallel vectors $\qquad$ (1)

Learning Outcomes

Identifies the scalar projection and vector projection, finds the sum and differences of two vectors, Solves different problems using vector product of two vectors.

Q.11

Given the position vectors of

three points A(i-j+2k),B(4i+5j+8k) and C(3i+3j+6k)

i)   Find the Projection of $\overline{AB}$ on $\overline{AC}$ (2)

ii)   Prove that A,B and C are Collinear   (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

I) $\overline{AB}$ =(4i+5j+8k)-(i-j+2k)=3i+6j+6k   (1)

$\overline{AC}$ =2i+4j+4k

Projection of $\overline{AB}$ on $\overline{AC}$ = $\dfrac{\overline{AB}.\overline{AC}}{\left|\overline{AC}\right|}$

$= \dfrac{3.2 + 6.4 + 6.4}{\sqrt{2^2 + 4^2 + 4^2}} = \dfrac{54}{6}$ =9units   (1)

ii) $\overline{AB} = \dfrac{3(2i + 4j + 4k)}{2}$ (1)

$\overline{AB} = \dfrac{3\overline{AC}}{2}$

A,B and C are collinear   (1)

Learning Outcomes

Identifies the scalar projection and vector projection, finds the sum and differences of two vectors, Solves different problems using vector product of two vectors

Q.12

i) If $\overline{a}$ is any vector,Prove that $\overline{a}$ = ($\overline{a}$ .i)i +($\overline{a}$ .j)j +($\overline{a}$ .k)k   (2)

ii) If $\overline{a}$ and $\overline{b}$ are unit vectors inclined at an angle θ ,then Prove that

$\sin\left(\dfrac{\theta}{2}\right) = \dfrac{1}{2}\left|\overline{a} - \overline{b}\right|$ (3)

(Scores: 5; Time: 10 mts)

Scoring Indicators

i)   Let $\overline{a}$ =ai+bj+ck   (1)

$\overline{a}$ .i=(ai+bj+ck).i=a

$\overline{b}$ .j =(ai+bj+ck).i=b   and $\overline{c}$ .k=(ai+bj+ck).i=c   (1)

Thus ($\overline{a}$ .i)i +($\overline{a}$ .j)j +($\overline{a}$ .k)k =ai+bj+ck   = $\overline{a}$

ii) $\left|\overline{a} - \overline{b}\right|^2$ =($\overline{a}$ -$\overline{b}$).($\overline{a}$ -$\overline{b}$)

$= \overline{a} . \overline{a} - \overline{a} . \overline{b} - \overline{b} . \overline{a} + \overline{b} . \overline{b}$ (1)

$= \left|\overline{a}\right|^2 - 2\overline{a} . \overline{b} + \left|\overline{b}\right|^2$, $\overline{a}$ and $\overline{b}$ are unit vectors

$= 2 - 2\overline{a} . \overline{b}$

$= 2(1 - \left|\overline{a}\right|\left|\overline{b}\right| \cos\theta)$ (1)

$= 2(1 - \cos\theta)$

$= (2)2 \sin^2\left(\dfrac{\theta}{2}\right)$

$\left|\overline{a} - \overline{b}\right| = 2\sin\left(\dfrac{\theta}{2}\right)$ (1)

CHAPTER – 11

THREE DIMENSIONAL GEOMETRY

Learning Outcomes

Establishes the relation between direction cosines and direction ratios. Forms the vector and Cartesian equations of the line. Form the equation of a line.

Q.1

Choose the correct answer from the bracket.
(i) If a line in the space makes angle α, β and γ with the coordinates axes, then $cos^2\alpha + cos^2\beta + cos^2\gamma$ is equal to
{(a) 1 , (b) 2, (c) 0 , (d) 3 } (1)

ii) The direction ratios of the line $\dfrac{x-6}{1} = \dfrac{2-y}{2} = \dfrac{z-2}{2}$ are

{ (a) (6 , - 2 , -2) , (b) (1 , 2 , 2) , (c) (6 ,1 , -2) , (d) (0 , 0 , 0) } (1)

iii) If the vector equation of a line is $\overline{r} = \hat{\imath} + \hat{\jmath} + \hat{k} + \mu(2\hat{\imath} - 3\hat{\jmath} - 4\hat{k})$, then the Cartesian equation of the line is

{ (a) $\dfrac{x+2}{2} = \dfrac{y+2}{4} = \dfrac{z+2}{1}$, (b) $\dfrac{x-1}{2} = \dfrac{y-1}{-3} = \dfrac{z-1}{-4}$, (c) $\dfrac{x+2}{1} = \dfrac{y+2}{4} = \dfrac{z+2}{1}$, (d) $\dfrac{x-1}{2} = \dfrac{y-1}{1} = \dfrac{z-1}{-4}$ }

(1)

iv) If the Cartesian equation of a plane is x + y + z =12 , then the vector equation of the lane is

{ (a) $\overline{r} . (2\hat{\imath} + \hat{\jmath} + \hat{k}) = 12$, (b) $\overline{r} . (\hat{\imath} + \hat{\jmath} + \hat{k}) = 12$, (c) $\overline{r} . (\hat{\imath} + \hat{\jmath} + 2\hat{k}) = 12$, (d) $\overline{r} . (\hat{\imath} + 3\hat{\jmath} + \hat{k}) = 12$ } (1)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) (a) 1 (1)
ii) (b) (1 , -2 , 2) (1)
(iii) (b) $\dfrac{x-1}{2} = \dfrac{y-1}{-3} = \dfrac{z-1}{-4}$ (1)
(iv) (b) $\overline{r} . (\hat{\imath} + \hat{\jmath} + \hat{k}) = 12$ (1)

Learning Outcomes

Forms the vector equation of a line. Applies the concept of shortest distance between two lines.

Q.2
Cartesian equation of two lines are $\dfrac{x+2}{2} = \dfrac{y+2}{4} = \dfrac{z+2}{1} , \dfrac{x-1}{2} = \dfrac{y-1}{-3} = \dfrac{z-1}{-4}$

i) Write the vector equation of the lines   (1)
ii) Find the shortest distance between the lines   (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $\bar{r}= (-2\hat{\imath} -2\hat{\jmath}-2\hat{k}) +\mu(2\hat{\imath} +4 \hat{\jmath} +\hat{k})$
$\bar{r} = (\hat{\imath} + \hat{\jmath} +\hat{k}) + \mu(2\hat{\imath} -3 \hat{\jmath} - 4\hat{k})$ (1)
Shortest distance $= \left| \frac{(b_1 \times b_2).(a_2-a_1)}{|b_1 \times b_2|} \right|$

$b_1 \times b_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & 1 \\ 2 & -3 & -4 \end{vmatrix}$
$= (-16 +3)\hat{\imath} - (-8 - 2)\hat{\jmath} + (-6 - 8)\hat{k} = -13\hat{\imath} +10\hat{\jmath} -14\hat{k}$ (1)

$|b_1 \times b_2|=\sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2} =\sqrt{(-13)^2 + (10)^2 + (-14)^2}=\sqrt{465}$
$a_2 - a_1 =(\hat{\imath} + \hat{\jmath} +\hat{k}) -(-2 \hat{\imath} -2 \hat{\jmath} -2\hat{k}) = 3\hat{\imath}+3\hat{\jmath}+3\hat{k}$ (1)
$(b_1 \times b_2). (a_2 - a_1) = (3\hat{\imath}+3\hat{\jmath}+3\hat{k}). (-13\hat{\imath} +10\hat{\jmath} -14\hat{k}) = -51$
Shortest distance $= \left| \frac{(b_1 \times b_2).(a_2-a_1)}{|b_1 \times b_2|} \right| = \left| \frac{-51}{\sqrt{465}} \right| = \frac{51}{\sqrt{465}}$ (1)

Learning Outcomes

Forms the Cartesian equation of a line. Finds the angle between two lines.

Q.3
Consider the lines $\bar{r} = (\hat{\imath} +2\hat{\jmath} +3\hat{k}) + \lambda (2\hat{\imath} -3 \hat{\jmath} - 4\hat{k})$ $\bar{r} = (\hat{\imath} + \hat{\jmath} +\hat{k}) + \mu(\hat{\imath}+3 \hat{\jmath} +2\hat{k})$
(i) Write the Cartesian equation   (1)
(ii) Find the angle between the line   (3)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{-4}$
$\frac{x-1}{1} = \frac{y-1}{3} = \frac{z-1}{2}$ (1)
ii) $\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 a_3}{\sqrt{a_1{}^2+a_2{}^2+a_3{}^2} \sqrt{b_1{}^2+b_2{}^2+b_3{}^2}}$ (1)

$= \frac{2.1+(-3)3+(-4)2}{\sqrt{2^2+(-3)^2+(-4)^2}\sqrt{1^2+3^2+2^2}}$ (1)

$= \frac{-15}{\sqrt{29}\sqrt{14}} = \frac{-15}{\sqrt{406}}$ (1)

Learning Outcomes

Forms the Cartesian and vector equations of a line. Applies the concept of distance

between a point and line.

Q.4

Let the vector equation of a plane be $\bar{r} \cdot (2\hat{i} + 4\hat{j} + 3\hat{k}) = 12$
i) write the Cartesian equation of the plane   (1)
ii) find the distance from the point (2,1,3) to the plane   (4)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i) $2x + 4y + 3z = 12$ (1)
ii) Distance of the point (x_1, y_1, z_1) to the plane

$Ax + By + Cz = D$ is $\left| \dfrac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|$ (1)

$= \left| \dfrac{2.2 + 4.1 + 3.3 - 12}{\sqrt{2^2 + 4^2 + 3^2}} \right|$ (1)

$= \left| \dfrac{4 + 4 + 9 - 12}{\sqrt{4 + 16 + 9}} \right| = \left| \dfrac{5}{\sqrt{29}} \right|$ (2)

Learning Outcomes

Forms the equation of any plane perpendicular to two planes. Applies the concept of distance between the planes.

Q.5

i)Find the equation of the plane through the point(1,2,3) and perpendicular to the plane x-y+z=2 and 2x+y-3z=5   (2)
ii)Find the distance between the planes x-2y+2z-8=0 and 6y-3x-6z-57=0   (2)

(Scores: 4; Time: 8 mts)

Scoring Indicators

i)Required equation is

$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 0$ (1)

(x-1)(3-1)-(y-2)(-3-2)+(z-3)(1+2)=0
2(x-1)+5(y-2)+3(z-3)=0
2x+5y+3z-2-10-9=0
2x+5y+3z-21=0   (1)

ii)The planes are

x-2y+2z-8=0 and3x-6y+6z+57=0
ie, 3x-6y+6z-24=0 and3x-6y+6z+57=0 (1)

$$\text{Distance}=\left|\frac{d1-d2}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{57-(-24)}{\sqrt{9+36+36}}\right|=\left|\frac{81}{\sqrt{81}}\right|=9 \quad (1)$$

Learning Outcomes

Forms the vector equation of a line, any point on a line and the intersecting point
.Applies the concept of angle between a plane and a line.

Q.6
Consider the Cartesian equation of a line
$$\frac{x-3}{2}=\frac{y+1}{3}=\frac{z-5}{-2}$$
i)Find the vector equation of the line (1)
ii)Find the intersecting point of the line with the plain 5x+2y-6z-7=0.   (2)
iii)Find the angle made by the line with the plane 5x+2y-6z-7=0 (2)

(Scores: 5; Time: 10 mts)
Scoring Indicators
i)The vector equation is
$$\bar{r}=(3i-j+5k)+\lambda(2i+3j-2k)$$ (1)

ii) Any point on the line is
$$\frac{x-3}{2}=\frac{y+1}{3}=\frac{z-5}{-2}=\lambda$$

x=2λ+3,y=3 λ -1 ,z=-2λ+5 (1)
Since this lies on the plane ,it satisfies the plane
5(2λ+3)+2(3λ-1)-6(-2λ+5)-7=0
10λ+6λ+12λ+15-2-30-7=0
28λ=24
λ=6/7
The point of intersection is (33/7,11/7,23/7) (1)
iii)Let θ be the angle between the line and the plane
The direction of the line and the plane
m1=2i+3j-2k and m2=5i+2j-6k (1)

$$\sin\theta=\left|\frac{m1.m2}{\|m1\|\|m2\|}\right|$$

$$=\left|\frac{2(5)+3(2)+(-2)(-6)}{\sqrt{4+9+4}\sqrt{25+4+36}}\right|=\left|\frac{28}{\sqrt{17}\sqrt{65}}\right|=\left|\frac{28}{\sqrt{1105}}\right| \quad (1)$$

Learning Outcomes

 Forms the Cartesian equation ,Applies the concept of equation of planes passing through the intersection of two planes

Q.7

Consider the vector equation of two planes
$$\bar{r}.(2i+j+k)=3 \qquad \bar{r}.(i-j-k)=4$$
i)Find the vector equation of any plane through the intersection of the above two planes   (2)
ii)Find the vector equation of the plane through the intersection of the above planes
    and the point (1,2,-1)   (1)

                                                                            (Scores: 3 ; Time: 6 mts)

Scoring Indicators

i)The cartesian equation are
   2x+y+z-3=0 and x-y-z-4=0   (1)
    Required equation of the plane is
   (2x+y+z-3) +λ(x-y-z-4)=0   (1)
   (2+ λ)x+(1- λ)y+(1- λ)z+(-3-4 λ)=0
$\bar{r}.[(2+\lambda)i+(1-\lambda)j+(1-\lambda)k]+(-3-4\lambda)=0$
ii)The above plane passes through (1,2,-1)
   (2+ λ)1+(1- λ)2+(1- λ)(-1)+(-3-4 λ)=0
   3=3+4 λ
   λ=0   (1)
Equation of the plane is
 2x+y+z-3=0
$\bar{r}.(2i+j+k)=3$

CHAPTER – 12

LINEAR PROGAMMING

Learning Outcomes

Identifies the concept of LPP. Recalls the method of solving linear inequalities. Identifies the objective function, constraints and non negativity restrictions.

Q.1

   i)   Choose the correct answer from the bracket. If an LPP is consistent, then its feasible region is always

{ (a) Bounded , (b) Unbounded,   (c) Convex region , (d) Concave region} (1)

   ii)   Maximize $Z = 2x + 3y$ subject to the constraints

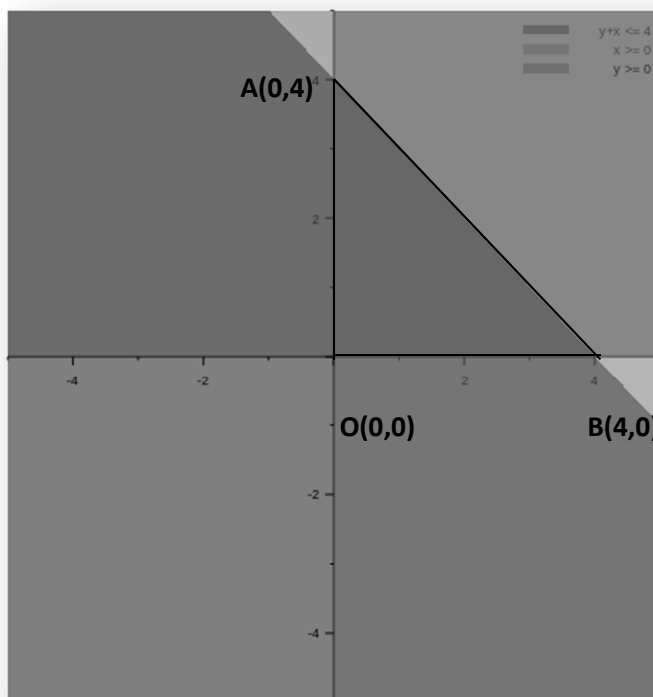$x + y \leq 4$, $x \geq 0$, $y \geq 0$

(3)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

   i)   (c) Convex region   (1)

(2)

Corner points of the feasible region are as follows

Corner points	Z=2x+3y

O(0,0)	0
A(0,4)	12 → Maximum
B(4,0)	8

∴ the maximum value of Z is 12 attained at (0,4)   (1)

Learning Outcomes

Formulates the linear programming problems
Identifies the decision variables, constraints, objective function, non negative constraints.
Recalls the solution of the linear inequalities. Infers solutions of LPP

Q.2
A manufacturer makes tea cups A and B. Three machines are needed for the manufacturing and the time in minutes required for each cups in the machine is given below

Machines	I	II	III
A	12	18	6
B	6	0	9

 Each machine is available for a maximum of 6 hours per day. If the profit on each cup A is 75 paise and that on each cup B is 50 paise. How many cups of types A and type B manufactured in a day to get maximum profit?   (5)

(Scores: 5 ; Time: 10 mts)
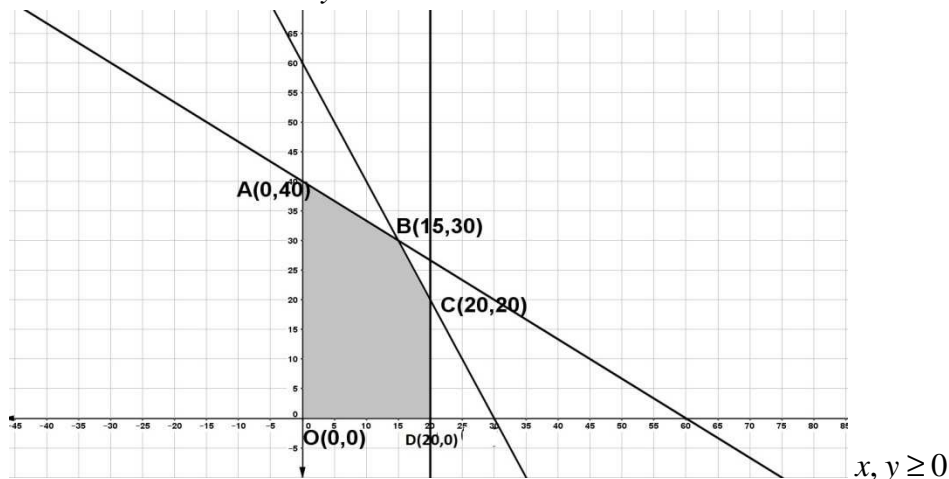
Scoring Indicators

Let x and y be the number of tea cups of types A and B
 Then the LPP is , Maximize $Z = 75x + 50y$ subject to the constraints
$$12x + 6y \leq 360$$
$$18x + 0y \leq 360$$
$$6x + 9y \leq 360$$

$$x, y \geq 0$$

(2)
Draw the graph   (2)

Corner Point	Z=75x+50y
O(0,0)	0
A(0,40)	2000
B(15,30)	2625 →Maximum
C(20,20)	2500
D(20,0)	1500

15 tea cups A and 30 tea cups B required to get maximum profit   (1)

Learning Outcomes

Recalls the method of solving the system of inequalities
Formulates LPP. Identifies the decision variables, constraints, objective function and the non negativity restriction. Identifies the solution of LPP

Q.3

A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum costs for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements (6)

(Scores: 6 ; Time: 12 mts)

Scoring Indicators

Let x units of food F_1 and y units of food F_2 be in the diet
Total cost $Z = 4x + 6y$

Then the LPP is

Minimize $Z = 4x + 6y$

Subject to the constraints
$$3x + 6y \geq 80$$
$$4x + 3y \geq 100$$
$$x, y \geq 0$$
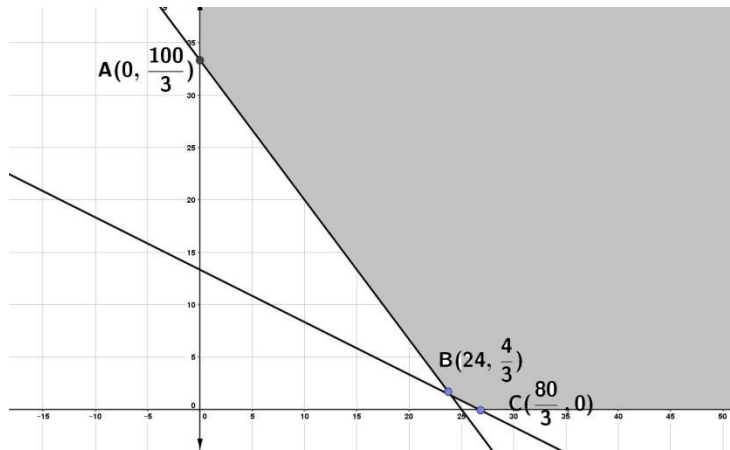
The feasible region is unbounded (2)

Corner Point	Z=4x+6y
$A(0,\dfrac{100}{3})$	108.67
$B(24,\dfrac{4}{3})$	104 →Smallest
$C(\dfrac{80}{3},0)$	200

As the feasible region is unbounded, 104 may or may not be the minimum value of Z. For this we draw a graph of the inequality $4x+6y<104$ or $2x+3y<52$ and check whether the resulting half plane has points in common with the feasible region or not. It can be seen that the feasible region has no common points with $2x+3y<52$ Therefore minimum cost of the mixture will be 104 (2)

Learning Outcomes

Recalls the method of solving the system of inequalities
Formulates LPP. Identifies the decision variables, constraints, objective function and the non negativity restriction. Identifies the solution of LPP

Q.4

The graph of a linear programming problem is given below. The shaded region is the feasible region. The objective function is $Z = px + qy$
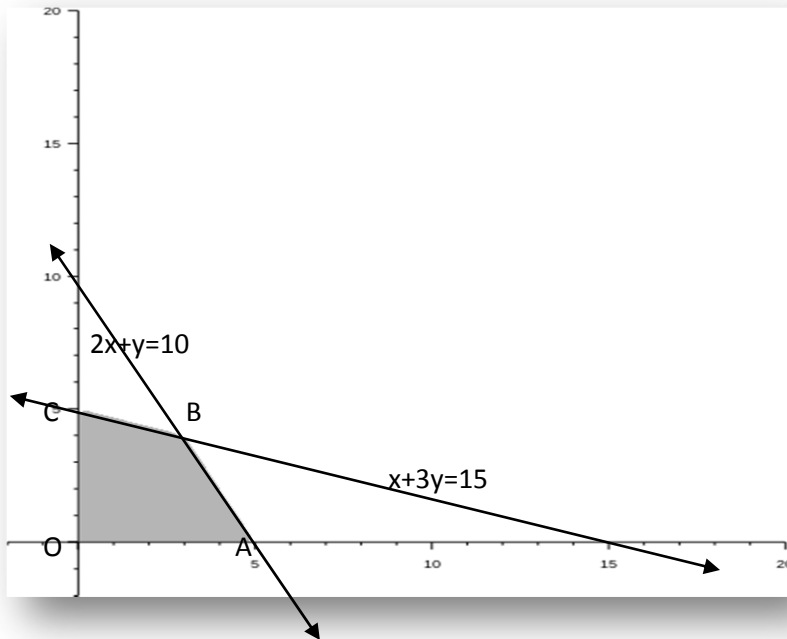
i)   What are the co-ordinates of the corners of the feasible region   (1)
ii)   Write the constraints   (1)
ii)   If the Max. Z occurs at A and B, what is the relation between p and q?
                                                                                (2)
iii)   If q= 1, write the objective function   (1)
iv)   Find the Max Z   (1)
                                            (Scores: 6 ; Time: 12 mts)

Scoring Indicators

i)   Corner points are   (0,0) , (5,0) , (3,4) , (0,5)   (1)
ii)   Constraints are $2x + y \leq 10$, $x + 3y \leq 15$, $x \geq 0$, $y \geq 0$ (1)

        At (3,4),   Z=3p+4q
        At (5,0),   Z=5p   (1)
        $\Rightarrow$ 3p+4q = 5p $\Rightarrow$ p = 2q   (1)
iii)   If q=1, p=2
        Then the objective function is , Maximize Z = 2x+y   (1)
iv)   At (3,4) Z=2x3+4 = 10 is the maximum value   (1)

CHAPTER – 13

PROBABILITY

Learning Outcomes

Applies the idea of conditional probability in life situation

Q.1

A coin is tossed three times , where the events A: occurring at most two heads , B : occurring at most one tail
(i) Write P(A) , P(B)   (1)
(i) Find P(A|B) and   P(B|A)   (2)

                                                                           (Scores: 3 ; Time: 6 mts)

Scoring Indicators

(i) S = { TTT,TTH,THT,HTT,THH,HHT, HTH,HHH }
   A = { TTH,THT,HTT,THH,HHT, HTH, TTT } ;   B = { HHH,HHT,THH,HTH }
$P(A) = \frac{7}{8}$, $P(B) = \frac{4}{8}$ (1)

(ii) A ∩ B = { HHT, HTH, THH }
   $P(A ∩ B) = \frac{3}{8}$
P(A|B) $=\frac{P(A ∩ B)}{P(B)} = \frac{3}{4}$ (1)
 and   P(B|A) $= \frac{P(A ∩ B)}{P(A)} = \frac{3}{7}$ (1)

Learning Outcomes

 Solves different life oriented problems in probability

Q.2

In a hostel 50 % of the girls like tea , 40 % like coffee and 20% like both tea and coffee .
A girl is selected at random.
(i) Find the probability that she likes neither tea nor coffee   (2)
(ii)   If the girl likes tea ,then find the probability that she likes coffee.   (1)
(iii)   If she likes coffee then find the probability she likes tea.   (1)

                                                                           (Scores: 4 ; Time: 8 mts)

SCORING INDICATORS

Let T denotes the set of girls who like tea and C denotes who like coffee .

i) $P(T) = 50\% = \frac{50}{100} = \frac{1}{2}$; $P(C) = 40\% = \frac{40}{100} = \frac{2}{5}$; $P(T \cap C) = 20\% = \frac{20}{100} = \frac{1}{5}$

$P(T \cup C)^| = 1 - P(T \cup C) = 1 - [P(T)) + (P(C) - P(T \cap C)]$ (1)

$= 1 - (\frac{1}{2} + \frac{2}{3} - \frac{1}{5}) = \frac{3}{10}$ (1)

ii) $P(C|T) = \frac{P(C \cap T)}{P(T)} = \frac{\frac{1}{5}}{\frac{1}{2}} = \frac{2}{5}$ (1)

iii) $P(T|C) = \frac{P(T \cap C)}{P(C)} = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$ (1)

Learning Outcomes

Solves problems using Baye's theorem

Q.3

Vineetha and Reshma are competing for the post of school leader . The probability Vineetha to be elected is 0.6 and Reshma is 0.4 Further if Vineetha is elected the probability of introducing a new pattern of election is 0.7 and the corresponding probability is 0.3 ,if Resma is elected .Find the probability that the new pattern of election is introduced by Reshma .   (4)

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

Let E_1 and E_2 be the respectively probability that Vineetha and
 Reshma will be elected . Let Abe the probability that a new pattern
of election is introduced .

$P(E_1) = 0.6$; $P(E_2) = 0.4$ (1)

$P(A|E_1) = 0.7$; $P(A|E_2) = 0.3$ (1)

$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$ (1)

$= \frac{0.4 \times 0.3}{0.6 \times 0.7 + 0.4 \times 0.3} = \frac{2}{9}$ (1)

Learning Outcomes

Forms probability distribution of random variables

Q.4

Find the probability distribution of the number of success in two tosses of a die where the success is defined as getting a number less than 5   (5)

(Scores: 5 ; Time: 10 mts)

Scoring Indicators

Here success refers to the number less than 5 .
Therefore the Random variable can take the values 0 , 1, 2
P(X = 0) = P(getting the number greater than or equal to 5 on both tosses)
$= \frac{2}{6} . \frac{2}{6} = \frac{1}{9}$ (1)
P(X = 1) = P(getting the number greater than or equal to 5 on first toss or
getting the number   less than 5 on second toss)
$= \frac{2}{6} . \frac{4}{6} + \frac{4}{6} . \frac{2}{6} = \frac{4}{9}$ (1)
P(X = 2) = P(getting the number   less than 5 on first toss
or getting the number greater than or equal to 5 on second toss)
$= \frac{4}{6} . \frac{2}{6} + \frac{2}{6} . \frac{4}{6} = \frac{4}{9}$ (1)
Probability distribution

X	0	1	2
P(X = x)	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$

(2)

Learning Outcomes

Recalls the concept of probability
Explains conditional probability and its representation
Applies the idea of conditional probability to life situations

Q.5

i)   Given that E and F are events such that P(E) = 0.6 , P(F) = 0.4

and $P(E \cap F) = 0.2$.   Then $\dfrac{P(E/_F)}{P(F/_E)}$ is

$(\dfrac{1}{2} , \dfrac{1}{3} , \dfrac{3}{2} , \dfrac{2}{3})$ (1)

ii)   A die is thrown and the sum of the numbers appearing is observed to be
greater than 9. What is the conditional probability that the number 5 is
appeared at least once   (2)

(Scores: 3 ; Time: 6 mts)

Scoring Indicators

i) $\dfrac{3}{2}$

(1)

ii)   Let E → Event that sum is greater than 9
            F → Event that 5 appears at least once
        $E \cap F = \{(5,5),(6,5),(5,6)\}$

        $P(E \cap F) = \dfrac{3}{36}$ (1)

        $F = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5),$
        $(5,1),(5,2),(5,3),(5,4),(5,6)\}$

        $P(F) = \dfrac{11}{36}$

        $P(E/_F) = \dfrac{3}{11}$ (1)

Learning Outcomes

Recalls the basic concept of probability
Solves problems related to probability using multiplication theorem

Q.6

i)   A and B are two events such that $P(A) = \dfrac{1}{5}$ and $P(A \cup B) = \dfrac{2}{5}$. Find

        $P(B)$ if they are mutually exclusive ($\dfrac{1}{5}$, $\dfrac{2}{5}$, $\dfrac{3}{5}$, $\dfrac{4}{5}$)   (1)

ii)   A box contains 3 red and 4 blue balls. Two balls are drawn one by one
       without replacement. Find the probability of getting both balls red   (2)

iii)   Three cards are drawn successively without replacement from a pack of 52
       cards. What is the probability that first two cards are queen and the third
       is king ?   (3)

                                                                    (Scores: 6 ; Time: 12 mts)

Scoring Indicators

i) $\dfrac{1}{5}$
                                                                                                                    (1)

ii)   Let A be the event that the first ball drawn is red and B be the event of
       drawing red ball in the second draw

        $P(A) = \dfrac{3}{7}$
                                                                                                                    (1)

        $P(B/_A) = $ Probability of getting one red ball in the second draw

$$= \frac{2}{6}$$

$$P(A \cap B) = P(A).P\left(\frac{B}{A}\right)$$

$$= \frac{3}{7}.\frac{2}{6} = \frac{1}{7}$$

(1)

iii)   Let Q denote the event that the card drawn is Queen and K denote the event of drawing a King

$$P(Q) = \frac{4}{52}$$

(1)

$$P\left(\frac{Q}{Q}\right) = \frac{3}{51}$$

$P\left(\frac{K}{QQ}\right)$ is the probability of drawing the third card is a king

$$P\left(\frac{K}{QQ}\right) = \frac{4}{50}$$

(1)

$$P(QQK) = P(Q)P\left(\frac{Q}{Q}\right)P\left(\frac{K}{QQ}\right)$$

$$= \frac{4}{52}.\frac{3}{51}.\frac{4}{50} = \frac{2}{5525}$$

(1)

Learning Outcomes

Explains independent events
Solves different life oriented problems in probability using independence of events

Q.7

i)   Find $P(A \cap B)$ if A and B are independent events with $P(A) = \frac{1}{5}$ and

$P(B) = \frac{5}{8}$ ($\frac{6}{13}$, $\frac{33}{40}$, $\frac{1}{8}$, $\frac{5}{8}$)

(1)

ii)   An unbiased die is thrown twice. Let the event A be getting prime number in the first throw and B be the event of getting an even number in the second throw. Check the independence of the events A and B

(2)

iii)   The probability of solving a problem independently by A and B are $\frac{1}{3}$

and $\frac{1}{4}$ respectively. Find the probability that exactly one of them solves the problem

(2)

(Scores: 5 ; Time: 10 mts)

Scoring Indicators

i) $\dfrac{1}{8}$

ii) $P(A) = \dfrac{18}{36} = \dfrac{1}{2}$

$P(B) = \dfrac{18}{36} = \dfrac{1}{2}$

$P(A \cap B) = $ P(prime number in first throw and even number in the second throw)   (1)

$= \dfrac{9}{36} = \dfrac{1}{4}$

$P(A)P(B) = \dfrac{1}{2} \cdot \dfrac{1}{2}$

$= \dfrac{1}{4} = P(A \cap B)$

$\therefore$ A and B are independent events   (1)

iii) $P(A) = \dfrac{1}{3}$

$P(B) = \dfrac{1}{4}$

$P(B') = 1 - P(B) = \dfrac{3}{4}$ (1)

$P(A') = 1 - P(A) = \dfrac{2}{3}$ (1)

Probability of exactly one of them solves the problem

$= P(A)P(B') + P(B)P(A')$

$= \dfrac{1}{3} \cdot \dfrac{3}{4} + \dfrac{1}{4} \cdot \dfrac{2}{3}$

$= \dfrac{1}{4} + \dfrac{1}{6} = \dfrac{5}{12}$ (1)

Learning Outcomes

Forms the partition of sample space
Applies the total probability theorem in life situations

Q.8

i)   A set of events $E_1, E_2, ... E_n$ are said to be a partition of the Sample Space , then which of the following conditions is always not true
$[E_1 \cup E_2 \cup .... \cup E_n = S,\ E_1 \cap E_n = \phi,\ P(E_1) > 0,\quad P(E_1) \geq P(E_n)]$ (1)

ii)   A person has undertaken a business. The probabilities are 0.80 that there will be a crisis, 0.85 that the business will be completed on time if there is no crisis and 0.35 that the business will be completed on time if there is a crisis. Determine the probability that the business will be completed on time.   (2)

iii)   A box contains 5 red and 10 black balls. A ball is drawn at random, its colour is noted and is returned to the box. More over 2 additional balls of the colour drawn are put in the box and then a ball is drawn. What is the probability that the second ball is red?   (3)

(Scores: 6 ; Time: 12 mts)

Scoring Indicators

i)

$$P(E_1) \geq P(E_n)$$
(1)

ii)   Let A be the event that the business will be completed on time and B be the event that there will be a crisis

$P(B) = 0.80$

$P(no\ crisis) = P(B') = 1 - P(B)$
$\qquad\qquad\qquad = 0.20$

$P(A/_B) = 0.35$

$P(A/_{B'}) = 0.85$
(1)

By theorem on total probability

$P(A) = P(B)P(A/_B) + P(B')P(A/_{B'})$
$\qquad = 0.8 \times 0.35 + 0.20 \times 0.85$
(1)
$\qquad = 0.45$

iii)   Let a red ball be drawn in the first attempt

$P(drawing\ a\ red\ ball)\ = \dfrac{5}{15} = \dfrac{1}{3}$

If two red balls are added to the box, then the box contains 7 red balls and 10 black balls

$P(drawing\ a\ red\ ball)\ = \dfrac{7}{17}$
(1)

Let a black ball be drawn in the first attempt

$P(drawing\ a\ black\ ball)\ = \dfrac{10}{15} = \dfrac{2}{3}$

If two black balls are added to the box, then the box contains 5 red and 12 black balls

$P(drawing\ a\ red\ ball)\ = \dfrac{5}{17}$
(1)

Probability of drawing the second ball red is

$$\frac{1}{3}\cdot\frac{7}{17}+\frac{2}{3}\cdot\frac{5}{17}=\frac{1}{3}$$

(1)

Learning Outcomes

Solves problems using Bayes' Theorem

Q.9

i)   Bag I contains 5 red and 6 black balls. Bag II contains 7 red and 5 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from bag I   (3)

ii)   A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being diamond.   (3)

(Scores: 6 ; Time: 12 mts)

Scoring Indicators

i)   Let E_1 be the event of choosing Bag I and E_2 be the event of choosing Bag II

A be the event of drawing a red ball

$$P(E_1) = P(E_2) = \frac{1}{2}$$

(1)

$$P(A/E_1) = \frac{5}{11}$$

$$P(A/E_2) = \frac{7}{12}$$

Using Bayes' Theorem   (1)

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{5}{11}}{\frac{1}{2}\cdot\frac{5}{11}+\frac{1}{2}\cdot\frac{7}{12}}$$

(1)

$$= \frac{60}{137}$$

ii)   Let E_1 be the event of choosing a diamond and E_2 be the event of choosing a non diamond card

A be the event that a card is lost

$$P(E_1) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_2) = \frac{39}{52} = \frac{3}{4}$$

When a diamond card is lost, there are 12 diamond cards in 52 cards.
Then

$$P(A/E_1) = \frac{^{12}C_2}{^{51}C_2} = \frac{22}{425}$$

(1)

When a non diamond card is lost, there are 13 diamond cards in 51 cards.
Then

$$P(A/E_2) = \frac{^{13}C_2}{^{51}C_2} = \frac{26}{425}$$

(1)

Then by Bayes' Theorem

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{4} \cdot \frac{22}{425}}{\frac{1}{4} \cdot \frac{22}{425} + \frac{3}{4} \cdot \frac{26}{425}}$$

(1)

$$= \frac{11}{50}$$

Learning Outcomes

Describes random variable
Forms probability distribution of a random variable

Q.10
   i)   If X denotes number of heads obtained in tossing two coins. Then which of the following is false
        (X(HH) = 2 , X(HT) = 1 , X(TH) = 0 , X(TT) = 0)   (1)
   ii)   Find the probability distribution of the number of heads in the simultaneous toss of two coins   (3)
   iii)   A coin is tossed so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails (3)

(Scores: 7 ; Time: 14 mts)

Scoring Indicators

i)   X(TH) = 0   (1)

ii)   Sample space is

$S = \{HH, HT, TH, TT\}$

Let X denote the number of heads, then

$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0$ (1)

Therefore X can take the values 0, 1 or 2

$P(HH) = P(HT) = P(TH) = P(TT) = \dfrac{1}{4}$

$P(X = 0) = \dfrac{1}{4}$

$P(X = 1) = \dfrac{1}{2}$

$P(X = 2) = \dfrac{1}{4}$ (1)

Then the Probability distribution is   (1)

X	0	1	2
P(X)	$\dfrac{1}{4}$	$\dfrac{1}{2}$	$\dfrac{1}{4}$

iii)   Let the probability of getting a tail in the biased coin be x .

$P(T) = x$

$P(H) = 3x$

$P(T) + P(H) = 1$

$\Rightarrow x + 3x = 1$ (1)

$\qquad x = \dfrac{1}{4}$

$P(T) = \dfrac{1}{4}, \qquad P(H) = \dfrac{3}{4}$

Let X denote the random variable representing the number of tails

$P(X = 0) = P(HH) = P(H).P(H)$

$\qquad\qquad = \dfrac{9}{16}$ (1)

$P(X = 1) = P(HT) + P(TH)$

$\qquad\qquad = \dfrac{3}{4}.\dfrac{1}{4} + \dfrac{1}{4}.\dfrac{3}{4}$

$$= \frac{3}{8}$$

$$P(X=2) = P(TT) = P(T).P(T)$$

$$= \frac{1}{16}$$

Then the Probability distribution is   (1)

X	0	1	2
P(X)	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

Learning Outcomes

Find the mean and variance of a probability distribution

Q.11

   i)   Fill in the blank of a probability distribution of a random variable X

X	0	1	2
P(X)	0.2	0.4	....

(0.2 , 0.3 , 0.4 , 0.6)   (1)

   ii)   Find the mean and variance of the number of heads in three tosses of a
           coin   (3)

(Scores: 7 ; Time: 14 mts)

Scoring Indicators

   i)   0.4   (1)

   ii)   Let X denote the number of heads in three tosses of a coin
           Sample space is
           $$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$ (1)

   Then X can take the values 0, 1, 2 or 3

   $$P(X=0) = P(TTT)$$
   $$= P(T)P(T)P(T)$$
   $$= \frac{1}{2}.\frac{1}{2}.\frac{1}{2}$$
   $$= \frac{1}{8}$$
   $$P(X=1) = P(HTT) + P(THT) + P(TTH)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{3}{8}$$

$$P(X = 2) = P(HHT) + P(HTH) + PTHH)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{3}{8}$$

$$P(X = 3) = P(HHH)$$

$$= \frac{1}{8}$$

Required probability distribution is   (1)

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean of X ,

$$E(X) = \sum x_i p(x_i)$$

$$= 0.\frac{1}{8} + 1.\frac{3}{8} + 2.\frac{3}{8} + 3.\frac{1}{8}$$

$$= \frac{3}{2} = 1.5$$

$$E(X^2) = \sum x_i^2 p(x_i)$$

$$= 0^2.\frac{1}{8} + 1^2.\frac{3}{8} + 2^2.\frac{3}{8} + 3^2.\frac{1}{8}$$

$$= 3$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 3 - (1.5)^2$$

$$= 0.75$$ (1)

Learning Outcomes

Recognizes Bernoulli trials
Solves problems using binomial distribution

Q.12

i)   The probability that a student is not a sportsman is   1/5. Then the
         probability that out of 6 students , 4 are sportsman is ....

$$(^{6}C_{4}(\frac{4}{5})^{4}(\frac{1}{5})^{2} \ , \ (\frac{4}{5})^{4}(\frac{1}{5})^{2} \ , \ ^{6}C_{4}(\frac{4}{5})^{2}(\frac{1}{5})^{4} \ , \ (\frac{4}{5})^{2}(\frac{1}{5})^{4})$$

(1)

ii) Find the probability of getting atmost 2 sixes in 6 throws of a single die

(Scores: 4 ; Time: 8 mts)

Scoring Indicators

i) $^{6}C_{4}(\frac{4}{5})^{4}(\frac{1}{5})^{2}$ (1)

ii) The repeated tossing of the die are Bernoullis trails

Let X represents the number of times of getting sixes in 6 throws of the die.

$$p = \frac{1}{6}$$

$$q = 1 - p = \frac{5}{6}$$

$$P(X = x) = {}^{n}C_{x}q^{n-x}p^{x}$$ (1)

$$= {}^{6}C_{x}(\frac{5}{6})^{6-x}(\frac{1}{6})^{x}$$

$$P(atmost \ 2sixes) = P(X \leq 2)$$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$ (1)

$$= {}^{6}C_{0}(\frac{5}{6})^{6} + {}^{6}C_{1}(\frac{5}{6})^{5}(\frac{1}{6}) + {}^{6}C_{2}(\frac{5}{6})^{4}(\frac{1}{6})^{2}$$

(1)

$$= \frac{35}{18}(\frac{5}{6})^{4}$$