DATE : 18/03/2019

SET-2

Code No. 65/1/2

Aakash
Medical | IIT-JEE | Foundations
(Divisions of Aakash Educational Services Limited)

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Time : 3 Hrs.

Class XII

Mathematics

(CBSE 2019)

Max. Marks : 100

GENERAL INSTRUCTIONS :

(i) All questions are compulsory.

*(ii) This question paper contains **29** questions divided into four sections **A, B, C** and **D**. Section **A** comprises of **4** questions of **one mark** each, Section **B** comprises of **8** questions of **two marks** each, Section **C** comprises of **11** questions of **four marks** each and Section **D** comprises of **6** questions of **six marks** each.*

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

*(iv) There is no overall choice. However, an internal choice has been provided in **1** question of Section A, **3** questions of Section B, **3** questions of Section C and **3** questions of Section D. You have to attempt only **one** of the alternatives in all such questions.*

(v) Use of calculator is not permitted. You may ask logarithmic tables, if required.

Section-A

Question numbers 1 to 4 carry 1 mark each.

1. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$. **[1]**

Sol. $x^2\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$

Order of the highest derivative = 2

and power of $\frac{d^2y}{dx^2}$ is 1

So, order of differential equation = 2 **[½]**

Degree of differential equation = 1 **[½]**

2. If f(x) = x + 7 and g(x) = x – 7, $x \in R$, then find $\frac{d}{dx}(fog)(x)$. **[1]**

Sol. Given f(x) = x + 7 and

g(x) = x – 7

(fog)(x) = f(g(x))

= f(x – 7)

= (x – 7) + 7

= x **[½]**

$$\frac{d}{dx}(fog)(x) = \frac{d}{dx}(x) = 1$$ **[½]**

3. Find the value of x – y, if

$$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}.$$ **[1]**

Sol. $$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2+y & 6+0\\0+1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$ **[½]**

$$\Rightarrow \begin{bmatrix}2+y & 6\\1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

Two matrices are equal and their corresponding entries are equal.

$\Rightarrow$ 2 + y = 5, 2x + 2 = 8

$\Rightarrow$ y = 3, x = 3

$\Rightarrow$ x – y = 3 – 3 = 0 **[½]**

4. If a line makes angles 90°, 135°, 45° with the x, y and z axes respectively, find its direction cosines. **[1]**

OR

Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$.

Sol. Let α = 90°

β = 135°

γ = 45°

$\because$ l = cosα = cos90° = 0 **[½]**

m = cosβ = cos135° $= -\frac{1}{\sqrt{2}}$

n = cosγ = cos45° $= \frac{1}{\sqrt{2}}$

Direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ **[½]**

OR

The equation of line passing through $A(\bar{a})$ and parallel to the vector $\bar{b}$ is

$\vec{r} = \vec{a} + \lambda\vec{b}$ where $\lambda \in R$ **[½]**

Here $\vec{a} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{b} = 2\hat{i} + 2\hat{j} - 3\hat{k}$

So equation of line will be

$\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k})$ **[½]**

Section-B

Question numbers 5 to 12 carry 2 marks each.

5. Examine whether the operation * defined on R by a * b = ab + 1 is (i) a binary or not. (ii) if a binary operation, is it associative or not? **[2]**

Sol. An operation * is called a binary operation

if for $\forall$ a, b $\in$ A, a * b $\in$ A.

Here a, b $\in$ R $\Rightarrow$ ab + 1 $\in$ R.

$\therefore$ * is a binary operation on R. **[1]**

For associative property :

Let a, b, c $\in$ R

We have to prove a * (b * c) = (a * b) * c

L.H.S. = a * (b * c)

= a * (bc + 1)

= a · (bc + 1) + 1

= abc + a + 1

and R.H.S. = (a * b) * c

= (ab + 1) * c

= (ab + 1) · c + 1

= abc + c + 1

$\because$ L.H.S. $\neq$ R.H.S. **[1]**

$\therefore$ '*' operation is not associative

6. If A = $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find $(A^2 - 5A)$. **[2]**

Sol. A = $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Now $A^2 = A.A$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$ [1]

Now value of $A^2 - 5A$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$ [½]

So $A^2 - 5A = \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}$ [½]

7. Find : $\int \sqrt{1-\sin 2x}\,dx, \frac{\pi}{4} < x < \frac{\pi}{2}$ [2]

OR

Find : $\int \sin^{-1}(2x)\,dx.$

Sol. Let I = $\int \sqrt{1-\sin 2x}\,dx, \frac{\pi}{4} < x < \frac{\pi}{2}$

$\therefore$ I = $\int \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\,dx$ [½]

$= \int \sqrt{(\cos x - \sin x)^2}\,dx$

$= \int |\cos x - \sin x|\,dx$

For $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, sinx > cosx

$\therefore$ |cosx – sinx| = sinx – cosx

$\Rightarrow$ I = $\int (\sin x - \cos x)\,dx$ [½]

= – cosx – sinx + C [1]

where C is the constant of integration.

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OR

Let $I = \int \sin^{-1}(2x)dx$

Let 2x = t

2dx = dt

$\therefore I = \frac{1}{2}\int 1.\sin^{-1}(t)dt$

Integrating by parts, we get

$= \frac{1}{2}\left[(\sin^{-1}t)t - \int \frac{t\,dt}{\sqrt{1-t^2}}\right]$ **[1]**

$= \frac{1}{2}\left[t\sin^{-1}t - \frac{1}{2}\int \frac{2t\,dt}{\sqrt{1-t^2}}\right]$

$= \frac{1}{2}\left[t\sin^{-1}t + \frac{1}{2}\int \frac{-2t}{\sqrt{1-t^2}}dt\right]$

$= \frac{1}{2}t\sin^{-1}t + \frac{1}{4}\frac{(1-t^2)^{-\frac{1}{2}+1}}{\frac{1}{2}} + C$

$= \frac{1}{2}t\sin^{-1}t + \frac{1}{2}\sqrt{1-t^2} + C$ **[1]**

$= \frac{1}{2}.2x.\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$

$= x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$

where C is the constant of integration.

8. Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants. **[2]**

Sol. $y = e^{2x}(a + bx)$

$\Rightarrow \quad e^{-2x}\cdot y = a + bx$

Differentiating w.r.t x

$$e^{-2x}\cdot\left[\frac{dy}{dx}\right] + y\cdot e^{-2x}\cdot(-2) = b$$

$\Rightarrow \quad e^{-2x}\cdot\left[\frac{dy}{dx} - 2y\right] = b$ **[1]**

Again differentiating w.r.t x

$$\Rightarrow \quad e^{-2x}\left[\frac{d^2y}{dx^2}-2\frac{dy}{dx}\right]+\left[\frac{dy}{dx}-2y\right]\cdot e^{-2x}\cdot(-2)=0$$

$$\Rightarrow \quad e^{-2x}\left[\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y\right]=0$$

$\Rightarrow \quad \frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$ is the required differential equation. **[1]**

9. A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) atmost 5 successes? **[2]**

OR

The random variable X has a probability distribution P(X) of the following form, where 'k' is some number.

$$P(X=x)=\begin{cases} k, & \text{if } x=0 \\ 2k, & \text{if } x=1 \\ 3k, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of 'k'.

Sol. Probability of success in one attempt is p.

$\therefore$ p = Probability of getting an odd number

$\Rightarrow \quad p=\frac{3}{6}=\frac{1}{2}$

Probability of unsuccess attempt is $q=1-p=\frac{1}{2}$

(i) Probability of 5 success in six attempts

$= {}^6C_5.p^5.q^1$

$= {}^6C_5.\left(\frac{1}{2}\right)^5.\left(\frac{1}{2}\right)^1$

$= \frac{6}{2^6}$

$= \frac{3}{32}$ **[1]**

(ii) Probability of atmost 5 success in six attempts

= 1 – Probability of 6 success

$= 1-{}^6C_6.p^6.q^0$

$= 1-{}^6C_6.\left(\frac{1}{2}\right)^6$

$= 1-\frac{1}{64}$

$= \frac{63}{64}$ **[1]**

OR

$$P(X=x)=\begin{cases} k, & \text{if } x=0 \\ 2k, & \text{if } x=1 \\ 3k, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

We know that,

$\sum P(x_i) = 1$ **[1]**

$\Rightarrow$ $k + 2k + 3k = 1$

$\Rightarrow$ $6k = 1$

$\Rightarrow$ $k=\frac{1}{6}$ **[1]**

10. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event "number is even" and B be the event "number is marked red". Find whether the events A and B are independent or not. **[2]**

Sol. The events are defined as

A = number is even

= {2, 4, 6}

and B = number is marked red

= {1, 2, 3}

$A\cap B$ = number which is even as well as marked red

= {2}

$\therefore$ Probability of occurrence of event A,

$P(A)=\frac{3}{6}=\frac{1}{2}$ **[1]**

Probability of occurrence of event B,

$P(B)=\frac{3}{6}=\frac{1}{2}$

and $P(A\cap B)=\frac{1}{6}$

Here $P(A\cap B) \neq P(A).P(B)$ **[1]**

$\therefore$ A and B are not independent events.

11. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. **[2]**

OR

If $\vec{a}=2\hat{i}+3\hat{j}+\hat{k}, \vec{b}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{c}=-3\hat{i}+\hat{j}+2\hat{k}$, find $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

Sol. Let any two unit vectors be $\hat{a}$ and $\hat{b}$.

Given that

Sum of two unit vectors is an unit vector.

So, $\hat{a}+\hat{b}=\hat{r}$ [where $\hat{r}$ is an unit vector]

$|\hat{a}+\hat{b}|=|\hat{r}|$

Now square both side, we get

$(\hat{a}+\hat{b}).(\hat{a}+\hat{b})=\hat{r}.\hat{r}$ **[½]**

$\Rightarrow$ $\hat{a}.\hat{a}+\hat{a}.\hat{b}+\hat{b}.\hat{a}+\hat{b}.\hat{b}=\hat{r}.\hat{r}$ [$\because \hat{n}.\hat{n}=1$]

$\Rightarrow$ $1+\hat{a}.\hat{b}+\hat{a}.\hat{b}+1=1$ [$\because \hat{a}.\hat{b}=\hat{b}.\hat{a}$]

$\Rightarrow \quad 2\hat{a}.\hat{b} = -1$

$\Rightarrow \quad \boxed{\hat{a}.\hat{b} = -\frac{1}{2}}$ ...(i) [½]

Now, magnitude of their difference = $|\hat{a}-\hat{b}|$

Let, $|\hat{a}-\hat{b}| = t$

Square on both side

$t^2 = |\hat{a}-\hat{b}|^2$ $[\because |\vec{a}|^2 = (\vec{a}).(\vec{a})]$

$\Rightarrow \quad t^2 = (\hat{a}-\hat{b}).(\hat{a}-\hat{b})$ [½]

$\Rightarrow \quad t^2 = \hat{a}.\hat{a} - \hat{a}.\hat{b} - \hat{b}.\hat{a} + \hat{b}.\hat{b}$

$\Rightarrow \quad t^2 = 1 - 2\hat{a}.\hat{b} + 1$

$\Rightarrow \quad t^2 = 2 - (-1)$ [By using Eqn. (i)]

$\Rightarrow \quad t^2 = 3$

$\Rightarrow \quad t = \sqrt{3}$

$\Rightarrow \quad \boxed{|\hat{a}-\hat{b}| = \sqrt{3}}$ Hence proved. [½]

OR

Given

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$,

$\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$

The value of $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ [1]

$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$

$= 2(-4-1) - 3(2+3) + 1(1-6)$

$= -30$ [1]

12. Find: $\int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x} dx$. [2]

Sol. Let $I = \int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x} dx$

Let $\tan^3 x = t$

$3\tan^2 x \sec^2 x\, dx = dt$

$\Rightarrow \quad I = \int \frac{1}{3} \frac{dt}{1-t^2}$ [½]

$= \frac{1}{3}\int \frac{dt}{1-t^2}$

$= \frac{1}{6} \ln\left|\frac{1+t}{1-t}\right| + C,$ [½]

where C is constant of integration

$I = \frac{1}{6} \ln\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right| + C$ [1]

Section-C

Question numbers 13 to 23 carry 4 marks each.

13. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$. **[4]**

Sol. $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$ ...(i)

$\tan^{-1}(2x) = \frac{\pi}{4} - \tan^{-1}(3x)$

$\Rightarrow \tan\left(\tan^{-1}(2x)\right) = \tan\left(\frac{\pi}{4} - \tan^{-1}(3x)\right)$ **[1]**

$\Rightarrow 2x = \frac{1-3x}{1+3x} \quad \Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow x = \frac{1}{6}$ or -1 **[1]**

$x = -1$ is rejected as $\tan^{-1}(-2)$ is negative and $\tan^{-1}(-3)$ is negative but RHS of (i) is positive

$\therefore x = \frac{1}{6}$ is the only solution. **[2]**

14. If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$. **[4]**

OR

If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

Sol. $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

Differentiating w.r.t. x

$$\frac{d}{dx}\left(\log(x^2+y^2)\right) = \frac{d}{dx}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right)$$

$$\Rightarrow \frac{1}{x^2+y^2}\cdot\frac{d}{dx}(x^2+y^2) = \frac{2}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{d}{dx}\left(\frac{y}{x}\right)$$ **[1]**

$$\Rightarrow \frac{2x+2y\frac{dy}{dx}}{x^2+y^2} = \frac{2\left[\frac{x\frac{dy}{dx}-y}{x^2}\right]}{(x^2+y^2)/x^2}$$ **[1]**

$$\Rightarrow x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$$ **[1]**

$$\Rightarrow (y-x)\frac{dy}{dx} = -y-x$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$ Hence proved. **[1]**

OR

Given $x^y - y^x = a^b$

Differentiating w.r.t. x

$$\frac{d}{dx}(x^y) - \frac{d}{dx}(y^x) = 0 \quad \text{...(i)}$$

Now let $x^y = v$

then $\log v = y \cdot \log x$

Differentiating w.r.t. x

$$\frac{1}{v}\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right]$$

$$\Rightarrow \frac{d}{dx}(x^y) = x^y\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right] \quad \text{...(ii)}$$ **[1]**

Similarly, let $y^x = u$

then $\log u = x \cdot \log y$

differentiating w.r.t. x

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y \cdot 1$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

$$\Rightarrow \frac{d}{dx}(y^x) = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad \text{...(iii)}$$ **[1]**

Replacing values from (ii) and (iii) in (i), we get

$$x^y\left[\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right] - y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow \frac{dy}{dx}\left[x^y \cdot \log x - y^x \cdot \left(\frac{x}{y}\right)\right] = y^x \cdot \log y - x^y\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^x \cdot \log y - x^{y-1} \cdot y}{x^y \cdot \log x - y^{x-1} \cdot x}$$ **[2]**

15. Find : $\int \frac{3x+5}{x^2+3x-18}dx.$ **[4]**

Sol. $I = \int \frac{3x+5}{x^2+3x-18}\,dx$

$$= \int \frac{\frac{3}{2}(2x+3) - \frac{9}{2} + 5}{x^2+3x-18}\,dx$$ **[1]**

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$$= \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \int \frac{\frac{1}{2}}{x^2+3x-18}dx$$

$$= \frac{3}{2}\int \frac{\frac{d}{dx}(x^2+3x-18)}{x^2+3x-18}dx + \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - 18 - \frac{9}{4}}$$ [1]

$$= \frac{3}{2}\ln\left|x^2+3x-18\right| + \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}$$

$$= \frac{3}{2}\ln\left|x^2+3x-18\right| + \frac{1}{2}\cdot\frac{1}{2\cdot\frac{9}{2}}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + c$$ [1]

$$= \frac{3}{2}\ln\left|x^2+3x-18\right| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + c$$ [1]

16. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. Hence evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$. [4]

Sol. To prove : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

R.H.S. $= \int_0^a f(a-x)dx$

Let $a - x = t$

On differentiating both sides w.r.t. x, we get

$-dx = dt$

And when $x = 0$ then $t = a$

And when $x = a$ then $t = 0$ [1]

$\therefore$ R.H.S. $= \int_a^0 f(t)(-dt)$

$= \int_0^a f(t)dt \quad \left\{\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right\}$

$= \int_0^a f(x)dx \quad \left\{\because \int_a^b f(x)dx = \int_a^b f(t)dt\right\}$

= L.H.S. Hence proved [1]

Now $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$ ...(i)

$I = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$ (above property)

$I = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$ ...(ii)

From equation (i) + equation (ii) we get,

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$$ [1]

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Let cos x = t

sin xdx = –dt

$$2I = \pi \int_1^{-1} \frac{-dt}{1+t^2}$$

$$2I = \pi \int_{-1}^{1} \frac{dt}{1+t^2}$$

$$I = \frac{\pi}{2}\left(\tan^{-1} t\right)_{-1}^{1}$$

$$= \frac{\pi}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{4}$$ [1]

17. If $\hat{i}+\hat{j}+\hat{k},\ 2\hat{i}+5\hat{j},\ 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not. [4]

Sol. Given $\overrightarrow{OA} = \hat{i}+\hat{j}+\hat{k},\ \overrightarrow{OB} = 2\hat{i}+5\hat{j},\ \overrightarrow{OC} = 3\hat{i}+2\hat{j}-3\hat{k},\ \overrightarrow{OD} = \hat{i}-6\hat{j}-\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \hat{i}+4\hat{j}-\hat{k}$$ [½]

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$= -2\hat{i}-8\hat{j}+2\hat{k}$$ [½]

Let angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is θ, then

$$\cos\theta = \left|\frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|}\right|$$ [1]

$$= \left|\frac{(\hat{i}+4\hat{j}-\hat{k})\cdot(-2\hat{i}-8\hat{j}+2\hat{k})}{\sqrt{1^2+4^2+(-1)^2}\cdot\sqrt{(-2)^2+(-8)^2+2^2}}\right|$$

$$= \left|\frac{-2-32-2}{\sqrt{18}\cdot\sqrt{72}}\right|$$

$$= \frac{36}{36}$$

⇒ cosθ = 1 [1]

∴ θ = 0°

Hence, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear. [1]

18. Using properties of determinants, prove the following : **[4]**

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

Sol. $\Delta = \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we get **[½]**

$$\Delta = \begin{vmatrix} a & b & c \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$, $R_2 \to R_2 - R_1$ **[1]**

$$\Delta = \begin{vmatrix} a & b & c \\ -(a+c) & (a+c) & -(a+c) \\ -(a+b) & -(a+b) & (a+b) \end{vmatrix}$$

Taking (a + c) and (a + b) common from R_2 and R_3 respectively. **[½]**

$$\Delta = (a+b)(a+c)\begin{vmatrix} a & b & c \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$ **[1]**

$$\Delta = (a+b)(a+c)\begin{vmatrix} a & b & c \\ -1 & 1 & -1 \\ -2 & 0 & 0 \end{vmatrix}$$

expanding along R_3

$\Delta = (a + b)(a + c)\cdot(-2)(-b - c) = 2(a + b)(b + c)(c + a)$ Hence proved. **[1]**

19. If $x = \cos t + \log \tan\left(\frac{t}{2}\right)$, $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **[4]**

Sol. Given y = sin t

Differentiating w.r.t. t

$$\frac{dy}{dt} = \cos t \qquad \text{...(i)}$$

Differentiating again w.r.t. t

$$\frac{d^2y}{dt^2} = -\sin t$$

$$\Rightarrow \frac{d^2y}{dt^2}_{\left(\text{at } t=\frac{\pi}{4}\right)} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$ **[1]**

Given $x = \cos t + \log\left(\tan\frac{t}{2}\right)$

Differentiating w.r.t. t

$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{\sin t} \qquad \left(\because 2\sin\frac{t}{2}\cdot\cos\frac{t}{2} = \sin t\right) \qquad \text{...(ii)}$$

From (i) and (ii) **[1]**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t + \frac{1}{\sin t}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin t \cdot \cos t}{1-\sin^2 t} = \frac{\sin t \cdot \cos t}{\cos^2 t}$$

$$\Rightarrow \frac{dy}{dx} = \tan t$$

Differentiating w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} \qquad \textbf{[1]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2 t}{\left(-\sin t + \frac{1}{\sin t}\right)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sec^2 t \cdot \sin t}{\cos^2 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sin t}{\cos^4 t}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{\left(\text{at } t=\frac{\pi}{4}\right)} = \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^4} = 2\sqrt{2} \qquad \textbf{[1]}$$

20. Show that the relation R on $\mathbb{R}$ defined as R = {(a, b) : a ≤ b}, is reflexive, and transitive but not symmetric. **[4]**

OR

Prove that the function f : N → N, defined by f(x) = x^2 + x + 1 is one-one but not onto. Find inverse of f : N → S, where S is range of f.

Sol. Given R = {(a, b) : a ≤ b }

Reflexivity

As a ≤ a ∀ a ∈ R

∵ 'R' is reflexive **[1]**

Symmetry

Let (a, b) ∈ R

so a ≤ b

from here it is not necessary that $b \le a$

So $(b, a) \in R$ is not true

Clearly R is not symmetric **[2]**

Transitivity

Let a R b and b R c

$\Rightarrow$ $a \le b$ and $b \le c$

It follows that $a \le c$

$\Rightarrow$ a R c

So a R b, b R c $\Rightarrow$ a R c

So 'R' is transitive **[1]**

Hence 'R' is reflexive as well as transitive but not symmetric

OR

$f : N \to N$ defined as $f(x) = x^2 + x + 1$

Let $x_1, x_2 \in N$

Then $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$\Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$

$\Rightarrow x_1 - x_2 = 0$ (as $x_1 + x_2 + 1 \ne 0$ for $x_1, x_2 \in N$)

$\Rightarrow x_1 = x_2$

so $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

$\Rightarrow$ 'f' is one -one. **[1½]**

'f' is not onto as $(x^2 + x + 1)$ does not attain all natural number values for $x \in N$.

(e.g., $x^2 + x + 1 \ne 1$ for any $x \in N$) **[½]**

so 'f' is one one but not onto

Now $f(x) = x^2 + x + 1$

Put $x = f^{-1}(x)$

$\therefore \quad x = (f^{-1}(x))^2 + f^{-1}(x) + 1$

$\Rightarrow (f^{-1}(x))^2 + (f^{-1}(x)) + (1 - x) = 0$

$$\Rightarrow \quad f^{-1}(x) = \frac{-1 \pm \sqrt{4x-3}}{2}$$

Taking + sign as S contains natural numbers only.

$$\text{so } f^{-1}(x) = \frac{-1 + \sqrt{4x-3}}{2}$$ **[2]**

21. Find the equation of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact. **[4]**

Sol. Given equation of curve $y = \sqrt{3x-2}$

Now differentiate it w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{2\sqrt{3x-2}}(3.1-0)$$

$$\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$$

Now slope of tangent to the curve $y=\sqrt{3x-2}$ at point (x_1, y_1) is

$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)}=\frac{3}{2\sqrt{3x_1-2}}$$ [½]

Given that tangent at (x_1, y_1) is parallel to tangent $4x - 2y + 5 = 0$ (slope = 2)

So slope $\frac{3}{2\sqrt{3x_1-2}}=2$ [½]

$\Rightarrow\quad 3=4\sqrt{3x_1-2}$

Now square both side

$\Rightarrow\quad 9 = 16(3x_1 - 2)$

$\Rightarrow\quad \frac{9}{16}+2=3x_1$

$\Rightarrow\quad 3x_1=\frac{41}{16}$

$x_1=\frac{41}{48}$

$\because$ Point (x_1, y_1) lies on the curve $y=\sqrt{3x-2}$

so $y_1=\sqrt{3\times\frac{41}{48}-2}=\sqrt{\frac{41}{16}-2}=\frac{3}{4}$ [½]

So equation of tangent at point $\left(\frac{41}{48},\frac{3}{4}\right)$ on the curve $y=\sqrt{3x-2}$ is which has slope = 2

Equation of tangent $\left(y-\frac{3}{4}\right)=2\left(x-\frac{41}{48}\right)$

$\Rightarrow\quad y-\frac{3}{4}=2x-\frac{41}{24}$

$\Rightarrow\quad 2x-y=\frac{41}{24}-\frac{3}{4}$

$\Rightarrow\quad 2x-y=\frac{23}{24}$

Equation of tangent is

$\Rightarrow\quad \boxed{48x-24y=23}$ [1]

Now equation of normal at point of contact to the curve, point of contact = $\left(\frac{41}{48},\frac{3}{4}\right)$ and slope of tangent at point of contact = 2

Let slope of normal = m

We know (m)(2) = – 1 (slope of normal. slope of tangent = – 1)

$\Rightarrow\quad \boxed{m=\frac{-1}{2}}$ [½]

Now equation of normal at point $\left(\frac{41}{48}, \frac{3}{4}\right)$

$$\Rightarrow \quad \left(y-\frac{3}{4}\right)=\frac{-1}{2}\left(x-\frac{41}{48}\right)$$

$$\Rightarrow \quad 2y-\frac{3}{2}=-x+\frac{41}{48}$$

$$\Rightarrow \quad x+2y=\frac{41}{48}+\frac{3}{2}$$

$$\Rightarrow \quad x+2y=\frac{113}{48}$$

Equation of normal is $\boxed{48x+96y=113}$ **[1]**

22. Solve the differential equation : x dy – y dx = $\sqrt{x^2+y^2}dx$, gven that y = 0 when x = 1. **[4]**

OR

Solve the differential equation : $(1+x^2)\frac{dy}{dx}+2xy-4x^2=0$, subject to the initial condition y(0) = 0.

Sol. Given differential equation is

$$x\,dy-y\,dx=\sqrt{x^2+y^2}\,dx$$

$$\frac{dy}{dx}=\frac{\sqrt{x^2+y^2}+y}{x},\ x\neq 0$$

Putting y = vx and $\frac{dy}{dx}=v+x\frac{dv}{dx}$, we get **[½]**

$$v+x\frac{dv}{dx}=\frac{\sqrt{x^2+v^2x^2}+vx}{x}$$

$$\Rightarrow \quad v+x\frac{dv}{dx}=\sqrt{1+v^2}+v$$

$$x\frac{dv}{dx}=\sqrt{1+v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$ **[½]**

On integrating both sides, we get

$$\log|v+\sqrt{1+v^2}|=\log|x|+\log c$$

$$\Rightarrow \quad |v+\sqrt{1+v^2}|=|cx|$$ **[1]**

$$\Rightarrow \quad \left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right|=|cx|$$

Given that x = 1, y = 0

| c | = 1

Hence, solution is $y+\sqrt{x^2+y^2}=x^2$. **[2]**

OR

Given $(1+x^2)\frac{dy}{dx}+2xy=4x^2$

$\Rightarrow \quad \frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where [½]

$P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$ [1]

I.F. $=e^{\int P dx}=e^{\int \frac{2x}{1+x^2}dx}=e^{\ln(1+x^2)}=1+x^2$ [½]

Solution is $y(1+x^2)=\int \frac{4x^2}{1+x^2}(1+x^2)dx+c$

$\Rightarrow \quad y(1+x^2)=\frac{4x^3}{3}+c$ [½]

Given that at x = 0, y = 0;

$\Rightarrow \quad 0 = 0 + c$

i.e. c = 0 [½]

$\Rightarrow \quad y=\frac{4x^3}{3(1+x^2)}$, is the required solution [1]

23. Find the value of λ, so that the lines $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. [4]

Sol. The equation of first line is

$L_1:\ \frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$

$\therefore \quad L_1=\frac{x-1}{-3}=\frac{y-2}{\frac{\lambda}{7}}=\frac{z-3}{2}(=k_1 \text{ say})$

$\therefore$ Direction ratio's of $L_1=\left\langle -3,\ \frac{\lambda}{7},\ 2\right\rangle$

and equation of second line is

$L_2:\ \frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$

$\therefore \quad L_2:\ \frac{x-1}{-\frac{3\lambda}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}(=k_2 \text{ say})$

$\therefore$ Direction ratio's of $L_2=\left\langle -\frac{3\lambda}{7}, 1, -5\right\rangle$ [1]

$\because$ Lines L_1 and L_2 are perpendicular

$\therefore$ $(-3)\left(-\frac{3\lambda}{7}\right)+\frac{\lambda}{7}\cdot 1+2\cdot(-5)=0$

$\frac{10\lambda}{7}=10$

$\lambda = 7$ **[1]**

Lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

intersect if $\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$ (as the lines are non-parallel)

Now, $\begin{vmatrix} 1-1 & 5-2 & 6-3 \\ -3 & 1 & 2 \\ -3 & 1 & -5 \end{vmatrix}$

$=\begin{vmatrix} 0 & 3 & 3 \\ -3 & 1 & 2 \\ -3 & 1 & -5 \end{vmatrix}$

= –63 (which is non-zero)

So lines do not intersect. **[2]**

Section-D

Question numbers 24 to 29 carry 6 marks each.

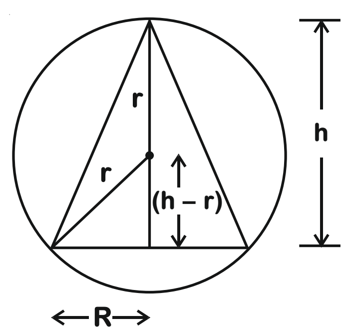
24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find the maximum volume of cone. **[6]**

Sol. Let the altitude of the cone be h.

$\because$ Radius of sphere is r, then the radius of base of the cone,

$R=\sqrt{r^2-(h-r)^2}$

$R=\sqrt{2hr-h^2}$ **[1]**



Volume of the cone,

$V=\frac{\pi}{3}R^2h$

$\Rightarrow$ $V=\frac{\pi}{3}(2hr-h^2)h$

$V=\frac{\pi}{3}[2h^2r-h^3]$ **[1]**

$\because$ Volume is maximum, then $\frac{dV}{dh}=0=\frac{\pi}{3}[4hr-3h^2]$

$\Rightarrow$ $\frac{\pi}{3}[4hr-3h^2]=0$

$\Rightarrow$ (4r – 3h)h = 0

$h = 0$ or $h = \dfrac{4r}{3}$ **[2]**

For maximum volume, $\dfrac{d^2V}{dh^2}$ should be negative.

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}[4r - 6h]$$

$\Rightarrow$ $\left.\dfrac{d^2V}{dh^2}\right|_{\left(h=\frac{4r}{3}\right)} = \dfrac{\pi}{3}\left[4r - 6\left(\dfrac{4r}{3}\right)\right] = -\dfrac{4\pi r}{3}$(negative) **[1]**

So for maximum volume, altitude of the cone should be $\dfrac{4r}{3}$.

Also $\quad V_{max} = \dfrac{\pi}{3}[2h^2r - h^3]$

$$= \frac{\pi}{3}\left[2\left(\frac{16}{9}\right)r^3 - \frac{64}{27}r^3\right]$$

$$= \frac{\pi}{3}r^3\left[\frac{32}{9} - \frac{64}{27}\right]$$

$$= \frac{32}{81}\pi r^3$$ **[1]**

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}. Hence solve the following system of equations:

2x – 3y + 5z = 11, 3x + 2y – 4z = –5, x + y – 2z = –3. **[6]**

OR

Obtain the inverse of the following matrix using elementary operations:

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol. $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$\Rightarrow$ |A| = 2(–4 + 4) + 3(–6 + 4) + 5(3 – 2) = –1 **[1]**

$$\text{Adj}(A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$ **[1]**

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ **[1]**

Given system of equations is

2x – 3y + 5z = 11

3x + 2y – 4z = –5

x + y – 2z = –3

$$\Rightarrow \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ [1]

$$\Rightarrow A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ [1]

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

∴ x = 1, y = 2 and z = 3 is the solution the given system of equations. [1]

OR

∵ A = IA

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$ [1]

$R_1 \to -R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 5 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}A$$ [1]

$R_2 \to -R_2 + R_3$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -1 & 1 \\ 3 & 0 & 1 \end{bmatrix}A$$ [1]

$R_1 \to R_1 + R_2$

$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ -5 & 4 & -3 \end{bmatrix}A$$ [1]

$R_3 \to -R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 5 & -4 & 3 \end{bmatrix} A$$ [1]

$R_2 \rightarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$

Here $I = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$

So $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$ [1]

26. A manufacturer has three machine operators A, B and C. The first operator A produces 1% of defective items, whereas the other two operators B and C produces 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job 30% of the time and C on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by A? [6]

Sol. Let E_1, E_2, E_3 & K be the following events

E_1 = operator A is chosen

E_2 = operator B is chosen

E_3 = operator C is chosen

K = Defective item is chosen [1]

$P(E_1) = \frac{50}{100}$, $P(E_2) = \frac{30}{100}$, $P(E_3) = \frac{20}{100}$

Now $P\left(\frac{K}{E_1}\right) = \frac{1}{100}$, $P\left(\frac{K}{E_2}\right) = \frac{5}{100}$, $P\left(\frac{K}{E_3}\right) = \frac{7}{100}$ [2]

We have to find $P\left(\frac{E_1}{K}\right)$

By Baye's theorem

$$P\left(\frac{E_1}{K}\right) = \frac{P(E_1)\cdot P\left(\frac{K}{E_1}\right)}{P(E_1)\cdot P\left(\frac{K}{E_1}\right) + P(E_2)P\left(\frac{K}{E_2}\right) + P(E_3)\cdot P\left(\frac{K}{E_3}\right)}$$ [2]

$$= \frac{\frac{50}{100} \times \frac{1}{100}}{\frac{50}{100} \times \frac{1}{100} + \frac{30}{100} \times \frac{5}{100} + \frac{20}{100} \times \frac{7}{100}}$$

$$= \frac{50}{50 + 150 + 140}$$

$$= \frac{50}{340} = \frac{5}{34}$$

$P\left(\frac{E_1}{K}\right) = \frac{5}{34}$ [1]

27. Find the vector and Cartesian equations of the plane passing through the points (2, 2, – 1), (3, 4, 2) and (7, 0, 6). Also find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above. [6]

OR

Find the vector equation of the plane that contains the lines $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and the point (– 1, 3, – 4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane thus obtained.

Sol. Let A, B, C be the points with position vectors $2\hat{i} + 2\hat{j} - \hat{k}$, $3\hat{i} + 4\hat{j} + 2\hat{k}$ and $7\hat{i} + 6\hat{k}$ respectively. The required plane passes through the point A (2, 2, – 1) and is normal to vector $\vec{n}$ given by $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

Clearly, $\overrightarrow{AB} = (3\hat{i} + 4\hat{j} + 2\hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k})$

$= \hat{i} + 2\hat{j} + 3\hat{k}$

And $\overrightarrow{AC} = (7\hat{i} + 6\hat{k}) - (2\hat{i} + 2\hat{j} - \hat{k})$

$= 5\hat{i} - 2\hat{j} + 7\hat{k}$ [1]

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}$$

$= \hat{i}(14 + 6) + \hat{j}(15 - 7) + \hat{k}(-2 - 10)$

$= 20\hat{i} + 8\hat{j} - 12\hat{k}$ [1]

Required plane is $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\vec{r} \cdot (20\hat{i} + 8\hat{j} - 12\hat{k}) = (2\hat{i} + 2\hat{j} - \hat{k}) \cdot (20\hat{i} + 8\hat{j} - 12k)$

$\Rightarrow \quad \vec{r} \cdot (20\hat{i} + 8\hat{j} - 12\hat{k}) = 40 + 16 + 12$

$\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$ [1]

Cartesian equation of the plane is

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

$\Rightarrow \quad 5x + 2y - 3z = 17$ [1]

Let equation of plane passing (4, 3, 1) and parallel to plane 5x + 2y – 3z = 17 is

5x + 2y – 3z = d [1]

5(4) + 2(3) – 3(1) = d

i.e. d = 23

$\Rightarrow \quad 5x + 2y - 3z = 23$

Vector equation is $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 23$ [1]

OR

The required plane passes through the point

A (–1, 3, –4) and contains the line $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z}{-1}$ which passes through the point B (1, 1, 0) and is parallel to the vector $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$.

Thus required plane passes through two points A(–1, 3, –4) and B(1, 1, 0) and is parallel to vector $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$.

Let $\vec{n}$ be the normal vector to the required plane

$\therefore \quad \vec{n} = \overrightarrow{AB} \times \vec{b}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 4 \\ 1 & 2 & -1 \end{vmatrix}$$ [1]

$$= \hat{i}(2-8) + \hat{j}(4+2) + \hat{k}(4+2)$$

$$= -6\hat{i} + 6\hat{j} + 6\hat{k}$$

Required plane passes through $\vec{\alpha} = -\hat{i} + 3\hat{j} - 4\hat{k}$ and is perpendicular to $\vec{x} = -6\hat{i} + 6\hat{j} + 6\hat{k}$.

So its vector equation is $(\vec{r} - \vec{\alpha}) \cdot \vec{x} = 0$

$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{\alpha} \cdot \vec{x}$ [1]

$$\vec{r} \cdot (-6\hat{i} + 6\hat{j} + 6\hat{k}) = (-\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (-6\hat{i} + 6\hat{j} + 6\hat{k})$$

$$\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = (-\hat{i} + 3\hat{j} - 4\hat{k}) \cdot (-\hat{i} + \hat{j} + \hat{k})$$ [2]

$$\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0$$

Length of perpendicular drawn from the point (2, 1, 4) to the above plane is given by

$$d = \left| \frac{(2\hat{i} + \hat{j} + 4\hat{k}) \cdot (-\hat{i} + \hat{j} + \hat{k})}{\sqrt{(-1)^2 + 1^2 + 1^2}} \right|$$

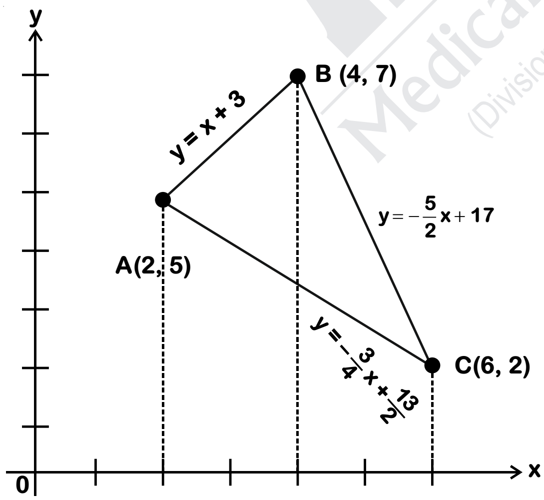
$$= \left| \frac{-2+1+4}{\sqrt{3}} \right| = \sqrt{3} \text{ units}$$ [2]

28. Using integration, find the area of triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2). [6]

OR

Find the area of the region lying about x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

Sol. [1]



Equation of the line AB is $y - 5 = \frac{7-5}{4-2}(x-2)$

$\Rightarrow \quad 2y - 10 = 2x - 4 \quad \Rightarrow \quad y = x + 3$

Equation of the line BC is $y - 7 = \frac{2-7}{6-4}(x-4)$

$\Rightarrow \quad 2y - 14 = -5x + 20 \quad \Rightarrow \quad y = \frac{-5}{2}x + 17$

Equation of the line CA is $y - 5 = \frac{2-5}{6-2}(x-2)$

$\Rightarrow \quad 4y - 20 = -3x + 6 \quad \Rightarrow \quad y = -\frac{3}{4}x + \frac{13}{2}$ [1]

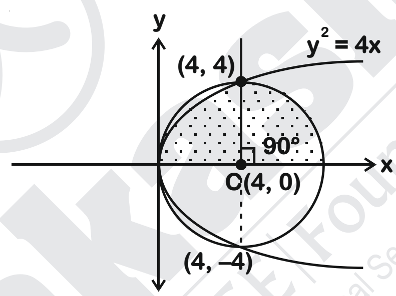
$\Rightarrow$ Area of triangle ABC = $\int_2^4 (x+3)\,dx + \int_4^6 \left(-\frac{5}{2}x + 17\right)dx - \int_2^6 \left(-\frac{3}{4}x + \frac{13}{2}\right)dx$ [1]

$$= \left[\frac{x^2}{2} + 3x\right]_2^4 + \left[\frac{-5x^2}{4} + 17x\right]_4^6 + \left[\frac{3x^2}{8} - \frac{13}{2}x\right]_2^6$$ [1]

$$= (8 + 12 - 2 - 6) + (-45 + 102 + 20 - 68) + \left(\frac{27}{2} - 39 - \frac{3}{2} + 13\right)$$

= 12 + 9 – 14 = 7 square units [2]

OR



[1]

$x^2 + y^2 - 8x = 0$, centre (4, 0), r = 4 ...(1)

$y^2 = 4x$ ...(2)

For intersection of (1) and (2)

$x^2 + 4x - 8x = 0 \quad \Rightarrow \quad x = 0, 4$

Intersection points (0, 0) (4, 4) (4, –4) [1]

$$A = \int_0^4 2\sqrt{x}\,dx + \int_4^8 \sqrt{8x - x^2}\,dx$$ [1]

$$= \left[2 \times \frac{2}{3}x^{3/2}\right]_0^4 + \frac{1}{2}\left[(x-4)\sqrt{8x - x^2} + 16\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$$ [1]

$$= \frac{4}{3}(8 - 0) + \frac{1}{2}\left[\frac{16\pi}{2}\right]$$

$$= \left(\frac{32}{3} + 4\pi\right) \text{square units}$$ [2]

29. A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making of one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹ 15 and on an item of model B is ₹ 10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit. [6]

Sol. Let x items of model A and y items of model B are made per day.

Time constraint for skilled men :

$2x + y \leq 5 \times 8$

$\Rightarrow \quad 2x + y \leq 40$ ...(i) [1]

Time constraint for semi skilled men:

$2x + 3y \leq 10 \times 8$

$\Rightarrow \quad 2x + 3y \leq 80$ ...(ii) [1]

Non-negative constraints :

$x \geq 0$ ...(iii)

$y \geq 0$ ...(iv)

Objective function (Profit)

$z = 15x + 10y$

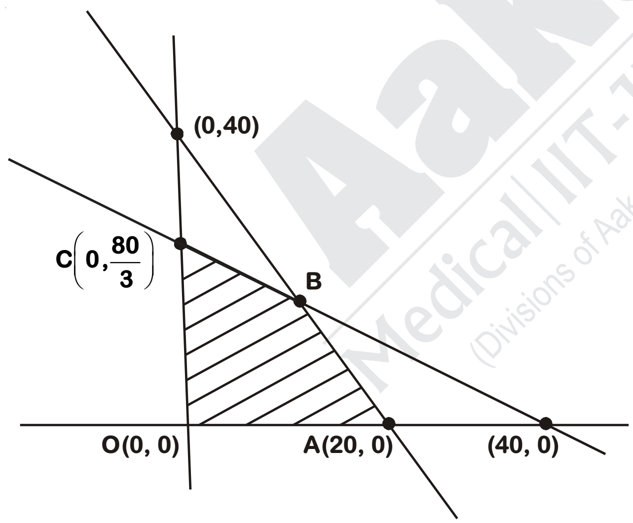
Solving the problem graphically

$2x + y = 40$

x	0	20
y	40	0

$2x + 3y = 80$

x	0	40
y	$\frac{80}{3}$	0



[3]

OABC is the feasible region

Corner Point	z = 15x + 10y
O(0, 0)	0
A(20, 0)	300
B(10, 20)	350 → Maximum
C$\left(0, \frac{80}{3}\right)$	$\frac{800}{3}$

The maximum profit is ₹ 350

For the maximum profit, 10 items of model A and 20 items of model B should be made. [1]

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