33

INTRODUCTION TO THREE DIMENSIONAL GEOMETRY

You have read in your earlier lessons that given a point in a plane, it is possible to find two numbers, called its co-ordinates in the plane. Conversely, given any ordered pair (x, y) there corresponds a point in the plane whose co-ordinates are (x, y).

Let a rubber ball be dropped vertically in a room The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor.

If the height of the ball above the floor is 2.5cm and the position of the point where it strikes the ground is given by (5, 4), one way of describing the position of ball in space is with the help of these three numbers (5, 4, 2.5).

Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers. In this lesson, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

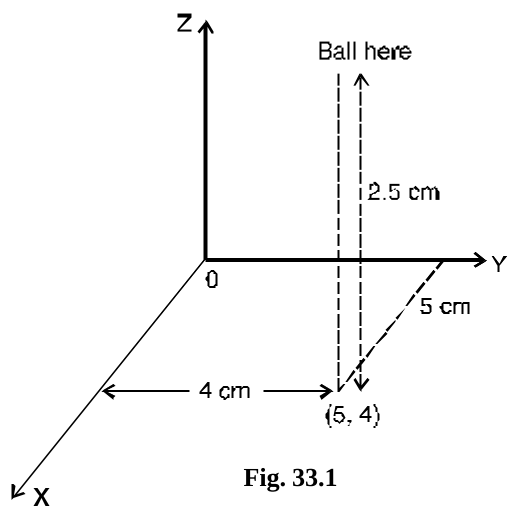


Fig. 33.1

OBJECTIVES

After studying this lesson, you will be able to :

- associate a point, in three dimensional space with given triplet and vice versa;
- find the distance between two points in space;

- find the coordinates of a point which divides the line segment joining twogiven points in a given ratio internally and externally;
- define the direction cosines/ratios of a given line in space;
- find the direction cosines of a line in space;
- find the projection of a line segment on another line; and
- find the condition of prependicularity and parallelism of two lines in space.

EXPECTED BACKGROUND KNOWLEDGE

- Two dimensional co-ordinate geometry
- Fundamentals of Algebra, Geometry, Trigonometry and vector algebra.

33.1 COORDINATE SYSTEM AND COORDINATES OF A POINT IN SPACE

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the co-ordinates of the point change. Thus, the origin can be taken arbitarily at any point of the room.

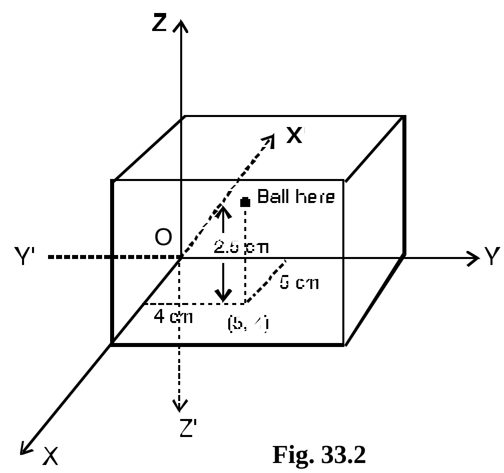


Fig. 33.2

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines X'OX, Y'OY and Z'OZ through O. The point O is called the origin of the co-ordinate system and the lines X'OX, Y'OY and Z'OZ are called the X-axis, the Y-axis and the Z-axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 33.2. The plane determined by the X-axis and the Y-axis is called XY-plane (XOY plane) and similarly, YZ-plane (YOZ-plane) and ZX-plane (ZOX-plane) can be determined. These three planes are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

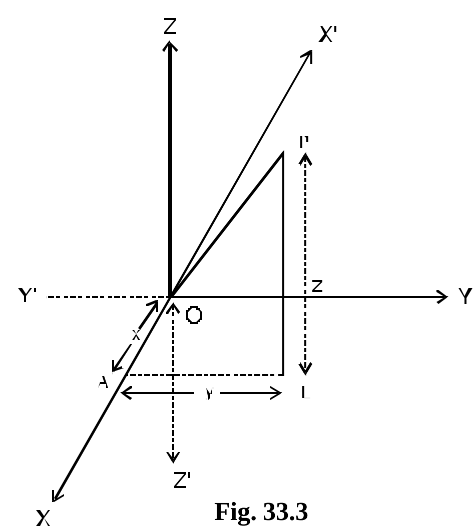


Fig. 33.3

Let P be any point is space. Through P draw perpendicular PL on XY-plane meeting this plane at L. Through L draw a line LA parallel to OY cutting OX in A. If we write OZ = x, AL = y and LP = z, then (x, y, z) are the co-ordinates of the point P.

Again, if we complete a reactangular parallelopiped through P with its three edges OA, OB and OC meeting each other at O and OP as its main diagonal then the lengths (OA, OB, OC) i.e., (x, y, z) are called the co-ordinates of the point P.

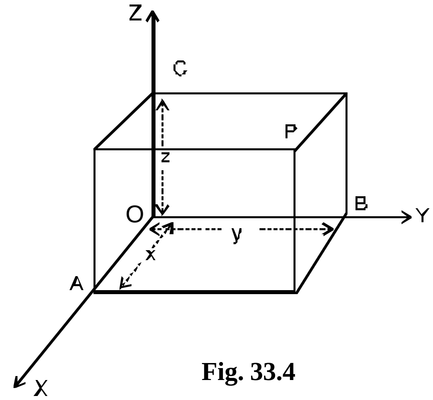


Fig. 33.4

Note : You may note that in Fig. 33.4

(i) The x co-ordinate of P = OA = the length of perpendicular from P on the YZ-plane.

(ii) The y co-ordinate of P = OB = the length of perpendicular from P on the ZX-plane.

(iii) The x co-ordinate of P = OC = the length of perpendicular from P on the XY-plane.

Thus, the co-ordinates x, y, and z of any point are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

Thus, given a point P in space, to it corresponds a triplet (x, y, z) called the co-ordinates of the point in space. Conversely, given any triplet (x, y, z), there corresponds a point P in space whose co-ordinates are (x, y, z).

Remarks

1. Just as in plane co-ordinate geometry, the co-ordinate axes divide the plane into four quadrants, in three dimentional geometry, the space is divided into eight octants by the co-ordinate planes, namely OXYZ, OX'YZ, OXY'Z, OXYZ', OXY'Z', OX'YZ', OX'Y'Z and OX'Y'Z'.

2. If P be any point in the first octant, there is a point in each of the other octants whose absolute distances from the co-ordinate planes are equal to those of P. If P be (a, b, c), the other points are (–a, b, c), (a, –b, c), (a, b, –c), (a, –b, –c), (–a, b, –c), (–a, –b, c) and (–a, –b, –c) respectively in order in the octants referred in (i).

3. The co-ordinates of point in XY-plane, YZ-plane and ZX-plane are of the form (a, b, 0), (0, b, c) and (a, 0, c) respectively.

4. The co-ordinates of points on X-axis, Y-axis and Z-axis are of the form (a, 0, 0), (0, b, 0) and (0, 0, c) respectively.

5. You may see that (x, y, z) corresponds to the position vector of the point P with reference to the origin O as the vector $\overrightarrow{OP}$.

Example 33.1 Name the octant wherein the given points lies :

(a) (2, 6, 8) (b) (–1, 2, 3) (c) (–2, –5, 1)

(d) (–3, 1, –2) (e) (–6, –1, –2)

Solution :

(a) Since all the co-ordinates are positive, $\therefore$ (2, 6, 8) lies in the octant OXYZ.

(b) Since x is negative and y and z are positive, $\therefore$ (−1, 2, 3) lies in the octant OX'YZ.

(c) Since x and y both are negative and z is positive $\therefore$ (−2, −5, 1) lies in the octant OX'Y'Z.

(d) (−3, 1, −2) lies in octant OX'YZ'.

(e) Since x, y and z are all negative $\therefore$ (−6, −1, −2) lies in the octant OX'Y'Z'.

CHECK YOUR PROGRESS 33.1

1. Name the octant wherein the given points lies :

(a) (−4, 2, 5) (b) (4, 3, −6) (c) (−2, 1, −3)

(d) (1, −1, 1) (f) (8, 9, −10)

33.2 DISTANCE BETWEEN TWO POINTS

Suppose there is an electric plug on a wall of a room and an electric iron placed on the top of a table. What is the shortest length of the wire needed to connect the electric iron to the electric plug ? This is an example necessitating the finding of the distance between two points in space.

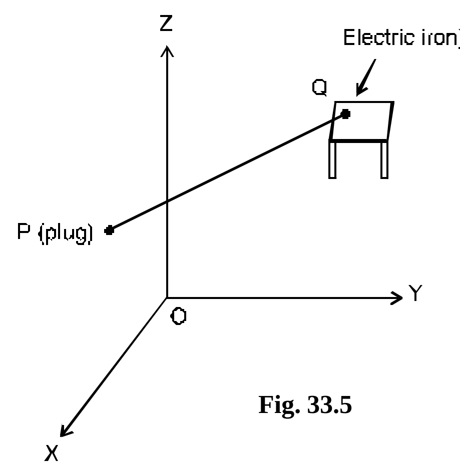


Fig. 33.5

Let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. With PQ as diagonal, complete the parallopiped PMSNRLKQ.

PK is perpendicular to the line KQ.

$\therefore$ From the right-angled triangle PKQ, right angled at K,

We have $PQ^2 = PK^2 + KQ^2$

Again from the right angled triangle PKL right angled at L,

$$PK^2 = KL^2 + PL^2 = MP^2 + PL^2 \quad (\because KL = MP)$$

$$\therefore \quad PQ^2 = MP^2 + PL^2 + KQ^2 \quad ...(i)$$

The edges MP, PL and KQ are parallel to the co-ordinate axes.

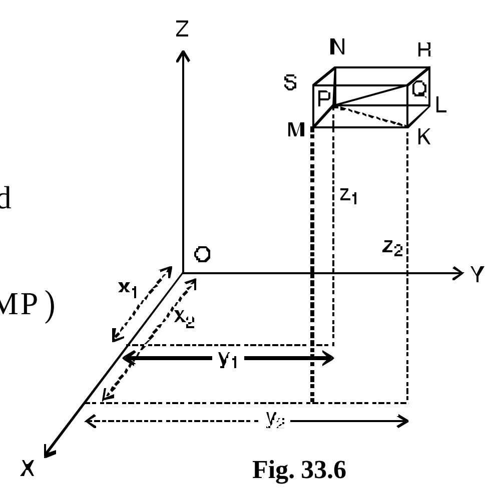


Fig. 33.6

Now, the distance of the point P from the plane $YOZ = x_1$ and the distance of Q and M from YOZ plane $= x_2$

$\therefore$ $\quad MP = |x_2 - x_1|$

Similarly, $\quad PL = |y_2 - y_1|$ and $KQ = |z_2 - z_1|$

$\therefore$ $\quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$ ....[From (i)]

or $\quad |\mathbf{PQ}| = \sqrt{(\mathbf{x_2} - \mathbf{x_1})^2 + (\mathbf{y_2} - \mathbf{y_1})^2 + (\mathbf{z_2} - \mathbf{z_1})^2}$

Corollary : Distance of a Point from the Origin

If the point $Q(x_2, y_2, z_2)$ coincides with the origin (0, 0, 0), then $x_2 = y_2 = z_2 = 0$

$\therefore$ The distance of P from the origin is

$$|OP| = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2 + (z_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2 + z_1^2}$$

In general, the distance of a point P(x, y, z) from origin O is given by

$$|OP| = \sqrt{x^2 + y^2 + z^2}$$

Example 33.2 Find the distance between the points $(2, 5, -4)$ and $(8, 2, -6)$.

Solution : Let $P\ (2, 5, -4)$ and $Q\ (8, 2, -6)$ be the two given points.

$$\therefore \quad |PQ| = \sqrt{(8-2)^2 + (2-5)^2 + (-6+4)^2}$$

$$= \sqrt{36 + 9 + 4}$$

$$= \sqrt{49}$$

$$= 7$$

$\therefore$ The distance between the given points is 7 units.

Example 33.3 Prove that the points $(-2, 4, -3)$, $(4, -3, -2)$ and $(-3, -2, 4)$ are the vertices of an equilateral triangle.

Solution : Let $A(-2, 4, -3)$, $B(4, -3, -2)$ and $C(-3, -2, 4)$ be the three given points.

Now $\quad |AB| = \sqrt{(4+2)^2 + (-3-4)^2 + (-2+3)^2}$

$$= \sqrt{36 + 49 + 1} = \sqrt{86}$$

$$|BC| = \sqrt{(-3-4)^2 + (-2+3)^2 + (4+2)^2} = \sqrt{86}$$

$$|CA| = \sqrt{(-2+3)^2 + (4+2)^2 + (-3-4)^2} = \sqrt{86}$$

Since $|AB| = |BC| = |CA|$, $\Delta\ ABC$ is an equilateral triangle.

Example 33.4 Verify whether the following points form a triangle or not :

(a) $A(-1, 2, 3)$ $B(1, 4, 5)$ and $C(5, 4, 0)$

(b) $(2, -3, 3)$, $(1, 2, 4)$ and $(3, -8, 2)$

Solution :

(a) $|AB| = \sqrt{(1+1)^2 + (4-2)^2 + (5-3)^2}$

$= \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$ = 3.464 (approx.)

$|BC| = \sqrt{(5-1)^2 + (4-4)^2 + (0-5)^2}$

$= \sqrt{16 + 0 + 25} = \sqrt{41}$ = 6.4 (approx.)

and $|AC| = \sqrt{(5+1)^2 + (4-2)^2 + (0-3)^2}$

$= \sqrt{36 + 4 + 9} = 7$

$\therefore$ $|AB| + |BC| = 3.464 + 6.4 = 9.864 > |AC|$, $|AB| + |AC| > |BC|$

and $|BC| + |AC| > |AB|$.

Since sum of any two sides is greater than the third side, therefore the above points form a triangle.

(b) Let the points $(2, -3, 3), (1, 2, 4)$ and $(3, -8, 2)$ be denoted by P, Q and R respectively,

then $|PQ| = \sqrt{(1-2)^2 + (2+3)^2 + (4-3)^2}$

$= \sqrt{1 + 25 + 1} = 3\sqrt{3}$

$|QR| = \sqrt{(3-1)^2 + (-8-2)^2 + (2-4)^2}$

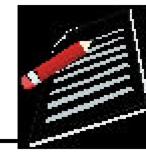
$= \sqrt{4 + 100 + 4} = 6\sqrt{3}$

$|PR| = \sqrt{(3-2)^2 + (-8+3)^2 + (2-3)^2}$

$= \sqrt{1 + 25 + 1}$

$= 3\sqrt{3}$

In this case $|PQ| + |PR| = 3\sqrt{3} + 3\sqrt{3} = 6\sqrt{3} = |QR|$. Hence the given points do not form a triangle. In fact the points lie on a line.



Example 33.5 Show that the points $A(1,2,-2), B(2,3,-4)$ and $C(3,4,-3)$ form a right angled triangle.

Solution : $AB^2 = (2-1)^2 + (3-2)^2 + (-4+2)^2 = 1+1+4 = 6$

$BC^2 = (3-2)^2 + (4-3)^2 + (-3+4)^2 = 1+1+1 = 3$

and $AC^2 = (3-1)^2 + (4-2)^2 + (-3+2)^2 = 4+4+1 = 9$

We observe that $AB^2 + BC^2 = 6+3 = 9 = AC^2$

$\therefore$ ΔABC is a right angled triangle.

Hence the given points form a right angled triangle.

Example 33.6 Prove that the points $A(0, 4, 1)$, $B(2, 3, -1)$, $C(4, 5, 0)$ and $D(2, 6, 2)$ are vertices of a square.

Solution : Here,

$$AB = \sqrt{(2-0)^2 + (3-4)^2 + (-1-1)^2}$$
$$= \sqrt{4+1+4} = 3 \text{ units}$$

$$BC = \sqrt{(4-2)^2 + (5-3)^2 + (0+1)^2}$$
$$= \sqrt{4+4+1} = 3 \text{ units}$$

$$CD = \sqrt{(2-4)^2 + (6-5)^2 + (2-0)^2}$$
$$= \sqrt{4+1+4} = 3 \text{ units}$$

and

$$DA = \sqrt{(0-2)^2 + (4-6)^2 + (1-2)^2}$$
$$= \sqrt{4+4+1} = 3 \text{ units}$$

$\therefore$ $AB = BC = CD = DA$

Now

$$AC^2 = \sqrt{(4-0)^2 + (5-4)^2 + (0-1)^2}$$
$$= 16+1+1 = 18$$

$\therefore$ $AB^2 + BC^2 = 3^2 + 3^2 = 18 = AC^2$

$\therefore$ $\angle B = 90^\circ$

$\therefore$ In quadrilateral A B C D, $AB = BC = CD = DA$ and $\angle B = 90^\circ$

$\therefore$ A B C D is a square.

CHECK YOUR PROGRESS 33.2

1. Find the distance between the following points :

   (a) $(4, 3, -6)$ and $(-2, 1, -3)$ (b) $(-3, 1, -2)$ and $(-3, -1, 2)$

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Notes

(c) $(0, 0, 0)$ and $(-1, 1, 1)$

2. Show that if the distance between the points $(5, -1, 7)$ and $(a, 5, 1)$ is 9 units, "a" must be either 2 or 8.

3. Show that the triangle formed by the points (a, b, c), (b, c, a) and (c, a, b) is equilateral.

4. Show that the the points $(-1, 0, -4)$, $(0, 1, -6)$ and $(1, 2, -5)$ form a right angled tringle.

5. Show that the points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of an isosceles right-angled triangle.

6. Show that the points $(3, -1, 2)$, $(5, -2, -3)$, $(-2, 4, 1)$ and $(-4, 5, 6)$ form a parallelogram.

7. Show that the points $(2, 2, 2)$, $(-4, 8, 2)$, $(-2, 10, 10)$ and $(4, 4, 10)$ form a square.

8. Show that in each of the following cases the three points are collinear :

(a) $(-3, 2, 4)$, $(-1, 5, 9)$ and $(1, 8, 14)$

(b) $(5, 4, 2)$, $(6, 2, -1)$ and $(8, -2, -7)$

(c) $(2, 5, -4)$, $(1, 4, -3)$ and $(4, 7, -6)$

33.3 COORDINATES OF A POINT OF DIVISION OF A LINE SEGMENT

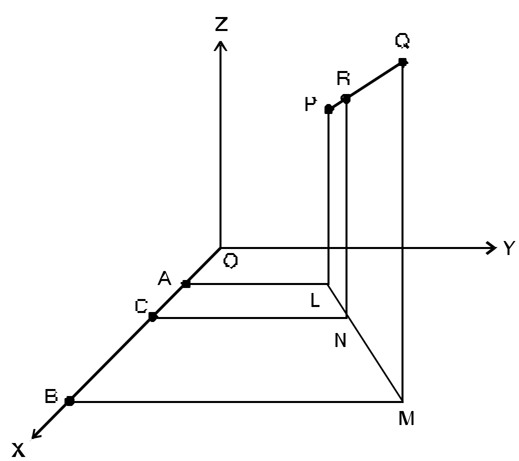

Fig. 33.7

Let the point R (x, y, z) divide PQ in the ratio $l : m$ internally.
Let the co-ordinates of P be (x_1, y_1, z_1) and the co-ordinates of Q be (x_2, y_2, z_2). From points P, R and Q, draw PL, RN and QM perpendiculars to the XY-plane.

Draw LA, NC and MB perpendiculars to OX.

Now $AC = OC - OA = x - x_1$ and $BC = OB - OC = x_2 - x$

Also we have, $\dfrac{AC}{BC} = \dfrac{LN}{NM} = \dfrac{PR}{RQ} = \dfrac{l}{m}$

$$\therefore \qquad \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

$$\text{or} \qquad mx - mx_1 = lx_2 - lx$$

$$\text{or} \qquad (l + m)\,x = lx_2 + mx_1$$

$$\text{or} \qquad x = \frac{lx_2 - mx_1}{l + m}$$

Similarly, if we draw perpendiculars to OY and OZ respectively,

we get $y = \dfrac{ly_2 + my_1}{l + m}$ and $z = \dfrac{lz_2 + mz_1}{l + m}$

$$\therefore \qquad \text{R is the point } \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}, \frac{lz_2 + mz_1}{l + m} \right)$$

If $\lambda = \dfrac{l}{m}$, then the co-ordinates of the point R which divides PQ in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1} \right), \lambda + 1 \neq 0$$

It is clear that to every value of λ, there corresponds a point of the line PQ and to every point R on the line PQ, there corresponds some value of λ. If λ is postive, R lies on the line segment PQ and if λ is negative, R does not lie on line segment PQ.

In the second case you may say the R divides the line segment PQ externally in the ratio $-\lambda : 1$.

Corollary 1 : The co-ordinates of the point dividing PQ externally in the ratio $l : m$ are

$$\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}, \frac{lz_2 - mz_1}{l - m} \right)$$

Corollary 2 : The co-ordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Example 33.7 Find the co-ordinates of the point which divides the line segment joining the points $(2, -4, 3)$ and $(-4, 5, -6)$ in the ratio 2 : 1 internally.

Solution : Let $A(2, -4, 3)$, $B(-4, 5, -6)$ be the two points.

Let P (x, y, z) divides AB in the ratio 2 : 1.

$$\therefore \qquad x = \frac{2(-4)+1.2}{2+1} = -2, \quad y = \frac{2.5+1(-4)}{2+1} = 2$$

$$\text{and} \qquad z = \frac{2(-6)+1.3}{2+1} = -3$$

Thus, the co-ordinates of P are $(-2, 2, -3)$

Example 33.8 Find the point which divides the line segment joining the points $(-1, -3, 2)$ and $(1, -1, 2)$ externally in the ratio 2 : 3.

Solution : Let the points $(-1, -3, 2)$ and $(1, -1, 2)$ be denoted by P and Q respectively.

Let R (x, y, z) divide PQ externally in the ratio 2 : 3. Then

$$x = \frac{2(1)-3(-1)}{2-3} = -5, \qquad y = \frac{2(-1)-3(-3)}{2-3} = -7$$

$$\text{and} \qquad z = \frac{2(2)-3(2)}{2-3} = 2$$

Thus, the co-ordinates of R are $(-5, -7, 2)$.

Example 33.9 Find the ratio in which the line segment joining the points $(2-3, 5)$ and $(7, 1, 3)$ is divided by the XY-plane.

Solution : Let the required ratio in which the line segment is divided be $l : m$.

The co-ordinates of the point are $\left(\frac{7l+2m}{l+m}, \frac{l-3m}{l+m}, \frac{3l+5m}{l+m}\right)$

Since the point lies in the XY-plane, its z-coordinate is zero.

i.e., $$\frac{3l+5m}{l+m} = 0 \quad \text{or} \quad \frac{l}{m} = -\frac{5}{3}$$

Hence the XY-plane divides the join of given points in the ratio 5 : 3 externally.

CHECK YOUR PROGRESS 33.3

1. Find the co-ordinates of the point which divides the line segment joining two points $(2, -5, 3)$ and $(-3, 5, -2)$ internally in the ratio 1 : 4.

2. Find the coordinates of points which divide the join of the points $(2, -3, 1)$ and $(3, 4, -5)$ internally and externally in the ratio 3 : 2.

3. Find the ratio in which the line segment joining the points $(2, 4, 5)$ and $(3, 5, -4)$ is divided by the YZ-plane.

4. Show that the YZ-plane divides the line segment joining the points $(3, 5, -7)$ and $(-2, 1, 8)$ in the ration 3 : 2 at the point $\left(0, \frac{13}{5}, 2\right)$.

5. Show that the ratios in which the co-ordinate planes divide the join of the points $(-2, 4, 7)$ and $(3, -5, 8)$ are 2 : 3, 4 : 5 (internally) and 7 : 8 (externally).

6. Find the co-ordinates of a point R which divides the line segment $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio 2 : 1. Verify that Q is the mid-point of PR.

33.4 ANGLE BETWEEN TWO LINES

You are already famililar with the concept of the angle between two lines in plane geometry. We will extend this idea to the lines in space.

Let there be two lines in space, intersecting or non-intersecting. We consider a point A in space and through it, we draw lines parallel to the given lines in space. The angle between these two lines drawn parallel to the given lines is defined as the angle between the two lines in space.

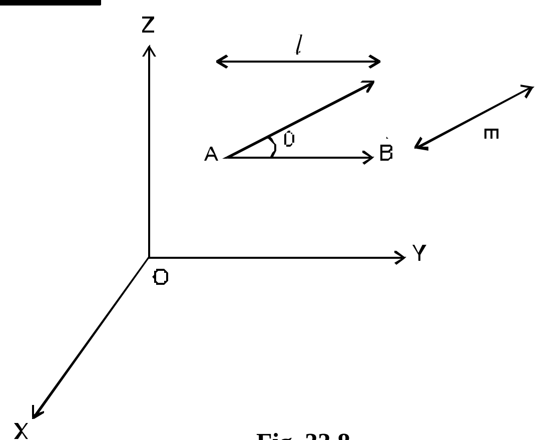

Fig. 33.8

You may see in the adjointing figure, that θ is the angle between the lines l and m.

33.5 DIRECTION COSINES OF A LINE

If α, β and γ are the angles which a line AB makes with the positive directions of X, Y and Z axes respectively, then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the direction cosines of the line AB and are usually denoted by the letters l, m and n respectively. In other words $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$. You may easily see that the direction cosines of the X-axis are 1, 0, 0, because the line coincides with the X axis and is perpendicular to Y and Z axes since $\cos 0° = 1$, $\cos 90° = 0$. Similarly direction cosines of Y and Z axes are 0, 1, 0 and 0, 0, 1 respectively.

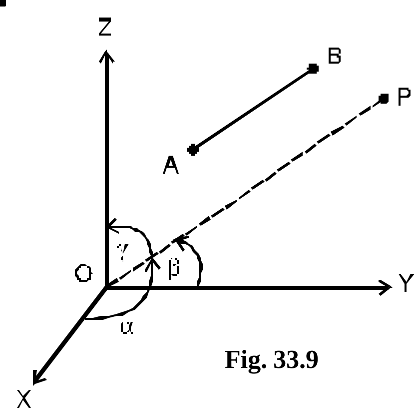

Fig. 33.9

33.5.1 RELATION BETWEEN DRECTION COSINES

Let OP be a line with direction cosines $\cos\alpha, \cos\beta$ and $\cos\gamma$ i.e. l, m and n.

Again since each of $\angle OLP$, $\angle OMP$ and $\angle ONP$ is a right angle.

We have, $\frac{OL}{OP} = \cos\alpha = l$

$$\frac{OM}{OP} = \cos\beta = m \text{ and } \frac{ON}{OP} = \cos\gamma = n$$

i.e. $l.OP = OL$, $m.OP = OM$ and $n.OP = ON$

$$\therefore \quad OP^2\left(l^2 + m^2 + n^2\right)$$

$$= OL^2 + OM^2 + ON^2 = OP^2$$

or $\quad l^2 + m^2 + n^2 = 1$

or $\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

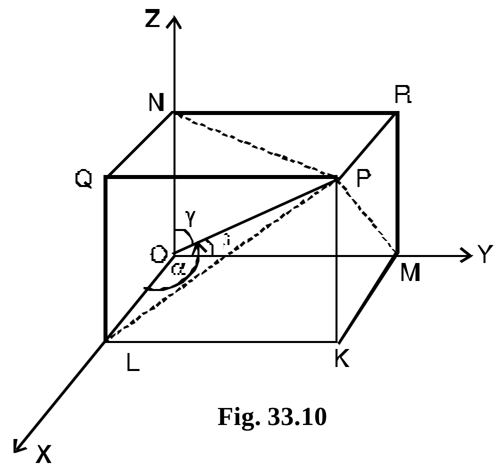

Fig. 33.10

This is the relation between the direction cosines of a line.

Corollary 1 : Any three numbers a, b and c which are proportional to the direction cosines l, m and n respectively of a given line are called the direction ratios or direction numbers of the given line. If a, b and c are direction numbers and l, m and n are direction cosines of a line, then l, m and n are found in terms of a, b and c as follows :

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \quad l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the same sign either positive or negative is to be taken throughout.

33.6 PROJECTION

Suppose you are standing under the shade of a tree. At a time when the sun is vertically above the tree, its shadow falling on the ground is taken as the projection of the tree on the ground at that instant.

This is called projection because the rays falling vertically on the tree create the image of the each point of the tree constituting its shadow (image).

Recall the example of a bouncing ball. When the ball falling freely from a point in space strikes the ground, the point where the ball strikes the ground is the projection of the point in space on the ground.

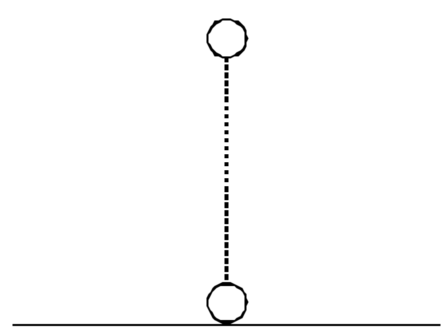

Fig. 33.11

33.6.1 Projection of a Point and of a Line Segment

The projection of a point on a plane can be taken as the foot of the perpendicular drawn from the point to the plane. Similarly, the line segment obtained by joining the feet of the perpendiculars in the plane drawn from the end points of a line segment is called the projection of the line segment on the plane.

We may similarly define the projection of a point and of a line segment on a given line.

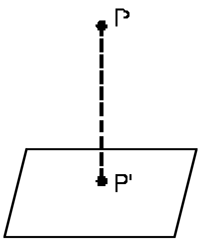

Fig. 33.12

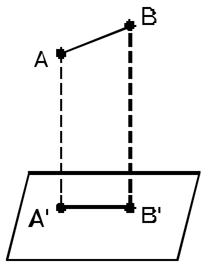

Fig. 33.13

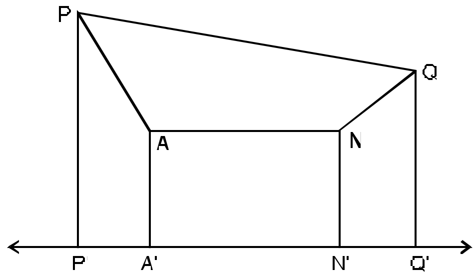

Fig. 33.14

Note : Projection of a line segment PQ on a line is equal to the sum of the projections of the broken line segments i.e., Projections of PQ = Sum of the projections of PA, AN and NQ.

33.6.2 Projection of a Line Segment Joning Two Points on a Line

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. To find the projection of PQ on a line with direction cosines l, m and n, through P and Q draw planes parallel to the co-ordinates planes to form a reactangular paralleloppied whose diagonal is PQ.

Now $\quad PA = x_2 - x_1,\ AN = y_2 - y_1$

and $\quad NQ = z_2 - z_1.$

The lines PA, AN and NQ are parallel to X-axis, Y-axis and Z-axis respectively.

Therefore, their respective projections on the line with direction cosines l, m and n are $(x_2 - x_1)\, l$, $(y_2 - y_1)\, m$ and $(z_2 - z_1)\, n$.

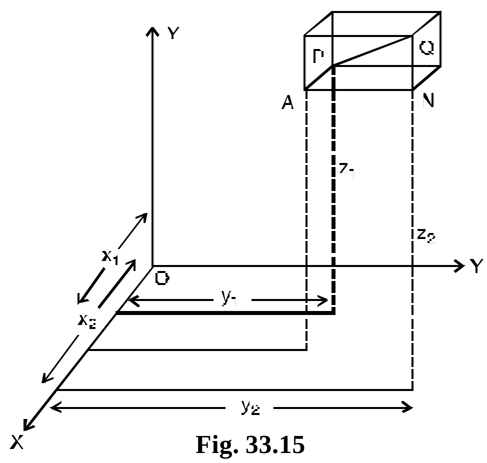

Fig. 33.15

Recall that projection of PQ on any line is equal to the sum of the projections of PA, AN and NQ on the line, therefore the required projection is

$$(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n.$$

33.7 DIRECTION COSINES OF THE LINE JOINING TWO POINTS

Let L and M be the feet of the perpendiculars drawn from $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on the X-axis respectively, so that $OL = x_1$ and $OM = x_2$.

Projection of PQ on X-axis

$$= LM = OM - OL$$

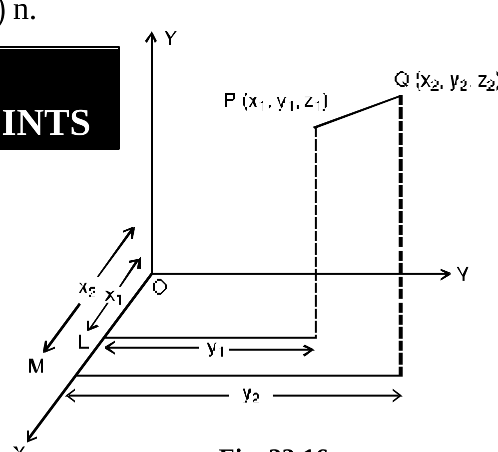

Fig. 33.16

$= x_2 - x_1$

Also, if l, m and n are the direction cosines of PQ, the projection of PQ on X-axis = l. PQ

$\therefore \quad l\ PQ = x_2 - x_1$

Similarly by taking projection of PQ on Y-axis and Z-axis respectively,

we get, $m.PQ = y_2 - y_1$ and $n.PQ = z_2 - z_1$

$$\therefore \quad \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ$$

Thus, the direction cosines of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

Example 33.10 Find the direction cosines of a line that makes equal angles with the axes.

Solution : Here $\alpha = \beta = $. We have, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\therefore \quad 3\cos^2\alpha = 1 \text{ or } \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Hence the required direction cosines are

$$\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$$

same sign (positive or negative) to be taken throughout.

Example 33.11 Verify whether it is possible for a line to make the angles 30°, 45° and 60° with the co-ordinate axes or not ?

Soluton : Let the line make angles α, β and γ with the co-ordinate axes.

$$\alpha = 30°,\ \beta = 45° \text{ and } \gamma = 60°$$

Since the relation between direction cosines is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we have

$\cos^2 30° + \cos^2 45° + \cos^2 60°$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{6}{4} \neq 1$$

In view of the above identity, it is not possible for a line to make the given angles with the co-ordinate axes.

Example 33.12 If 6, 2 and 3 are direction ratios of a line, find its direction cosines.

Solution : Let l, m and n be the direction cosines of the line.

$$\therefore \quad l = \pm\frac{6}{\sqrt{6^2 + 2^2 + 3^2}} = \pm\frac{6}{7},\quad m = \pm\frac{2}{\sqrt{6^2 + 2^2 + 3^2}} = \pm\frac{2}{7}$$

Notes

and $\qquad n = \pm\frac{3}{\sqrt{6^2+2^2+3^2}} = \pm\frac{3}{7}$

Hence the required direction cosines are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ or $-\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}$.

Example 33.13 Find the projections (feet of the perpendiculars) of the point $(2, 1, -3)$ on the (a) Co-ordinate planes (b) Co-ordinate axes.

Solution : (a) The projections of the point on the co-ordinate planes YZ, ZX and XY are $(0, 1, -3)$, $(2, 0, -3)$ and $(2, 1, 0)$ respectively.

(b) The projections on the co-ordinate axes are $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, -3)$ respectively.

Example 33.14 Find the direction cosines of the line-segment joining the points $(2, 5, -4)$ and $(8, 2, -6)$.

Solution : Let l, m and n be the direction cosines of the line joining the two given points $(2, 5, -4)$ and $(8, 2, -6)$.

Then the direction cosines are proportional to $8-2,\ 2-5$ and $-6+4$

i.e., $6, -3, -2$ are direction ratios of the line.

$\therefore$ The required direction cosines of the line are $\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}$ or $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$

Since $\qquad \sqrt{(6)^2+(-3)^2(-2)^2} = 7.$

Example 33.15 Find the projection of the line segment joining the points (3, 3, 5) and (5, 4, 3) on the line joining the points $(2, -1, 4)$ and (0, 1, 5).

Solution : The direction cosines of the line joining the points $(2, -1, 4)$ and (0, 1, 5) are

$$-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$$

because $\qquad \sqrt{(2-0)^2+(-1-1)^2+(4-5)^2} = 3$

Thus, the projection of this line segment on the given line is

$$(5-3)\left(-\frac{2}{3}\right)+(4-3)\left(\frac{2}{3}\right)+(3-5)\left(\frac{1}{3}\right) = \frac{4}{3}$$

Hence the required projection is $\frac{4}{3}$ because the projection is the length of a line segment which is always taken as positive.

OPTIONAL - I
Vectors and three dimensional Geometry

Notes

CHECK YOUR PROGRESS 33.4

1. Find the direction cosines of the line having direction ratios

   (a) 3, −1, 2 (b) 1, 1, 1

2. Find the projections of the point (−3, 5, 6) on the

   (a) Co-ordinate palnes (b) Co-ordinate axes

3. Find the direction cosines of the line segment joining the points

   (i) (5, −3, 8) and (6, −1, 6)

   (ii) (4, 3, −5) and (−2, 1, −8)

4. Find the projection of a line segment joining the points P (4, −2, 5) and Q (2, 1, −3) on the line with direction ratios 6, 2 and 3.

5. Find the projection of a line segment joining the points (2, 1, 3) and (1, 0, -4) on the line joining the points (2, -1, 4) and (0, 1, 5).

33.8 ANGLE BETWEEN TWO LINES WITH GIVEN DIRECTION COSINES

Let OP and OQ be the two lines through the origin O parallel to two lines in space whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Let θ be the angle between OP an OQ and let the co-ordinates of P be (x_1, y_1, z_1).

Draw PL perpendicular to XY-plane and LA perpendicular to X-axis. Then the projection of OP on OQ = Sum of projections of OA, AL and LP on OQ.

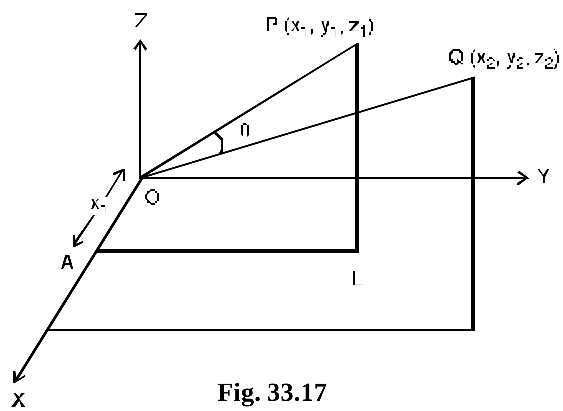

Fig. 33.17

i.e. $\quad OP \cos\theta = x_1 l_2 + y_1 m_2 + z_1 n_2$ ....(i)

But $\quad x_1$ = projection of OP om X-axis = $OP . l_1$

Similarly, $\quad y_1 = OP.m_1$ and $z_1 = OP.n_1$ ....(ii)

Thus, we get, $OP \cos\theta = OP\,(l_1 l_2 + m_1 m_2 + n_1 n_2)$ ....[From (i) and (ii)]

giving $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos$ ....(iii)

Corollary 1 : If the direction ratio of the lines are a_1, b_1, c_1 and a_2, b_2, c_2 then the angle θ between the two lines is given by

$$\cos\theta = \pm\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here positive or negative sign is to be taken depending upon θ being acute or obtuse.

Corollary 2 : If OP an OQ are perpendicular to each other, (i.e., if $\theta = 90^c$ then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos 90$$
$$= 0$$

Corollary 3 : If OP and OQ are parallel, then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Since OP || OQ and O is a common point, OP lies on OQ.

Hence $\sin\theta = 0$

Now $\quad \sin^2\theta = 1 - \cos^2\theta = 1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$

$$= \left(l_1^2 + m_1^2 + n_1^2\right)\left(l_1^2 + m_1^2 + n_1^2\right) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$

$$= (l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2$$

and hence $l_1 m_2 - l_2 m_1 = 0$, $m_1 n_2 - m_2 n_1 = 0$ and $n_1 l_2 - n_2 l_1 = 0$

These gives $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

Remarks

(a) Two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are

(i) perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(ii) parallel if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

(b) The condition of perpendicularty of two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$$\left(\textbf{Hint : } l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}},\ m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} \text{ and } n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}\right)$$

(c) The condition of parallelism of two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Example 33.16 Find the angle between the two lines whose direction ratios are $-1,\ 0, 1$ and $0, -1, 1$.

Solution : Let θ be the angle between the given lines.

$$\therefore \qquad \cos\theta = \pm\frac{(-1)\times 0 + 0\times(-1) + 1\times(1)}{\sqrt{(-1)^2 + 0^2 + 1^2}\,\sqrt{0^2 + (-1)^2 + 1^2}} = \left(\pm\frac{1}{2}\right)$$

$$\therefore \qquad \theta = 60^\circ \text{ or } 120^\circ$$

Example 33.17 Find the acute angle between the lines whose direction ratios are $5,\ -12, 13$ and $-3,\ 4, 5$.

Solution : Let θ be the angle between the two given lines, then

$$\cos\theta = \pm\frac{5(-3) + (-12)4 + (13)5}{\sqrt{5^2 + (-12)^2 + 13^2},\ \sqrt{(-3)^2 + 4^2 + 5^2}}$$

$$= \pm\frac{-15 - 48 + 65}{\sqrt{25 + 144 + 169},\ \sqrt{9 + 16 + 25}}$$

$$= \pm\frac{2}{\sqrt{169 + 169},\ \sqrt{50}}$$

$$= \frac{2}{13\sqrt{2}\times 5\sqrt{2}} = \pm\frac{1}{65}$$

Since, θ is acute it is given by $\cos\theta = \dfrac{1}{65}$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{1}{65}\right)$$

CHECK YOUR PROGRESS 33.5

1. Find the angle between the lines whose direction ratios are 1, 1, 2 and $\sqrt{3} - 1,\ -\sqrt{3} - 1,$ 4.
2. Show that the points A (7, 6, 3), B (4, 10, 1), C (−2, 6, 2) and D (1, 2, 4) are the vertices of a reactangle.
3. By calculating the angle of the triangle with vertices (4, 5, 0), (2, 6, 2) and (2, 3, −1), show that it is an isosceles triangle.
4. Find whether the pair of lines with given direction cosines are parallel or perpendicular.

   (a) $\dfrac{2}{3}, -\dfrac{2}{3}, \dfrac{1}{3}; -\dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3}$ (b) $\dfrac{3}{5}, \dfrac{4}{5}, \dfrac{0}{5}\ ;\ -\dfrac{4}{5}, \dfrac{3}{5}, \dfrac{0}{5}$

LET US SUM UP

- For a given point P (x, y, z) in space with reference to reactangular co-ordinate axes, if we draw three planes parallel to the three co-ordinate planes to meet the axes (in A, B and C say), then

  OA = x, OB = y and OC = z where O is the origin.

  Converswly, given any three numbers, x, y and z we can find a unique point in space whose co-ordinates are (x, y, z).

- The distance PQ between the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

  $$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

  In particular the distance of P from the origin O is $\sqrt{x_1^2 + y_1^2 + z_1^2}$.

- The co-ordinates of the point which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $l : m$

  (a) internally are $\left(\dfrac{lx_2 + mx_1}{l+m}, \dfrac{ly_2 + my_1}{l+m}, \dfrac{lz_2 + mz_1}{l+m}\right)$

  (b) externally are $\left(\dfrac{lx_2 - mx_1}{l-m}, \dfrac{ly_2 - my_1}{l-m}, \dfrac{lz_2 - mz_1}{l-m}\right)$

  In particular, the co-ordinates of the mid-point of PQ are

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

- If l, m and n are the direction cosines of the line, then $l^2 + m^2 + n^2 = 1$.
- The three numbers which are proportional to the direction cosines of a given line are called its direction ratios.
- Direction cosines of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$.
- The projection of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line with direction cosines l, m and n is $(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$.
- The direction cosines l, m and n of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

  $$\frac{x_2 - x_1}{l} = \frac{y_2 - x_1}{m} = \frac{z_2 - x_1}{n} = PQ$$

- The angle θ between two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

  If the lines are

  (a) perpindicular to each other then, $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

  (b) parallel to each other then $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then the angle θ between them is given by

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

- The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

SUPPORTIVE WEB SITES

http:// www.wikipedia.org.
http:// mathworld.wolfram.com

TERMINAL EXERCISE

1. Show that the points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) form an isosceles right-angled triangle.

2. Prove that the points P, Q and R, whose co-ordinates are respectively $(3, 2, -4)$, (5, 4, –6) and (9, 8, –10) are collinear and find the ratio in which Q divides PR.

3. A (3, 2, 0), B (5, 3, 2), C (–9, 6, –3) are three points forming a triangle. AD, the bisector of the angle $\angle BAC$ meets BC at D. Find the co-ordinates of D.

   (**Hint :** D divides BC in the ratio AB : AC)

4. Find the direction cosines of the line joining the point (7, –5, 4) and (5, –3, 8).

5. What are the direction cosines of a line equally inclined to the axes ? How many such lines are there ?

6. Determine whether it is possible for a line to make the angle 45°, 60° and 120° with the co-ordinate axes.

7. Show that the points (0, 4, 1), (2, 3, −1), (4, 5, 0) and (2, 6, 2) are the vertices of a square.

8. Show that the points (4, 7, 8), (2, 3, 4), (−1, −2, 1) and (1, 2, 5) are the vertices of a parallelogram.

9. A (6, 3, 2), B(5, 1, 4), C (3, −4, 7) and D (0, 2, 5) are four points. Find the projections of (i) AB on CD, and (ii) CD on AB.

10. Three vertices of a parallelogram ABCD are A (3, −4, 7), B (5, 3, −2) and C (1, 2, −3). Find the fourth vertex D.

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Notes

ANSWERS

CHECK YOUR PROGRESS 33.1

1. (a) OX' YZ (b) OXYZ' (c) OX' YZ'
   (d) OXY'Z (e) OXYZ'

CHECK YOUR PROGRESS 33.2

1. (a) 7 (b) $2\sqrt{5}$ (c) $\sqrt{3}$

CHECK YOUR PROGRESS 33.3

1. $(1, -3, 2)$ 2. $\left(\frac{13}{5}, \frac{6}{5}, -\frac{13}{5}\right)$; $(5, 18, -17)$

3. $-2:3$ 6. $(2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$

CHECK YOUR PROGRESS 33.4

1. (a) $\pm\frac{3}{\sqrt{14}}, \pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}$ (b) $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$

2. (a) $(0, 5, 6)$, $(-3, 0, 6)$ and $(-3, 5, 0)$

   (b) $(-3, 0, 0)$, $(0, 5, 0)$ and $(0, 0, 6)$

3. (a) $\pm\frac{1}{3}, \pm\frac{2}{3}, \mp\frac{2}{3}$ (b) $\pm\frac{6}{7}, \pm\frac{2}{7}, \pm\frac{3}{7}$

4. $\frac{30}{7}$. 5. $\frac{7}{3}$.

CHECK YOUR PROGRESS 33.5

1. $\frac{\pi}{3}$ 4. (a) Parallel (b) Perpendicular

TERMINAL EXERCISE

2. $1:2$ 3. $\left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$ 4. $\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ or $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

5. $\pm\frac{1}{\sqrt{3}}, \pm\frac{-1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$; 4 6. Yes.

9. (i) $\frac{13}{7}$ (ii) $\frac{13}{3}$ 10. $(-1, -5, 6)$