Differential Equations

- Theory and Applications -
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CHAPTER 0

Introduction

This textbook covers the material for the undergraduate Differential Equations course at California State University Sacramento. Although there might be issues related just to a particular campus, I believe that the presentation shown here is useful to a general audience.

First, let's see the particular issues. This is a 3 unit class, taught 3 times 50 minutes (or 2 times 75 minutes) per week during a semester of 15 weeks. Most of the students are science majors, including mathematics, physics and engineering. Many of the students are transfer students, who took the prerequisite classes - Precalculus, Calculus 1 and 2 - at other campuses, so there is a wide range of mathematical knowledge and maturity.

At the beginning of every semester a week of review of calculus, especially differentiation and integration rules, proved to be necessary.

The Linear Algebra course is not a prerequisite for this class, and within the time frame allowed, it is difficult to spend time on covering the complete fundamentals regarding operations with matrices, eigenvalues and eigenvectors. Also, there is no computer lab component for this course. These are not optimal starting points for this class and I hope that the coming years will bring some changes.

Secondly, let's talk about some general issues. Almost all of my students were used to getting the 1000+ pages textbooks for their earlier courses. Over the years these huge textbooks killed the habits of taking time to read them, focus on the details and understanding the definitions and theorems describing the main ideas. The most frequent question I do get is: "We see what is the material, but how much of it we have to know for the exam?" The answer - "All of it." - usually brings out a big sign of disbelief.

Differential Equations is a very important mathematical subject from both theoretical and practical perspectives.

The theoretical importance is given by the fact that most pure mathematics theories have applications in Differential Equations. For students, all the prerequisite knowledge is tested in this class.

The practical importance is given by the fact that the most important time dependent scientific, social and economical problems are described by differential, partial differential and stochastic differential equations. The bridge between Nature or Universe and us is provided by mathematical modeling, which is the process of finding the correct mathematical equations describing a certain problem. This process might start with experimental measurements and analysis, which lead to certain equations, in our case differential equations. Then, these differential equations are solved and their solutions tested for agreement to experimental results. In this process we generate some solutions, which have the role to predict the **future behavior** of the analyzed problem.

In general, regarding the future, there is no solution manual and here comes another issue. Most of my students were used to having solution manuals for their mathematics classes and checking whether the solution is right or wrong was reduced to comparison with the answers in the solution manual. However, this eliminates the need to completely understand what we are doing and whether the answer really makes sense.

Differential Equations is probably one of the best classes which can make us understand that Nature does not provide us with a complete solution manual. We usually have to find some approximate answers and we are also left with the task of predicting how accurate these answers are, without knowing the correct answer.

For this reason, there will be NO SOLUTION MANUAL posted. I request the students to check the correctness of their answers by applying the theoretical methods shown in class, but also by using a computer software in the campus computer labs. The available software is Mathematica, which could be substituted off campus by Wolfram Alpha. There are many mathematical softwares, like Maple, Matlab, Octave, and you are free to use whichever is available to you. The most important thing is to actively participate in the teaching-learning process and based on the information presented in class, create your own way of checking your answers. The answers given by computers might be in a different form than the ones obtained on paper, but it is a good challenge to compare them. You must develop intuition, theoretical and computer knowledge to be able to test and decide whether a solution is correct or wrong.

CHAPTER 1

Calculus review. Differentiation and integration rules.

1.1. Derivatives

DEFINITION 1.1.1. *Consider a function $y : I \to \mathbb{R}$, where I is an interval on the real line $\mathbb{R}$. We say that the function y has a derivative at $t_0 \in I$ if the limit*

$$\lim_{t \in I,\, t \to t_0} \frac{y(t) - y(t_0)}{t - t_0}$$

exists and it is finite. In the case when the derivative exists, we use the notation

$$y'(t_0) = \lim_{t \in I,\, t \to t_0} \frac{y(t) - y(t_0)}{t - t_0}\,.$$

Other notations for the derivative of function y at t_0 can be $\frac{dy}{dt}(t_0)$ or $\frac{d}{dt}y(t_0)$.
In case t_0 is one of the endpoints of the interval I, then the above limits become one sided limits.
If the derivative exists at every $t_0 \in I$, then $y'(t)$ is a new function, called the derivative function.
If $y'(t)$ has a derivative function, then we call it the second derivative of the function $y(t)$ and denote it by $y''(t)$.
For higher order derivatives we use the notations $y'''(t), y^{(4)}(t)$, ... , $y^{(n)}(t)$, or $\frac{d^n}{dt^n}y(t)$.

Interpretations and applications of the derivative:

(1) $y'(t_0)$ is the instantaneous rate of change of the function y at t_0.
(2) $y'(t_0)$ is the slope of the tangent line to the curve $y = y(t)$, $t \in I$ at the point $(t_0, y(t_0))$.
(3) If the function y has a local maximum (minimum) at t_0, which is in the interior of I, and y is differentiable at t_0, then $y'(t_0) = 0$. However, $y'(t_0)$ might not be zero if t_0 is one of the endpoints.
(4) If $y'(t) \geq 0$ for every $t \in I$, then the function y is increasing on I.
(5) If $y'(t) \leq 0$ for every $t \in I$, then the function y is decreasing on I.
(6) If $y''(t) \geq 0$ for every $t \in I$, then the function y is concave-up on I.
(7) If $y''(t) \leq 0$ for every $t \in I$, then the function y is concave-down on I.

Derivatives of the most used elementary functions:

$$(t^n)' = nt^{n-1}$$

$$(a^t)' = a^t \ln a\,, \quad (e^t)' = e^t\,, \quad (\ln t)' = \frac{1}{t}$$

$$(\sin t)' = \cos t\,, \quad (\cos t)' = -\sin t\,, \quad (\tan t)' = \sec^2 t$$

$$(\arcsin t)' = \frac{1}{\sqrt{1-t^2}}\,, \quad (\arccos t)' = \frac{-1}{\sqrt{1-t^2}}\,, \quad (\arctan t)' = \frac{1}{1+t^2}\,.$$

Differentiation Rules: In the following rules y and z are differentiable functions on an interval I, $t \in I$ and $c \in \mathbb{R}$.

(1)

$$\Big(y(t) + z(t)\Big)' = y'(t) + z'(t)\,.$$

(2)

$$\Big(c \cdot y(t)\Big)' = c \cdot y'(t)\,.$$

(3)

$$\Big(y(t) \cdot z(t)\Big)' = y'(t) \cdot z(t) + y(t) \cdot z'(t)\,.$$

(4)

$$\left(\frac{y(t)}{z(t)}\right)' = \frac{y'(t) \cdot z(t) - y(t) \cdot z'(t)}{z^2(t)}\,, \quad \text{if} \;\; z(t) \neq 0\,.$$

(5)

$$\Big(y\big(z(t)\big)\Big)' = y'\big(z(t)\big) \cdot z'(t)\,.$$

Examples:

$$(t^2 - 3t + 5)' = 2t - 3$$

$$\Big(t^3 \cdot e^{2t}\Big)' = 3t^2 \cdot e^{2t} + t^3 \cdot 2e^{2t}$$

$$\Big(\tan t\Big)' = \left(\frac{\sin t}{\cos t}\right)' = \frac{\cos t \cdot \cos t - \sin t \cdot (-\sin t)}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$\Big(\sqrt{1+t^2}\Big)' = \frac{1}{2}(1+t^2)^{-\frac{1}{2}} \cdot 2t = \frac{t}{\sqrt{1+t^2}}$$

$$\Big(\arctan(t^2)\Big)' = \frac{1}{1+t^4} \cdot 2t = \frac{2t}{1+t^4}$$

Note: To define functions, calculate derivatives and plot graphs with Mathematica, see Chapter 8.

Homework exercises:

(1) Find the derivatives of the following functions:

$$
\begin{aligned}
(a)\quad & f(t) = 2t^3 + 5t^2 - 3t - 4 \\
(b)\quad & f(t) = t^2\, e^{t^3} \\
(c)\quad & f(t) = \sin t \cdot \cos t \\
(d)\quad & f(t) = \frac{t^2 - 1}{t^3 + 8} \\
(e)\quad & f(t) = \sqrt[3]{4t^2 + 1} \\
(f)\quad & f(t) = t \arcsin 3t \\
(g)\quad & f(t) = \frac{t}{\sqrt{t^2 + 1}} \\
(h)\quad & f(t) = (2t + 1) \ln t \\
(i)\quad & f(t) = (\tan t)^2 \cdot \sec t\,.
\end{aligned}
$$

(2) Graph the following functions. Find the domain, the horizontal and vertical asymptotes, local minima and maxima and intervals where the following functions are decreasing or increasing, convex or concave.
Check your answers by graphing the functions with Mathematica.

$$
\begin{aligned}
(a)\quad & f(t) = t^3 - 4t\,. \\
(b)\quad & f(t) = \frac{2t - 4}{t^2 - 6t + 5}\,. \\
(c)\quad & f(t) = \ln t - 2t\,. \\
(d)\quad & f(t) = \frac{e^t}{t}\,. \\
(e)\quad & f(t) = te^{-t^2}\,. \\
(f)\quad & f(t) = \arctan t\,. \\
(g)\quad & f(t) = 3\sin(2t) + 1\,.
\end{aligned}
$$

1.2. Antiderivatives and Indefinite Integrals

DEFINITION 1.2.1. *Let $y : I \to \mathbb{R}$ be a function. A differentiable function $Y : I \to \mathbb{R}$ is called an* **antiderivative** *of the function y on the interval I if*

$$Y'(t) = y(t)\,, \quad \textit{for all } \; t \in I\,.$$

The set (or collection) of all the antiderivatives of y is denoted by

$$\int y(t)\,dt$$

and called the **indefinite integral** *of the function y.*

Examples:

(a)

$$y : \mathbb{R} \to \mathbb{R}\,,\; y(t) = 2t\,,\; Y(t) = t^2\,,\; \int 2t\,dt = t^2 + c\,.$$

(b)

$$y : (-1,1) \to \mathbb{R}\,,\; y(t) = \frac{1}{\sqrt{1-t^2}}\,,\; Y(t) = \arcsin t\,,$$

$$\int \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin t + c\,.$$

Integration Rules:

(1) Linearity, the sum rule.

$$\int y(t) + z(t)\,dt = \int y(t)\,dt + \int z(t)\,dt = Y(t) + Z(t) + c\,.$$

(2) Linearity, the constant multiple rule.

$$\int a \cdot y(t)\,dt = a \int y(t)\,dt = a\,Y(t) + c\,.$$

(3) Integrals of some elementary functions:

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + c, \; n \neq -1.$$

$$\int \frac{1}{t}\,dt = \ln|t| + c$$

$$\int e^t\,dt = e^t + c$$

$$\int \sin t\,dt = -\cos t + c$$

$$\int \cos t\,dt = \sin t + c$$

$$\int \tan t\,dt = \ln|\sec t| + c$$

$$\int \sec t\, dt = \ln|\sec t + \tan t| + c$$

$$\int \frac{1}{t^2 + a^2}\, dt = \frac{1}{a} \arctan\left(\frac{t}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - t^2}}\, dt = \arcsin\left(\frac{t}{a}\right) + c\,.$$

(3) The substitution rule : $u = z(t)$, $du = z'(t)\, dt$,

$$\int y\big(z(t)\big) \cdot z'(t)\, dt = \int y(u)\, du = Y(u) + c = Y(z(t)) + c\,.$$

Example: Use $u = t^3 + 1$ and $du = 3t^2 dt$ to get

$$\int \frac{t^2}{\sqrt{t^3 + 1}}\, dt = \int \frac{1}{\sqrt{u}} \frac{1}{3}\, du = \frac{2}{3}\sqrt{u} + c = \frac{2}{3}\sqrt{t^3 + 1} + c\,.$$

(4) The integration by parts.

$$\int y(t)z(t)\, dt = y(t)Z(t) - \int y'(t)Z(t)\, dt\,.$$

Example:

$$\int te^{2t}\, dt = t\frac{e^{2t}}{2} - \int 1\frac{e^{2t}}{2}\, dt = \frac{te^{2t}}{2} - \frac{e^{2t}}{4} + c\,.$$

(5) Trigonometric substitutions.

(a) For integrals containing $\sqrt{a^2 + t^2}$ use $t = a \cdot \tan\theta$.
Example. Use $t = 2\tan\theta$ and $dt = 2\sec^2\theta\, d\theta$ to get

$$\int \frac{1}{t^2\sqrt{t^2 + 4}}\, dt = \int \frac{\cos\theta}{4\sin^2\theta}\, d\theta = -\frac{1}{4\sin\theta} + c = -\frac{\sqrt{t^2 + 4}}{4t} + c\,.$$

(b) For integrals containing $\sqrt{a^2 - t^2}$ use $t = a \cdot \sin\theta$.

(c) For integrals containing $\sqrt{t^2 - a^2}$ use $t = a \cdot \sec\theta$.

(6) Trigonometric integrals.

(a) For integrals of the form $\int \sin^n(t)\cos^{2k+1}(t)\, dt$ use the substitution $u = \sin t$.
Example. Use $u = \sin t$ and $du = \cos t\, dt$ to get

$$\int \sin^2 t\, \cos^3 t\, dt = \int u^2(1 - u^2)\, du = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 t}{3} - \frac{\sin^5 t}{5} + c\,.$$

(b) For integrals of the form $\int \cos^n(t)\sin^{2k+1}(t)\, dt$ use the substitution $u = \cos t$.

(c) For integrals of the form $\int \sin^{2n}(t)\cos^{2k}(t)\, dt$ use the double angle formulas $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$ and $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$.

The double angle formulas follow from the following two trigonometric identities:

$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t - \sin^2 t = \cos(2t)\,.$$

(d) For integrals of the form $\int \tan^n(t)\sec^{2k}(t)\,dt$ use the substitution $u = \tan t$.

(e) For integrals of the form $\int \tan^{2k+1}(t)\sec^n(t)\,dt$ use the substitution $u = \sec t$.
Example. Use $u = \sec t$ and $du = \sec t\,\tan t\,dt$ to get

$$\int \tan^3(t)\,\sec^2(t)\,dt = \int (u^2-1)u\,du = \frac{u^4}{4} - \frac{u^2}{2} + c = \frac{\sec^4(t)}{4} - \frac{\sec^2(t)}{2} + c\,.$$

(7) Integration by partial fraction decompositions. Some examples:

(a)

$$\frac{2t+3}{(t-1)(t+2)} = \frac{A}{t-1} + \frac{B}{t+2}\,,\ A = \frac{5}{3},\ B = \frac{1}{3}$$
$$\int \frac{2t+3}{(t-1)(t+2)}\,dt = \frac{5}{3}\ln|t-1| + \frac{1}{3}\ln|t+2|\,.$$

(b)

$$\frac{t^2+t+2}{t(t+1)^2} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{(t+1)^2},\ A = 2,\ B = -1,\ C = -2$$
$$\int \frac{t^2+t+2}{t(t+1)^2}\,dt = 2\ln|t| - \ln|t-1| + \frac{2}{t+2}\,.$$

(c)

$$\frac{2t-19}{(t+3)(t^2+16)} = \frac{A}{t+3} + \frac{Bt+C}{t^2+16},\ A = -1,\ B = 1,\ C = -1$$

$$\int \frac{2t-19}{(t+3)(t^2+16)}\,dt = -\ln|t+3| + \frac{1}{2}\ln(t^2+16) - \frac{1}{4}\arctan\left(\frac{t}{4}\right) + c\,.$$

Note: To calculate integrals with Mathematica, see Chapter 8.

Homework exercises: Calculate the following integrals. Check your answers by differentiation and also by using Mathematica. For instructions, see Chapter 8.

$$(1) \quad \int (2t^3 - 3t^2 + 2t - 5)\, dt$$

$$(2) \quad \int \frac{t}{1+t^2}\, dt$$

$$(3) \quad \int t^2 e^{t^3}\, dt$$

$$(4) \quad \int (t^2 + t + 1)e^t\, dt$$

$$(5) \quad \int t\, \sin t\, dt$$

$$(6) \quad \int \frac{1}{t^2\sqrt{9-t^2}}\, dt$$

$$(7) \quad \int \frac{1}{\sqrt{t^2-25}}\, dt$$

$$(8) \quad \int \frac{1}{\sqrt{4t^2+1}}\, dt$$

$$(9) \quad \int \sin^5 t \cdot \cos^2 t\, dt$$

$$(10) \quad \int \tan^3 t \cdot \sec^4 t\, dt$$

$$(11) \quad \int \cos^4 t\, dt$$

$$(12) \quad \int \frac{1}{t^2-1}\, dt$$

$$(13) \quad \int \frac{t+1}{t^2+4t+3}\, dt$$

$$(14) \quad \int \frac{t^2-1}{t^3+t}\, dt$$

$$(15) \quad \int \frac{5t^2+20t+6}{t^3+2t^2+t}\, dt$$

$$(16) \quad \int \ln t\, dt$$

$$(17) \quad \int t \ln t\, dt$$

1.3. Definite Integrals

DEFINITION 1.3.1. *Consider a bounded function* $y : [a, b] \to \mathbb{R}$*. For a partition of the interval* $[a, b]$*,*

$$P = \left\{ a = t_0 < t_1 < ... < t_n = b \right\},$$

and sample points $t_{k-1} \le t_k^* \le t_k\,,\ \ 1 \le k \le n$*, define the Riemann-sum*

$$S(y, P) = \sum_{k=1}^{n} y\big(t_k^*\big)\,\big(t_k - t_{k-1}\big)\,.$$

The norm of the partition P *is defined as the length of the largest subinterval* $[t_{k-1}, t_k]$*. If the Riemann-sums have a well-defined finite limit as the norm of the partition* P *tends to* 0*, then we say that the function* y *is Riemann-integrable on* $[a, b]$ *and we denote this definite integral by*

$$\int_a^b y(t)\,dt\,.$$

The set of Riemann-integrable functions on $[a, b]$ includes, among others, the continuous functions and, also the bounded functions with finitely many jump discontinuities.

Geometrical interpretation of the definite integral:

$\int_a^b y(t)\,dt$ is the **net area** bounded by the t-axis, $t = a$, $t = b$ and the graph of the function y. Net area means the difference of the area above and the area below the t-axis. If we want the **total area** bounded by the t-axis, $t = a$, $t = b$ and the graph of the function y, we have to calculate $\int_a^b |y(t)|\,dt$. In particular, if $y(t) \ge 0$ for all $t \in [a, b]$, then the total area is given by $\int_a^b y(t)\,dt$.

The Fundamental Theorem of Calculus (FTC):

THEOREM 1.3.1. *If* $y : [a, b] \to \mathbb{R}$ *is a Riemann-integrable function on* $[a, b]$ *and* Y *is an antiderivative function of* y *on* $[a, b]$*, then*

$$\int_a^b y(t)\,dt = Y(b) - Y(a)\,.$$

Corollary to the FTC:

COROLLARY 1.3.1. *If* y *is a continuous function on* $[a, b]$*, then the function*

$$Y(t) = \int_a^t y(s)\,ds$$

is an antiderivative of y*, and hence*

$$\frac{d}{dt}\left(\int_a^t y(s)\,ds\right) = y(t)\,,\quad a \le t \le b\,.$$

Note. The integration rules for indefinite integrals apply for definite integrals. Just, we have to take care of the lower and upper limits of integrations.

Examples. (a) We can use the substitution $u = t^2$ with $du = 2tdt$ to calculate the following definite integral:

$$\int_0^2 2te^{t^2}\,dt = \int_0^4 e^u\,du = e^u\Big|_0^4 = e^4 - 1\,.$$

(b)

$$\int_0^{\frac{\pi}{2}} \sin^2 t \cdot \cos^3 t\,dt = \int_0^{\frac{\pi}{2}} \sin^2 t \cdot \cos^2 t \cdot \cos t\,dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 t \cdot (1 - \sin^2 t) \cdot \cos t\,dt$$

$$u = \sin t\,, \quad du = \cos t\,dt$$

$$= \int_0^1 u^2(1 - u^2)\,du = \int_0^1 u^2 - u^4\,du =$$

$$= \frac{u^3}{3} - \frac{u^5}{5}\Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}.$$

Homework exercises: Calculate the following definite integrals. Check your answers with Mathematica. For instructions, see Chapter 8.

(1) $\int_0^2 (t^3 - t + 1)\,dt$

(2) $\int_2^3 \frac{1}{t^2}\,dt$

(3) $\int_3^4 \frac{1}{t\ln t}\,dt$

(4) $\int_0^1 \frac{t}{1 + t^2}\,dt$

(5) $\int_0^1 \frac{1}{\sqrt{4 - t^2}}\,dt$

(6) $\int_0^{\pi} t\,\sin(2t)\,dt$

(7) $\int_0^1 t^2\,e^t\,dt$

(8) $$\int_{\sqrt{3}}^{2} \frac{t^2 - 3}{t}\, dt$$

(9) $$\int_{\pi/6}^{\pi/3} \frac{\cos^3 t}{\sqrt{\sin t}}\, dt$$

(10) $$\int_{1}^{2} \frac{t + 1}{t(t^2 + 1)}\, dt$$

(11) $$\int_{-2}^{0} \frac{t}{t^2 - 6t + 8}\, dt$$

(12) $$\int_{0}^{1} \frac{1}{t^2 + 2t + 5}\, dt$$

(13) $$\int_{1}^{2} \frac{1}{t^3 + 2t^2 + t}\, dt$$

CHAPTER 2

Introduction to Differential Equations

2.1. Definitions

Definition 2.1.1. *A* **differential equation (DE)** *is an equation in which an unknown function $y(t)$ appears together with some of its derivatives.*

In general, a DE can be written as

$$F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0\,, \quad t \in I\,.$$

Examples:

(a)

$$y''(t) - 2y'(t) + y(t) - t^2 = 0\,, \quad t \in (-1, 1)\,.$$

(b)

$$y^{(4)}(t) \cdot y'(t) - y(t) = 2t + 1\,, \quad t \geq 0\,.$$

(c)

$$\frac{e^t y'(t)}{1 + y^2(t)} = 5\,, \ t \in \mathbb{R}\,.$$

(d) Calculating the indefinite integral $\int 2t\,dt$ is the same as solving the DE $y'(t) = 2t$. Both problems ask for those functions, which have derivative equal to $2t$.

Definition 2.1.2. *The* **order of a DE** *is defined by the highest derivative present in the equation.*

Examples.
(a) The DE $y''(t) - (y'(t))^3 + 5y^6(t) = e^t$ has order 2.
(b) The DE $y^{(4)}(t) - y'(t) = 0$ has order 4.

Normal form of a DE. If the DE can be solved in the highest order derivative, then we say that we have obtained its normal form, which can be written as:

$$y^{(n)}(t) = f(t, y(t), y'(t), ..., y^{(n-1)}(t))\,, \quad t \in I\,.$$

Examples.

(a) The DE

$$t^2 y''(t) - ty'(t) + y(t) = e^t\,, \quad t \in [1, 2]$$

can be written in the following normal form:

$$y''(t) = \frac{1}{t} y'(t) - \frac{1}{t^2} y(t) + \frac{1}{t^2} e^t\,, \quad t \in [1, 2]\,.$$

This normal form was obtained by dividing the DE by t^2. However, if we consider the interval $[-1,1]$, dividing by t^2, which becomes 0 for $t=0$, makes the normal form not defined on the entire interval $[-1,1]$.

(b) The DE

$$e^{y'(t)} + y'(t) = (t+1)y(t)\,,\ t \in [0,1]$$

cannot be solved in $y'(t)$, so it cannot be written in normal form.

Definition 2.1.3. *A* **system of differential equations (SDEs)** *is formed by a number of differential equations involving more than one unknown functions and their derivatives.*

Example of a SDEs:

$$\begin{cases} y'(t) = y(t) + z(t) \\ z'(t) = y(t) - z(t)\,,\ t \in \mathbb{R}\,. \end{cases}$$

Note. Every higher order DE can be rewritten as a first order SDEs. This is very important for studying the existence of solutions and their numerical approximations.

Example.
Consider the second order DE $y''(t) = y(t)$ and introduce the function $z(t) = y'(t)$. Now we can write the SDEs

$$\begin{cases} y'(t) = z(t) \\ z'(t) = y(t)\,, \end{cases}$$

which has a pair of solutions $(y(t), z(t))$, in which the first component is the same as the solution of the original second order DE and the second component is the derivative of it. Solving the SDEs is equivalent to solving the DE.

Definition 2.1.4. *A* **solution** *of a DE on an interval I is a function $y = y(t)$ which, when substituted into the DE, satisfies the equation identically on the interval I.*

Examples of solutions.

(a) $y(t) = \cos t$ is a solution of $y''(t) + y(t) = 0$ on $(-\infty, +\infty)$. To verify this we have to observe that $y''(t) = -\cos t$, and hence we get

$$-\cos t + \cos t = 0\,, \quad \text{for each } \ t \in (-\infty, +\infty)\,,$$

which means that the $y(t) = \cos t$ satisfies the DE identically on $(-\infty, +\infty)$.
But, observe also that it is not the only solution. $y_2(t) = \sin t$ is another solution. Moreover, any function of the form $y(t) = a\cos t + b\sin t$ is a solution.

(b) $y(t) = \sqrt{1-t^2}$ is a solution of the DE $y'(t)\cdot y(t) + t = 0$ on the interval $(-1,1)$, but it is not a solution on any interval larger than $(-1,1)$.

Explicit and implicit solutions. Functions can be defined explicitly or implicitly. Therefore, solutions of DEs, which are functions, can be obtained explicitly or implicitly and,

hence, we can talk about explicit or implicit solutions. The above examples are all explicit solutions.
For an example of an implicit solution consider the equation

$$t^2 + y(t) + y^3(t) = 5\,,$$

which defines the function $y(t)$ implicitly. If we use implicit differentiation, we get the DE

$$2t + y'(t) + 3y^2(t)\,y'(t) = 0\,,$$

which has the same function $y(t)$, as an implicitly defined solution.

Indefinite integrals: When we calculate the indefinite integral $\int 2t\,dt$, we actually solve the DE $y'(t) = 2t$. All the solutions are in the form $t^2 + c$, where the parameter c can be any real number. We can write this as $y(t) = t^2 + c$, and the meaning is that we have a one-parameter family of solutions, which is the same as the family of all the antiderivatives of $2t$.

In general, DEs tend to have infinitely many solutions, but the general situation is much more complex.

Families of solutions:
If the solutions of a DE depend on parameters $c_1, ..., c_k$, then we call them a k-parameter family of solutions.

Singular solutions of DE.
A solution of a DE, which is not part of any family of solutions is called singular solution.

Examples of solutions for DEs.

(a) $y'(t) - y(t) = 0$ has solutions of the form $y(t) = ce^t$. Therefore, we have a one-parameter family of solutions and, as we will see later, all solutions are part of this family.

(b) $y''(t) - y(t) = 0$ has a two-parameter family of solutions of the form $y(t) = c_1e^t + c_2e^{-t}$.

(c) $y'(t) = t\sqrt{y(t)}$ has a one-parameter family of solutions $y(t) = \left(\frac{1}{4}t^2 + c\right)^2$, but also a solution $y(t) = 0$, which is not part of this family.

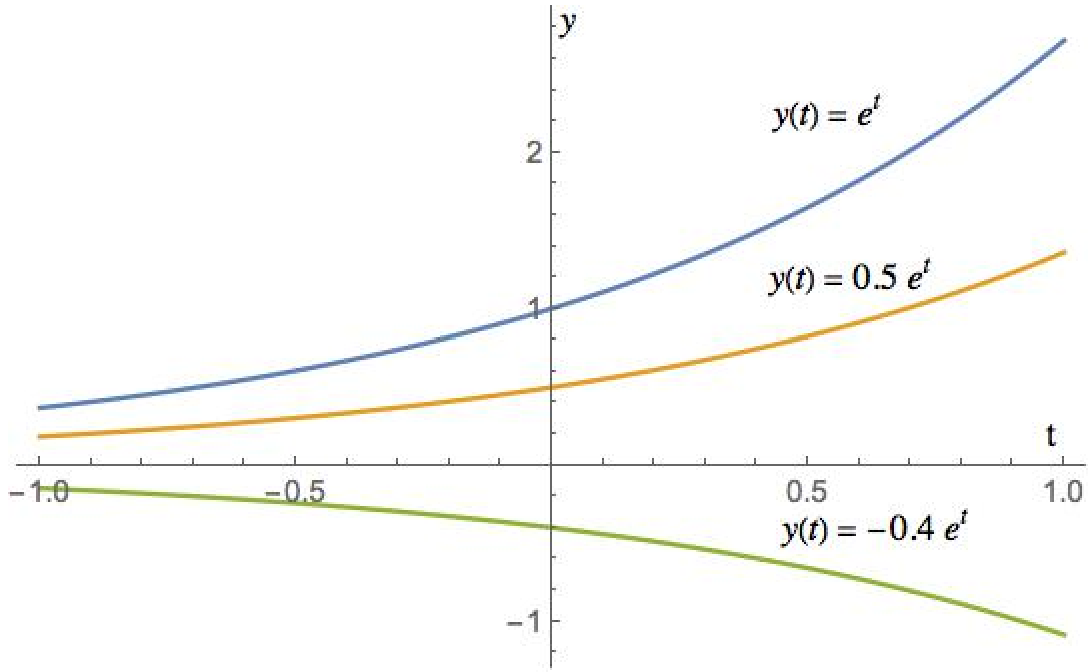
(d) $(y'(t))^2 + (y(t))^2 = 0$ has exactly one solution, the constant function $y(t) = 0$.

(e) $(y'(t)^2 + (y(t))^2 = -1$ does not have any solutions.

Solution curve of a DE.

The graph of a solution of a DE is called a solution curve.
For example, $y_1(t) = e^t$, $y_2(t) = 0.5e^t$ and $y_3(t) = -0.4e^t$ are solutions of $y'(t) - y(t) = 0$, so their graphs, which are the curves with equations $y = e^t$, $y = 0.5e^t$ and $y = -0.4e^t$ are

solution curves.



Homework exercises.

1. Find the order of the following DEs:

$$(a) \quad y'''(t) + t^2 y''(t) - y(t) = t^4 .$$
$$(b) \quad y^{(4)}(t) + y'(t) - y^5(t) = 0 .$$
$$(c) \quad (1 - t^3)y''(t) + e^t y'(t) - \sqrt{1+t}\,y(t) = 4 .$$
$$(d) \quad t^3\, y'(t) + y(t) = \sin t .$$
$$(e) \quad \frac{y(t)}{1 + (y'(t))^2} = 2 .$$
$$(f) \quad \sqrt{y''(t) + t^2} = y'(t) .$$

2. Find the normal form of the following DEs:

$$(a) \quad (1 + t^2)y''(t) + t y'(t) - 5y(t) = t^3 + 4 .$$
$$(b) \quad y(t)y'(t) + t = 1 .$$
$$(c) \quad \sqrt{y'(t) + 4} + y(t) - t = 0 .$$

3. Rewrite the following DEs as systems of first order DEs.

$$(a)\quad y''(t)+2y'(t)+y(t)=t\,.$$
$$(b)\quad t^3y'''(t)-2t^2y''(t)+3t^3y'(t)-4t^4y(t)=0\,.$$
$$(c)\quad y''(t)-y(t)=t\,.$$
$$(d)\quad y'''(t)+2y'(t)+t^2y(t)=e^t\,.$$

4. Verify whether the indicated function is a solution of the given DE or not.

$$(a)\quad y''(t)+4y'(t)+3y(t)=0\,,\ y(t)=e^{-3t}\,,\quad t\in\mathbb{R}\,.$$
$$(b)\quad y''(t)-4y'(t)+3y(t)=0\,,\ y(t)=e^{-3t}\,,\quad t\in\mathbb{R}\,.$$
$$(c)\quad (4-t^2)y'(t)+2ty(t)=0\,,\ y(t)=\frac{1}{4-t^2}\,,\quad -2<t<2\,.$$
$$(d)\quad (4-t^2)y'(t)-2ty(t)=0\,,\ y(t)=\frac{1}{4-t^2}\,,\quad -2<t<2\,.$$
$$(e)\quad t^2y''(t)-6y(t)=0\,,\ y(t)=\frac{1}{t^2}\,,\ t>0\,.$$
$$(f)\quad t^2y''(t)-6y(t)=0\,,\ y(t)=\frac{1}{t^2}\,,\ t<0\,.$$
$$(g)\quad t^2y''(t)+6y(t)=0\,,\ y(t)=\frac{1}{t^2}\,,\ t>0\,.$$
$$(h)\quad t^2y''(t)-6y(t)=0\,,\ y(t)=\frac{1}{t^2}\,,\ -1<t<1\,.$$

5. Verify whether the indicated family of functions is a family of solutions of the given DE or not. In case of solutions, plot three different integral curves.

$$(a)\quad y''(t)+y(t)=1\,,\ y(t)=c\cos t+d\sin t+1\,.$$
$$(b)\quad y''(t)-y(t)=2\,,\ y(t)=c\,e^t+d\,e^{-t}-2\,.$$
$$(c)\quad y''(t)+6y'(t)+9y(t)=0\,,\ y(t)=c\,e^{3t}+d\,te^{3t}\,.$$
$$(d)\quad y''(t)-6y'(t)+9y(t)=0\,,\ y(t)=c\,e^{3t}+d\,te^{3t}\,.$$
$$(e)\quad y'(t)-y(t)+y^2(t)=0\,,\ y(t)=\frac{c\,e^t}{1+c\,e^t}\,.$$
$$(f)\quad y'(t)+y(t)+y^2(t)=0\,,\ y(t)=\frac{c\,e^t}{1+c\,e^t}\,.$$

6. Verify that the equation

$$y^3-t^2y=5$$

forms a implicit solution of the DE

$$y'(t)=\frac{2ty(t)}{3y^2(t)-t^2}\,.$$

2.2. Initial value problems

Consider an n$^{\text{th}}$-order DE, $F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0\,, \quad t \in I\,$, and fix $t_0 \in I$.

A **system of initial conditions** is a system of the form

$$y(t_0) = \alpha_0\,,\ y'(t_0) = \alpha_1\,, ..., y^{(n-1)}(t_0) = \alpha_{n-1}\,,$$

where $\alpha_0, \alpha_1, ..., \alpha_{n-1}$ are n given numbers.

Initial Value Problems (IVP). The problem which combines a DE and a system of initial conditions is called an Initial Value Problem:

$$(IVP) \quad \begin{cases} F(t, y(t), y'(t), ..., y^{(n)}(t)) = 0\,, \quad t \in I \\ y(t_0) = \alpha_0 \\ y'(t_0) = \alpha_1 \\ \ldots\ldots\ldots \\ y^{(n-1)}(t_0) = \alpha_{n-1} \end{cases}$$

General solution of a DE: A n-parameter family of solutions of a n$^{\text{th}}$-order DE is called a general solution if for every system of initial conditions a member of that family solves the corresponding IVP.

Example. Consider the Initial Value Problem:

$$(IVP) \quad \begin{cases} y''(t) - y(t) = 0\,, \quad -\infty < t < \infty \\ y(0) = 1 \\ y'(0) = 2\,. \end{cases}$$

The initial condition $y(0) = 1$ tells that the solution must go through the point $(0, 1)$, while the condition $y'(0) = 2$ indicates that the slope of the tangent line to the solution curve at (0,1) must be 2.

The 2-parameter family of solutions

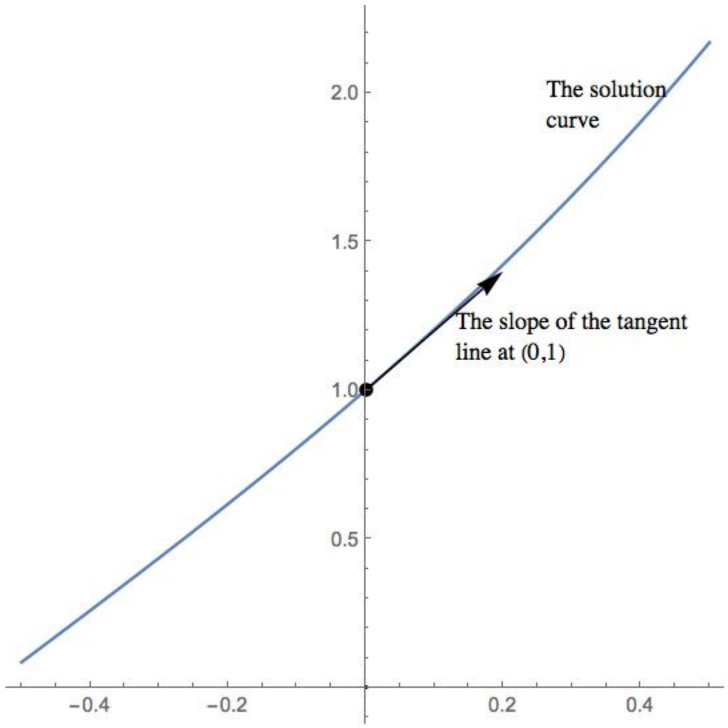
$$y(t) = ce^t + de^{-t}\,,$$

is a general solution of the DE. The initial conditions lead to the linear system of equations

$$\begin{cases} c + d = 1 \\ c - d = 2\,. \end{cases}$$

Solving this system of linear equations gives $c = 3/2$ and $d = -1/2$. Therefore, this IVP has a unique solution of the form

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t}\,.$$



Homework exercises:

1. Consider the general solution

$$y(t) = c\cos(2t) + d\sin(2t)$$

of the DE

$$y''(t) + 4y(t) = 0\,, \quad t \in \mathbb{R}\,.$$

Determine the values of the parameters using the following initial conditions:

$$(a) \quad y(0) = 0\,,\ y'(0) = 0\,.$$
$$(b) \quad y(0) = 1\,,\ y'(0) = 0\,.$$
$$(c) \quad y(0) = 0\,,\ y'(0) = 1\,.$$
$$(d) \quad y(\frac{\pi}{4}) = 2\,,\ y'(\frac{\pi}{4}) = 1\,.$$
$$(e) \quad y(\frac{\pi}{3}) = -1\,,\ y'(\frac{\pi}{3}) = 1\,.$$

2. Consider the family of solutions

$$y(t) = \tan(t^2 + c)\,,$$

of the DE

$$y'(t) = 2t(1 + y^2(t))\,.$$

Determine the values of the parameters using the following initial conditions and determine the domain of the corresponding function. How many solutions do you have?

$$(a) \quad y(0) = 0\,.$$
$$(b) \quad y(0) = 1\,.$$
$$(c) \quad y(1) = -1\,.$$

3. Consider the family of solutions

$$y(t) = -\frac{1}{t+c}$$

of the DE

$$y'(t) = y^2(t)\,, \quad -2 < t < 2\,.$$

Determine the values of the parameters using the following systems of initial conditions and compare the domain of the corresponding function to the interval $(-2, 2)$.

$$(a) \quad y(0) = 0\,.$$
$$(b) \quad y(0) = 1\,.$$
$$(c) \quad y(1) = -1\,.$$
$$(d) \quad y(1.5) = 3\,.$$
$$(e) \quad y(-0.5) = 4\,.$$

2.3. Classifications of DEs

We will use the following two classifications of DEs:

- By order: As we discussed in the previous section, the order of a DE is the order of the highest derivative present in the equation. So, we can talk about DEs of order one, two, three and so on.

- By linearity: A DE of the form

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + ... + a_1(t)\, y'(t) + a_0(t)\, y(t) = f(t)\,,$$

where the functions $a_n(t), ..., a_0(t)$ are given and act as coefficients of the derivatives of the unknown function and $f(t)$ is the function on the right hand side, is called a **linear DE** of order n.
DEs in any other form are called **non-linear**.

Examples.

(1) The DE

$$(t^3 + 1)y''(t) + \sin(t) \cdot\ y'(t) - 5y(t) = e^t$$

is a linear DE of order 2.

(2) The DE

$$(t^3 + 1)y''(t) + \sin(y'(t)) - 5y(t) = e^t$$

is a non-linear DE of order 2.

(3) The DE

$$y'(t) + y^2(t) = t + 1$$

is non-linear and of first order.

(4) The DE

$$y''' + 3y''(t) \cdot y'(t) - ty(t) = 1$$

is non-linear and of third order.

Homework exercises:
Determine whether the following DEs are linear or nonlinear and find their orders.

$$(1)\quad \sqrt{t^2+4}\, y''(t) - 5y'(t) + \frac{1}{t}y(t) = t^3 + 1\,.$$

$$(2)\quad y(t)\cdot y'(t) - 2t = 0\,.$$

$$(3)\quad y'(t) = \frac{y(t)}{t}\,.$$

$$(4)\quad y'(t) = \frac{t}{y(t)}\,.$$

$$(5)\quad y'''(t) - y'(t) = 1\,.$$

$$(6)\quad y''(t) + 4y'(t) + 3y(t) = 2t + 1\,.$$

$$(7)\quad \sqrt{y'(t)+1} - y(t) = 0\,.$$

$$(8)\quad y''(t) + (t-1)y'(t) + \tan(y(t)) = 0\,.$$

$$(9)\quad \cos(t)\cdot y'''(t) - y'(t) = t^2\,.$$

$$(10)\quad y'(t) + e^{y(t)} = t\,.$$

2.4. Examples of DEs modelling real-life phenomena

(1) Radioactive decay
It is known that a radioactive material decomposes at a rate proportional to the amount present at the current time. This can be expressed as a DE

$$M'(t) = kM(t)\,, \quad 0 \le t\,,$$

where $M(t)$ is the mass of the radioactive material present after time t.
As we will see later, the solutions of this first order, linear DE are of the form

$$M(t) = ce^{kt}\,.$$

The constant k is determined experimentally by the half-life of the radioactive material, while the parameter c is determined by the initial condition

$$M(0) = M_0\,,$$

which describes the amount of the material present at time $t = 0$.

(2) Population dynamics.
In 1798 the English economist Thomas Malthus proposed that a population grows at a rate proportional to its size. This leads to the same DE as in the case of radioactive decay:

$$N'(t) = kN(t)\,, \quad t \ge 0\,.$$

Notice that the radioactive decay has the same DE as this model of population dynamics. However, in the case of the radioactive decay the solution is accurate on long time periods, while in the case of the population dynamics only on a short term, except an idealistic situation of an isolated population with unlimited resources.

For a demonstration of this model see:

http://demonstrations.wolfram.com/ContinuousExponentialGrowth/

In a more realistic scenario, the growth rate depends on the size of the populations as well as on external environmental factors, like limited resources. One possible scenario leads to the logistic DE

$$N'(t) = \alpha N(t)\Big(\beta - N(t)\Big)\,,$$

where $\beta > 0$ is the carrying capacity of the environment.

For a demonstration of this model see:
http://demonstrations.wolfram.com/LogisticEquation/

If more than one species interact within the same environment, then we need systems to describe their behavior. In case of two animal species, where the first species eats only vegetation and the second species eats the first species, we are lead to the Lotka-Volterra prey-predator model:

$$\begin{cases} x'(t) = -ax(t) + bx(t)\,y(t) \\ y'(t) = dy(t) - cx(t)\,y(t)\,, \end{cases}$$

where a, b, c, d are positive constants and the functions $x(t), y(t)$ describe the number of the population of the two species.

For a demonstration of the two species model check:
http://demonstrations.wolfram.com/PredatorPreyModel/

For a more realistic model see:
http://demonstrations.wolfram.com/PredatorPreyEcosystemARealTimeAgentBasedSimulation/

(3) Series RLC electric circuits.
The DE describing the state of an electric circuit comes from Kirchhoff's second law of electricity, which says that the sum of the voltage drops around the circuit must add up to the electromotive force. In case of a circuit containing an inductor, a capacitor and a resistor, we denote by L, R, C the inductance, resistance and capacitance. The DE describing this circuit is

$$L\, q''(t) + R\, q'(t) + \frac{1}{C}\, q(t) = E(t)\,,$$

where $q(t)$ is the charge on the capacitor and $E(t)$ is the impressed voltage at time t.

For a demonstration of a series RLC circuit check:
http://demonstrations.wolfram.com/SeriesRLCCircuits/

(4) Mass-Spring systems.
The DE describing a vertical, free mass-spring system follows from Hooke's law and has the form

$$my''(t) + ky(t) = 0\,, \quad t \geq 0\,,$$

where $y(t)$ is the the vertical displacement measured from the natural length of the spring, m is the mass attached to the spring and k is the proportionality constant of the spring.
However, if we assume that damping forces proportional to the velocity act on the mass-spring system, then we have the DE

$$my''(t) + \delta y'(t) + ky(t) = 0\,,$$

where $\delta > 0$ is the damping constant.
To have unique solutions, we have to give, as initial conditions, the initial height and the initial velocity at which the spring is released.

For a demonstration on this problem check:
http://demonstrations.wolfram.com/FreeVibrationsOfASpringMassDamperSystem/

CHAPTER 3

First order differential equations solvable by analytical methods

In this chapter we present several types of first order DEs, which can be solved by algebraic manipulations and integrations.

3.1. Differential equations with separable variables

DEs with separable variables have the form

$$y'(t) = f(t) \cdot g(y(t)) \,.$$

We simplify the way we write these equations in order to separate the variables:

$$y' = f(t) \cdot g(y) \,.$$

Then replace y' by $\frac{dy}{dt}$

$$\frac{dy}{dt} = f(t) \cdot g(y) \,,$$

and get

$$\frac{dy}{g(y)} = f(t)\, dt \,.$$

Integrate the left side with respect to y and the right side with respect to t to obtain an equation of the form

$$G(y) = F(t) + c \,.$$

This is the **implicit form** of the solution. Solving this equation in y gives the solution in **explicit form**.

Examples.

(1) Solve the DE

$$y' = \frac{t}{y} \,, \quad -5 < t < 5 \,.$$

Solution:

$$\frac{dy}{dt} = \frac{t}{y}$$

$$y\, dy = t\, dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + c$$

$$y^2 = t^2 + c \,, \quad \text{solution in implicit form}$$

$$y(t) = \pm\sqrt{t^2 + c} \,, \quad \text{two families of solutions.}$$

(2) Solve the IVP

$$y' = \frac{t}{y}, \quad y(0) = -2\,.$$

First we solve the DE as in Example 1 and get

$$y(t) = \pm\sqrt{t^2 + c}\,.$$

The initial condition shows that we have to use the family of solutions with negative sign and get

$$y(0) = -\sqrt{c} = -2\,,$$

which gives $c = 4$. Therefore, the solution is

$$y(t) = -\sqrt{t^2 + 4}\,.$$

(3) Solve the DE

$$y' = t\sqrt{y}\,,\ t \in \mathbb{R}\,.$$

For separating the variables we need to divide the DE by $\sqrt{y}$, which possibly excludes the constant function $y(t) \equiv 0$ from the family of solutions we get. However, if we substitute the constant 0 function into the DE, we get the identity $0 = 0$, which shows that $y(t) \equiv 0$ is a solution. Later we will see that it is a singular solution.

$$\frac{dy}{\sqrt{y}} = t\,dt$$

$$2\sqrt{y} = \frac{t^2}{2} + c$$

$$y(t) = \left(\frac{t^2}{4} + \frac{c}{2}\right)^2$$

Observing that $\frac{c}{2}$ is just playing the role of an arbitrary constant, to simplify the form of the solutions, we can replace it by c. In conclusion, we have the one-parameter family of solutions

$$y(t) = \left(\frac{t^2}{4} + c\right)^2\,.$$

In this family no particular value of c gives the constant 0 function, hence $y(t) \equiv 0$ is not member of this family, and therefore it is a singular solution.

Solving DEs and IVPs with "Mathematica".

In this section we solve the DE $y'(t) = 2ty(t)$ analytically. The solutions of DEs by numerical methods will be shown in Section 4.4.

Start with the Mathematica input line:

```
DSolve[y'[t] == 2*t*y[t], y[t], t]
```

The answer is given as

y[t] -> e^{t^2} C[1],

which means that the family of solutions is

$$y(t) = ce^{t^2} \,.$$

If we want to solve the IVP

$$y'(t) = 2ty(t)\,, \quad y(1) = 2\,,$$

then we use the input line

```
DSolve[{y'[t] == 2*t*y[t],y[1]==2}, y[t], t] .
```

The answer is

y[t] -> $2e^{-1+t^2}$

which means that the solution is

$$y(t) = 2e^{-1+t^2} = \frac{2}{e}e^{t^2}\,,$$

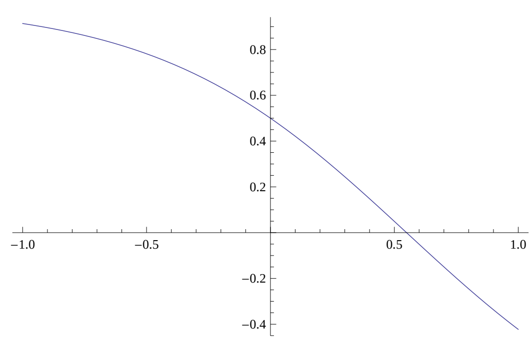
and hence $c = \frac{2}{e}$.
If we want to solve and graph the solution of the IVP

$$y'(t) = y^2(t) - 1\,, \quad y(2) = 1\,,$$

then we use the lines:

```
sol = DSolve[{y'[t] == (y[t])^2 - 1, y[0] == 0.5}, y[t], t]
Plot[Evaluate[y[t] /.  sol], {t, -1, 1}]
```

The graph is:



Homework Exercises.

1. Solve the following DEs and IVPs. For the IVPs, give the largest interval on which the solution is defined and graph the solution curve.

$$
\begin{aligned}
&(1)\quad y' = \frac{y}{t}\,,\ t > 0\,.\\
&(2)\quad y' = ty\,,\quad y(0) = 1\,.\\
&(3)\quad y' = y^2 - 9\,, t \in \mathbb{R}\,.\\
&(4)\quad y' = t\sqrt{4 - y^2}\,,\ t \in \mathbb{R}\\
&(5)\quad y' + 2ty^2 = 0\,,\quad y(1) = \frac{1}{5}\,.\\
&(6)\quad y' = \frac{ty}{t^2 - 1}\,, t > 1\,.\\
&(7)\quad y' = \frac{ty}{t^2 - 1}\,, -1 < t < 1\,.\\
&(8)\quad y' = \frac{ty}{t^2 - 1}\,,\quad y(2) = 0.5\,.\\
&(9)\quad y' = y\tan t\,, -\frac{\pi}{2} < t < \frac{\pi}{2}\,.\\
&(10)\quad y' = \frac{2t}{\ln y}\,,\ y(2) = 1\,.
\end{aligned}
$$

2. Assume that an epidemic spreads in a city with population 100,000 at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. This can be modeled by the logistic DE

$$y'(t) = 10^{-6}\, y(t)(50{,}000 - y(t))\,, t \geq 0\,,$$

where $y(t)$ is the number of people already infected and t is the number of hours. Assuming that at $t = 0$, the number of people already infected was 1,000, estimate the number of the infected people after 10 hours. Graph the solution curve. What is $\lim_{t\to\infty} y(t)$?

3.2. First order linear differential equations

The first order linear differential equations have the general form of

$$a(t)y'(t) + b(t)y(t) = f(t)\,. \tag{3.2.1}$$

If the function f on the right hand side is constantly 0, then we say that the equation is homogeneous. Otherwise, it is non-homogeneous.
The following steps are required to solve a first order linear DE:

Step 1.
Given a non-homogeneous linear DE (3.2.1), first we solve the corresponding homogeneous DE

$$a(t)y'(t) + b(t)y(t) = 0\,. \tag{3.2.2}$$

We solve it as a separable DE.

$$\begin{aligned} a(t)y' &= -b(t)y \\ \frac{dy}{y} &= -\frac{b(t)}{a(t)}\,. \end{aligned} \tag{3.2.3}$$

Let's stop for a moment. The division by y, shows that, as in the previous section, we have to check, by substitution into (3.2.2), that the constant function $y(t) \equiv 0$ is a solution. Indeed, it is, but as we will see later that it is not a singular solution, because it is a member of the family of solutions we get.

Also, the division by $a(t)$, shows that the domain of the solutions has to exclude the numbers t for which $a(t)$ becomes 0.

Using the notation

$$u(t) + c = \int -\frac{b(t)}{a(t)}\,dt\,,$$

the integration of (3.2.3) leads to

$$\ln|y| = u(t) + c\,.$$

By exponentiating both sides we get that

$$e^{\ln|y|} = e^{u(t)+c} = e^{u(t)} \cdot e^c\,,$$

and by replacing the positive constant e^c to a general constant c, we get that

$$y(t) = c\,e^{u(t)}\,.$$

In conclusion, the family of solutions of the homogeneous linear DE (3.2.2) always has the general form

$$y_h(t) = c\,z(t)\,.$$

Note that, for $c = 0$, the constant 0 function is a member of this family of solutions.
Step 2.
We need a so-called particular solution of the non-homogeneous linear DE, which will be found by the variation of parameters method. We search for the particular solution as

$$y_p(t) = c(t)z(t)\,,$$

where $c(t)$ is an unknown function and $z(t)$ is taken from Step 1.
Substitute $y_p(t)$ into the non-homogeneous equation (3.2.1):

$$a(t)\Big(c'(t)z(t) + c(t)z'(t)\Big) + b(t)c(t)z(t) = f(t)\,.$$

Rearrange this equation as

$$a(t)c'(t)z(t) + c(t)\Big[a(t)z'(t) + b(t)z(t)\Big] = f(t)\,,$$

and use the fact that $z(t)$ is a solution of the homogeneous equation, which makes the expression inside the square brackets be 0. Hence,

$$c'(t) = \frac{f(t)}{a(t)z(t)}\,,$$

and therefore $c(t)$ is an antiderivative of $\frac{f(t)}{a(t)z(t)}$. Once $c(t)$ is determined, we get $y_p(t)$.

Step 3.
Finally, the solution of the non-homogeneous linear DE (3.2.1) looks like

$$y(t) = y_h(t) + y_p(t)\,.$$

Note. This method is not valid for non-linear differential equations. In particular, it cannot be used to solve the DE $y' + ty^2 = t$.

Example. Solve the DE

$$y' - 2ty = t\,.$$

Step 1.

$$\begin{aligned}
&y' - 2ty = 0\\
&\frac{dy}{dt} = 2ty\\
&\frac{dy}{y} = 2t\,dt\\
&\ln|y| = t^2 + c\\
&|y| = e^{t^2+c}\\
&y_h(t) = ce^{t^2}
\end{aligned}$$

Step 2.

$$\begin{aligned}
&y_p(t) = c(t)e^{t^2} \\
&y_p'(t) = c'(t)e^{t^2} + c(t)2te^{t^2} \\
&c'(t)e^{t^2} + c(t)2te^{t^2} - 2tc(t)e^{t^2} = t \\
&c'(t)e^{t^2} = t \\
&c'(t) = te^{-t^2} \\
&c(t) = \int te^{-t^2}\, dt = -\frac{1}{2}e^{-t^2} \\
&y_p(t) = -\frac{1}{2}e^{-t^2}e^{t^2} = -\frac{1}{2}
\end{aligned}$$

Step 3.

$$y(t) = ce^{t^2} - \frac{1}{2}\,.$$

Homework Exercises.

1. Solve the following DEs and IVPs. For the IVPs, give the largest interval on which the solution is defined and graph the solution curve.

$$\begin{aligned}
&(1) \quad y' - 4y = 0\,,\ t \in \mathbb{R}\,. \\
&(2) \quad y' - 4y = 0\,,\ y(0) = -1\,. \\
&(3) \quad y' + 2y = e^t\,,\ t \in \mathbb{R}\,. \\
&.(4) \quad y' + 3y = e^{5t}\,,\quad y(0) = 5. \\
&(5) \quad y' + \frac{2}{t+1}y = 3t\,,\ t > -1\,.
\end{aligned}$$

(6) $y' + \tan t \, y = 2 \sin t \, \cos t \,, \quad y(0) = 1.$

(7) $y' + 3t^2 y = t^2 \,, \; t \in \mathbb{R} \,.$

(8) $t^2 y' + ty = 1 \,, \; t < 0 \,.$

(9) $\cos t \, y' + \sin t \, y = 1 \,, \; 0 < t < \frac{\pi}{2} \,.$

(10) $\cos t \, y' + \sin t \, y = 1 \,, \; y(\frac{\pi}{4}) = 1 \,.$

(11) $y' + 2ty = te^{-t^2} \,, \; t \in \mathbb{R} \,.$

(12) $(1 - t^2)y' - 2ty = e^{-t} \,, \; t > 1 \,.$

(13) $(1 - t^2)y' - 2ty = e^{-t} \,, \; -1 < t < 1 \,.$

(14) $y' + \tan t \, y = \cos t \,, \; y(0) = 0 \,.$

(15) $(1 + t^2)y' + 4ty = \frac{2}{1 + t^2} \,, \; y(0) = 1 \,.$

2. The plutonium 239 disintegrates according to the DE:

$$A'(t) = k \, A(t) \,,$$

where $k = -0.0000286728$, and $A(t)$ is the amount of plutonium 239 present after t number of years. If at the present time we have an amount of 10kg, then estimate the amount left after 100 years.

3. The C-14 carbon isotope - which is used in carbon dating of fossils - disintegrates according to

$$A'(t) = k \, A(t) \,,$$

where $k = -0.00012378$, and $A(t)$ is the amount present after t number of years. If we measure that 50% of the $C - 14$ is left, how old is the fossil?

4. A population of bacteria in a culture grows according to the differential equation

$$N'(t) = k \, N(t) \,,$$

where $k = 0.5$, and $N(t)$ is the number of bacteria present after t hours. If at present time we approximately 5000 bacteria, estimate their number after 10 hours.

5. Consider the problem of a free falling object with mass M. Assume that only gravity and air resistance act upon the object. Let us suppose that the air resistance is proportional to the velocity $v(t)$ of the object. Newton's second law of motion gives the DE

$$Mv'(t) = Mg - kv(t) \,, \; t \geq 0 \,.$$

More exactly, this is a first order linear DE with constant coefficients:

$$Mv'(t) + kv(t) = Mg \,, \; t \geq 0 \,.$$

Suppose that 2 objects with mass $M_1 = 10$ kg and $M_2 = 20$kg are released from an altitude of 3000 meters with initial vertical velocity 0. Suppose that the constant $k = 0.5$ for both

objects. Answer the following questions:
(a) Calculate the velocities $v_1(t)$ and $v_2(t)$ of the two objects.
(b) What is their terminal (highest) velocity?
(c) Which object is falling faster?
(d) What is their speed after 5 seconds?

6. Visit:
http://demonstrations.wolfram.com/LinearFirstOrderDifferentialEquation/

3.3. Bernoulli's differential equations

Bernoulli's differential equations have the form

$$y' + a(t)y = b(t)y^k\,,$$

where $k \neq 0$ and $k \neq 1$. This is a non-linear equation, which will be changed to a linear one.

Changing the non-linear DE into a linear DE.
Divide the equation by y^k and get

$$y^{-k}\,y' + a(t)y^{1-k} = b(t)\,.$$

Introduce a new function

$$z(t) = y^{1-k}(t)\,,$$

for which

$$z'(t) = (1-k)\cdot y^{-k}(t)\cdot y'(t)\,.$$

Therefore, the non-linear Bernoulli's DE is changed to

$$\frac{1}{1-k}z' + a(t)z = b(t)\,,$$

which is a first order linear DE in the unknown function $z(t)$.

Solve the first order linear DE in $z(t)$.
This is done according to the Steps 1, 2 and 3 from the previous section.

Return to y(t). Write

$$y(t) = z(t)^{\frac{1}{1-k}}\,,$$

which is the solution of the Bernoulli's DE.

Example. Solve the DE

$$y' + \frac{1}{t}y = t^2y^2\,,\quad t > 0\,.$$

Solution:
Changing the non-linear DE into a linear DE.
Divide the DE by y^2:

$$y^{-2}y' + \frac{1}{t}y^{-1} = t^2\,.$$

Introduce

$$z(t) = (y(t))^{-1} = \frac{1}{y(t)}\,.$$

Then, $z' = (-1)y^{-2}\,y'$ and the linear DE in z looks like

$$-z' + \frac{1}{t}z = t^2\,.$$

Solve the first order linear DE in $z(t)$.
Step 1.

$$-z' + \frac{1}{t}z = 0$$

$$\frac{dz}{z} = \frac{dt}{t}$$
$$\ln|z| = \ln|t| + c$$
$$z_h(t) = c\,t\,.$$

Step 2. Search the particular solution in the form $z_p(t) = c(t) \cdot t$.
By substituting $z_p(t)$ into the DE of $z(t)$ gives $c'(t) = -t$, which gives $c(t) = -\frac{t^2}{2}$ and hence

$$z_p(t) = -\frac{t^3}{2}\,.$$

Step 3.

$$z(t) = ct - \frac{t^3}{2}\,.$$

Return to $y(t)$.

$$y(t) = \frac{1}{ct - \frac{t^3}{2}}\,.$$

Homework Exercises. Solve the following DEs and IVPs. For the IVPs, give the largest interval on which the solution is defined and graph the solution curve.

$$(1)\quad ty' - y = \frac{-t^3}{y^2}\,,\quad t > 0\,.$$

$$(2)\quad ty' - y = \frac{-t^3}{y^2}\,,\quad y(1) = 2\,.$$

$$(3)\quad y' + y = \frac{1}{\sqrt{y}}\,,\quad y(0) = 4\,.$$

$$(4)\quad y' + y = \frac{1}{\sqrt{y}}\,,\quad y(0) = -4\,.$$

$$(5)\quad ty' + y = t^2y^2\,,\ t < 0\,.$$

$$(6)\quad t^2y' - 2ty = 3y^4\,,\quad y(1) = \frac{1}{2}\,.$$

$$(7)\quad ty' - (1+t)y = ty^2\,,\ t > 0.$$

$$(8)\quad 3y^2y' + 2y^3 = e^t\,,\ -1 < t < 1.$$

$$(9)\quad -2t^2y' + ty = 5y^3\,,\ t < 0.$$

$$(10)\quad \frac{-2t^2y'}{y^3} + \frac{t}{y^2} = 5\,,\ t > 0.$$

$$(11)\quad -2t^2y' + ty = 5y^3\,,\ y(-1) = 0\,.$$

$$(12)\quad y' - ty = t\sqrt{y^3}\,,\ y(1) = 4\,.$$

3.4. Non-linear homogeneous differential equations

The non-linear part of the title has the meaning to distinguish between the earlier studied linear homogeneous DEs and the ones in this section. Note, that, while most of the DEs in this section are non-linear, there are linear DEs which are homogeneous in this non-linear sense.

The non-linear homogeneous differential equations have the form

$$y' = f\left(\frac{y}{t}\right) .$$

We can solve them by introducing a new function

$$z(t) = \frac{y(t)}{t} .$$

Hence,

$$y(t) = tz(t)$$

and

$$y' = z + tz' .$$

The new DE in z is

$$z + tz' = f(z) ,$$

which is always a DE with separable variable. After solving this DE in z, we can get $y(t)$ from the equation $y(t) = t\,z(t)$.

Example. Solve the DE

$$t^2y' - y^2 - yt = 0 , \quad t > 0 .$$

Solution:
Dividing the equation by t^2 gives:

$$y' = \left(\frac{y}{t}\right)^2 + \frac{y}{t} .$$

Then,

$$\begin{aligned} z &= \frac{y}{t} \\ y &= tz \\ y' &= z + tz' \\ z + tz' &= z^2 + z \\ t\frac{dz}{dt} &= z^2 \\ \frac{dz}{z^2} &= \frac{dt}{t} , \ z \neq 0 \end{aligned}$$

Note: $z(t) = 0$ is excluded from the solutions, so we have to check, by substitution, whether it is a solution or not. It turns out that it is a solution.

$$\frac{-1}{z} = \ln t + c$$

$$z = \frac{-1}{\ln t + c}$$

Not that $z(t) = 0$ is not part of this family, so it is a singular solution.
Therefore, the solutions of this problem can be organized in a one-parameter family of solutions

$$y = \frac{-t}{\ln t + c},$$

and a singular solution

$$y(t) \equiv 0 .$$

Homework Exercises. Solve the following DEs and IVPs. For the IVPs, give the largest interval on which the solution is defined and graph the solution curve.

(1) $ty' - y + t = 0, \; 0 < t < 2 .$

(2) $ty' - y + t = 0, \; y(1) = 2 .$

(3) $ty' - y + t = 0, \; y(0) = 2 .$

(4) $(y - 2t)y' + t = 0, \; -1 < t < 1 .$

(5) $t^2y' + y^2 + yt = 0, \; t < 0.$

(6) $y' = \dfrac{t + 3y}{3t + y}, \; t > 0$

(7) $ty' = y + \sqrt{t^2 - y^2}, \quad t > 0$

(8) $ty^2y' = y^3 - t^3, \quad y(1) = 3 .$

(9) $(t^2 + 2y^2)y' = ty, \quad y(-1) = 1 .$

(10) $ty^3\, y' = y^4 + t^4, \; t > 0 .$

(11) $y' = \dfrac{t^3 + y^3}{ty^2}, \; y(1) = 3 .$

3.5. Differential equations of the form $y'(t) = f(at + by(t) + c)$.

In these equations a, b, c are constant and we introduce the function

$$z(t) = at + by(t) + c\,.$$

Then

$$z' = a + by'\,,$$

and, in z, we get a DE with separable variables:

$$z' = bf(z) + a\,.$$

We solve this equation and get $z(t)$, from which we obtain $y(t)$.

Example.

Solve the DE

$$y' = (4t + y + 3)^2\,.$$

Solution:

$$z = 4t + y + 3$$

$$z' = 4 + y'$$

$$y' = z' - 4$$

$$z' - 4 = z^2$$

$$\frac{dz}{dt} = 4 + z^2$$

$$\frac{dz}{z^2 + 4} = dt$$

$$\frac{1}{2}\arctan\frac{z}{2} = t + c$$

$$\arctan\frac{z}{2} = 2t + c$$

$$\frac{z}{2} = \tan(2t + c)$$

$$z = 2\tan(2t + c)$$

$$4t + y + 3 = 2\tan(2t + c)$$

$$y = 2\tan(2t + c) - 4t - 3$$

Homework Exercises.

Solve the following DEs and IVPs. For the IVPs, give the largest interval on which the solution is defined and graph the solution curve.

(1) $y' = \cos(t+y)\,, \quad -\pi < t < \pi.$

(2) $y' = \cos(t+y)\,, \quad y(0) = \dfrac{\pi}{4}$

(3) $y' = 1 + e^{y-t+5}\,, \ t > 0\,.$

(4) $y' = \dfrac{1-t-y}{t+y}\,, \ y(0) = -1.$

(5) $y' = \dfrac{1-t-y}{t+y}\,, \ y(1) = -1.$

(6) $y' = \dfrac{3t+2y}{3t+2y+2}\,, \quad y(-1) = -1$

(7) $y' = \dfrac{3t+2y}{3t+2y+2}\,, \quad y(0) = -1$

3.6. Second order differential equations reducible to first order differential equations

We will solve second order differential equations which contain just y'' and y', and no y. These equations have the general form form $f(t, y', y'') = 0$.
If we introduce the function $z = y'$, then we get a first order DE in z: $f(t, z, z') = 0$. Once we get z, the solution y is found by integration.

Example.
Solve the IVP:

$$y'' + 3y' = e^{2t}\,, \;\; y(0) = 1,\; y'(0) = 0\,.$$

Solution:
Introducing the function $z = y'$ we get the linear DE in z:

$$z' + 3z = e^{2t}\,.$$

Solving this equation in z gives:

$$z(t) = ce^{-3t} + \frac{1}{5}e^{2t}\,.$$

Integrating z leads to

$$y(t) = \frac{-c}{3}e^{-3t} + \frac{1}{10}e^{2t} + d\,.$$

The initial conditions give the system

$$\begin{cases} \frac{-c}{3} + \frac{1}{10} + d = 1 \\ c + \frac{1}{5} = 0\,. \end{cases}$$

Solving this system in c and d gives $c = -\frac{1}{5}$ and $d = \frac{5}{6}$.
Therefore, the solution of the IVP is

$$y(t) = \frac{1}{15}e^{-3t} + \frac{1}{10}e^{2t} + \frac{5}{6}\,.$$

Homework Exercises. Solve the following DEs and IVPs.

(1) $ty'' + 3y' = 0\,, \quad t > 0\,.$

(2) $ty'' + 3y' = 0\,, \quad y(1) = 1\,,\ y'(1) = 2\,.$

(3) $y'' = (y')^2\,, \quad y(0) = 1\,,\ y'(0) = -\dfrac{1}{e}\,.$

(4) $t^4y'' + t^3y' = 4\,, t > 0\,.$

(5) $t^4y'' + t^3y' = 4\,, t < 0\,.$

(6) $y'' + 3y' = e^{2t}\,,\ y(0) = 4\,,\ y'(0) = 0\,.$

(7) $2y'y'' = 1 + (y')^2\,.$

(8) $y'' = \dfrac{3t^2y'}{1+t^3}\,.$

CHAPTER 4

General theory of differential equations of first order

4.1. Slope fields (or direction fields)

Consider a first order DE in normal form

$$y'(t) = f(t, y(t))\,,\ t \in I\,.$$

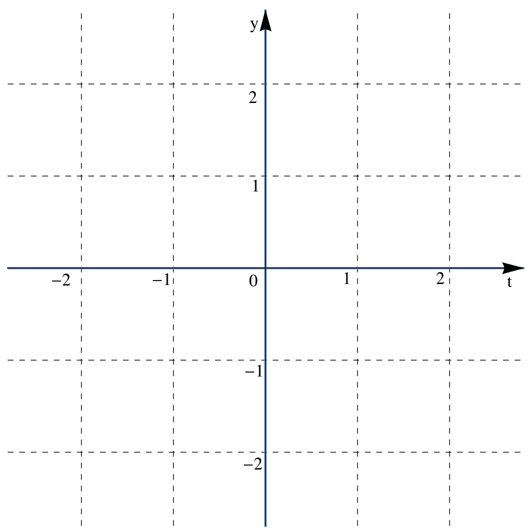
If $y : I \to \mathbb{R}$ is a solution to this DE, then at any point $t_0 \in I$, the value of $f(t_0, y(t_0))$ is the slope to the graph of the function y, which is a solution curve to the DE.
Therefore, if we show a rectangular grid in the ty-coordinate system and evaluate $f(t, y)$ at the points in the grid, then we have graphical information about where solution curves are heading, without actually solving the DE.

Definition 4.1.1. *A slope field of a DE is a rectangular grid with slopes, as arrows pointing left, drawn at each point of the grid.*

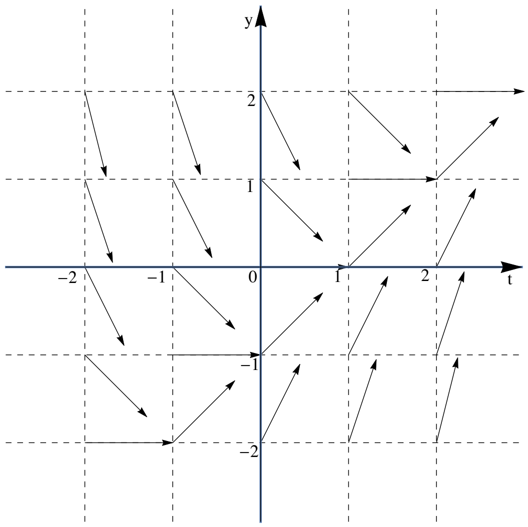
Example. This example shows how to draw a slope field manually. Consider the DE

$$y' = t - y\,.$$

Draw first a grid in the ty-coordinate system for $t = -2, -1, 0, 1, 2$ and $y = -2, -1, 0, 1, 2$

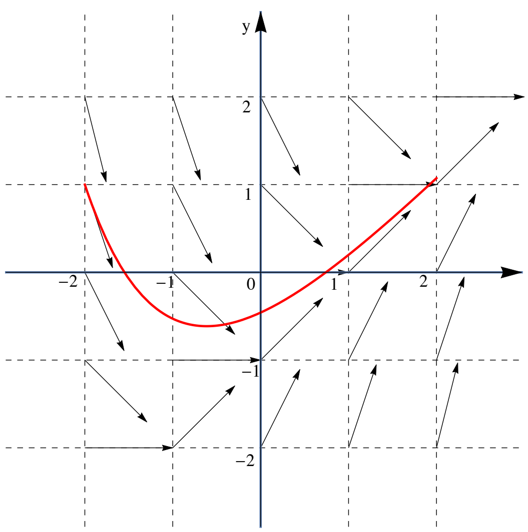


The right hand side to the DE gives the function $f(t, y) = t - y$. Evaluate this function at each point of the grid and show the results as slopes at the corresponding points. For example, $f(2, 1) = 1$ gives a slope 1 at the point $(2, 1)$. Continuing in this way we get the following slope field.



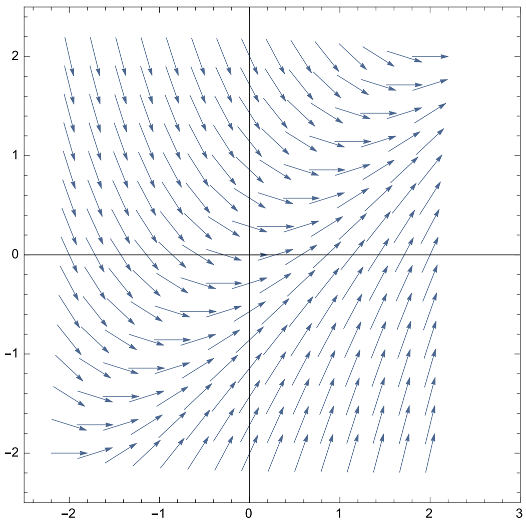
Based on the slope field we can get graphical information about solution curves. If we choose an initial point, then we can draw an approximative solution curve on the graph by following the slopes in the slope field. The following graph shows the slope field and solution curve for the IVP

$$\begin{cases} y' = t - y \\ y(-1) = 0.5 \; . \end{cases}$$



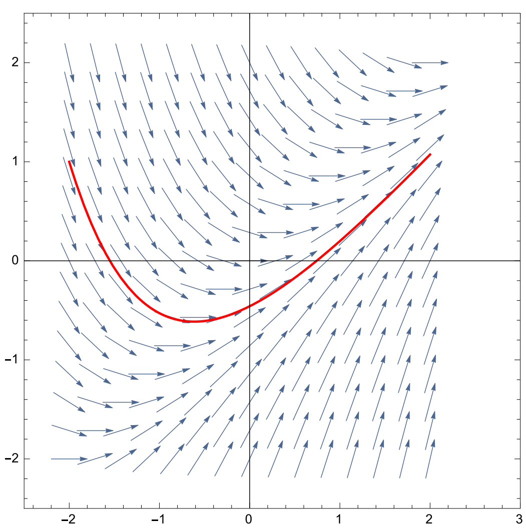
Of course, if the slope field is filled with more slopes, our information about solution curves is more complete.
Mathematica can graph a slope field in the following way. The role of the cosine arctangent and the sine arctangent is to restrict the length of each vector to one.

```
VectorPlot[{Cos[ArcTan[t - y]], Sin[ArcTan[t - y]]}, {t, -2, 2},
{y, -2, 2}, PlotRange->{{-2.5, 3}, {-2.5, 2.5}}, Axes -> True,
VectorStyle -> Arrowheads[0.02]]
```



We can add to the slope field the solution curve starting at $(-2, 1)$, which shows how solution curves follow the slopes.

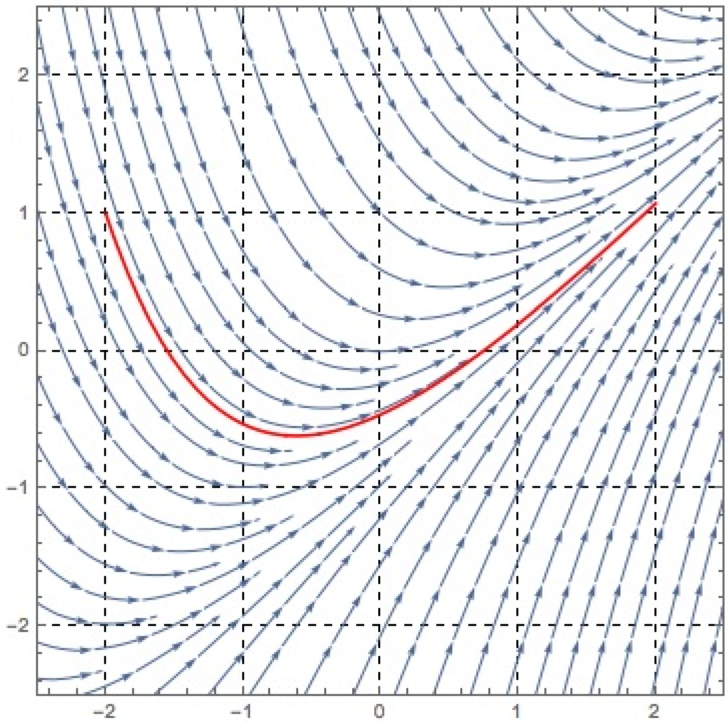
```
Show[VectorPlot[{Cos[ArcTan[t - y]], Sin[ArcTan[t - y]]}, {t, -2, 2},
{y, -2, 2}, PlotRange -> {{-2.5, 3}, {-2.5, 2.5}}, Axes->True,
VectorStyle -> Arrowheads[0.015]], Plot[4*Exp[-t - 2] + t - 1, {t, -2, 2},
PlotStyle -> Red]]
```



Also, there is the option of using `StreamPlot`.

```
In[50]:= Show[StreamPlot[{1, t - y}, {t, -2.5, 2.5}, {y, -2.5, 2.5}, GridLines → Automatic,
           GridLinesStyle → Directive[Dashed], PlotRange → {{-2.5, 2.5}, {-2.5, 2.5}}],
          Plot[4 * Exp[-t - 2] + t - 1, {t, -2, 2}, PlotStyle → Red]]
```

Out[50]=



More slope fields can be found at
http://demonstrations.wolfram.com/SlopeFields/.

4.1.1. Autonomous first order differential equations.

First order DEs in the form

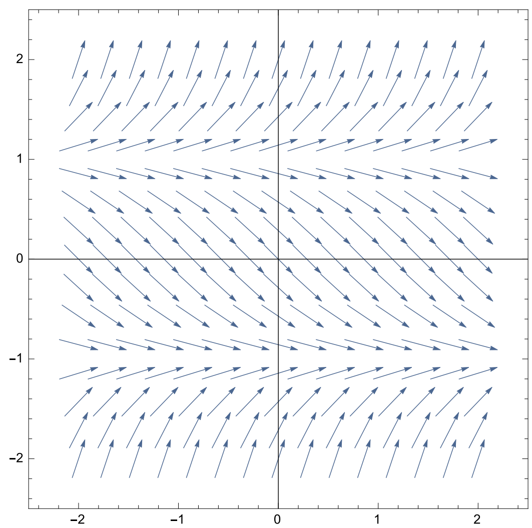
$$y'(t) = f(y(t)),$$

or shortly

$$y' = f(y),$$

are called autonomous first order DEs. Their slope fields show equal slopes along horizontal grid lines. For example, lets have a look at the slope field of

$$y' = y^2 - 1.$$



Definition 4.1.2. *A* **phase portrait** *for a first order DE is a slope field with several solution curves, showing the most important qualitative properties of solutions.*

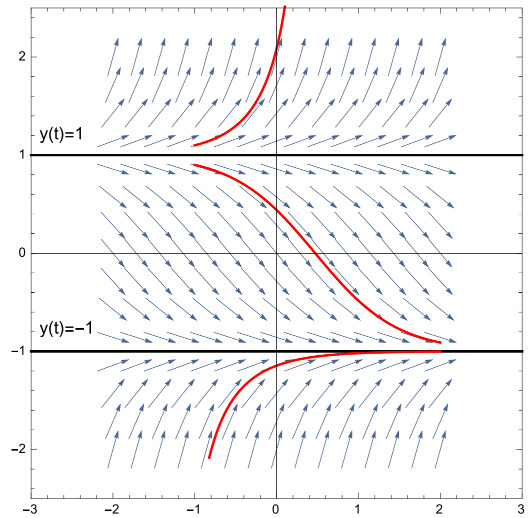
Definition 4.1.3. **Critical numbers (or points)** *for an autonomous first order DE are numbers c such that $f(c) = 0$.*

Definition 4.1.4. **Equilibrium solutions** *are the constant functions $y(t) = c$, corresponding to the critical numbers c.*

Example. Consider the DE

$$y' = y^2 - 1.$$

In this case $f(y) = y^2 - 1$ and the critical numbers correspond to the solutions of $y^2 - 1 = 0$, which are ± 1. Hence the critical numbers are $c = -1$ and $c = 1$, while the equilibrium solutions are $y(t) = -1$ and $y(t) = 1$. The phase portrait in this case looks like:



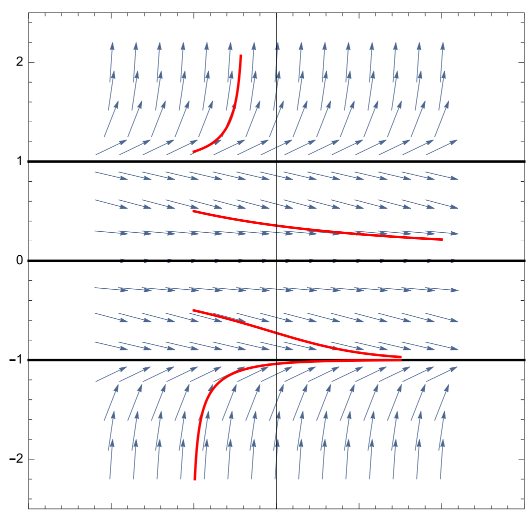
Classifications of equilibrium solutions:

(a) We call an equilibrium solution $y(t) = c$ **attractor** (or asymptotically stable) if for any other solution $z(t)$ which starts from a position sufficiently close to c, we have $\lim_{t\to\infty} z(t) = c$.

(b) We call an equilibrium solution $y(t) = c$ **repeller** (or unstable) if any other solution $z(t)$ starting any close to c moves away from it as $t \to \infty$.

(c) We call an equilibrium solution $y(t) = c$ **semi-stable** if it is an attractor from one side and repeller from the other side.

Example. Let us look at the phase portrait of $y' = y^2(y^2 - 1)$.



The $y(t) = 1$ is a repeller, $y(t) = 0$ is semi-stable and $y(t) = -1$ is an attractor.

Homework Exercises.

1. Sketch slope fields and approximate solution curves for the given DEs and initial conditions:

(a)
$$y' = t + y\,,\quad y(-1) = 2\,,\ y(0) = -1\,.$$
(b)
$$y' = t - y\,,\quad y(-1) = 2\,,\ y(0) = -1\,.$$
(c)
$$y' = \frac{t}{y}\,,\quad y(1) = 1\,,\ y(0) = -1\,.$$
(d)
$$y' = |t| - |y|\,,\quad y(-1) = 0\,,\ y(0) = 1\ .$$
(e)
$$y' = y^2 - t, y(0) = 0\,,\ y(0) = 0.6\,,\ y(0) = 0.8\,.$$
(f)
$$y' = t(y+1)\,,\quad y(0) = 0\,,\ y(1) = -1\,.$$
(g)
$$y' = y\sin t\,,\quad y(0) = 0\,,\ y(\pi) = 1\,.$$
(h)
$$y' = \frac{t}{t^2+1}\,,\quad y(0) = 0\,,\ y(0) = 1\,.$$
(i)
$$y' = \frac{1}{t^2+y^2}\,,\quad y(1) = 0\,,\ y(-1) = 0\,.$$
(j)
$$y' = \frac{1}{|t|+|y|}\,,\quad y(1) = 0\,,\ y(-1) = 0\,.$$
(k)
$$y' = \frac{1}{t+y}\,,\quad y(1) = 0\,,\ y(-1) = 0\,.$$

2. For the following autonomous DEs sketch a phase-portrait, find the critical numbers, equilibrium solutions and classify them:

(a)
$$y' = y^2 - y^4\,.$$
(b)
$$y' = (y-1)^2\,.$$
(c)
$$y' = y^4 - y\,.$$
(d)
$$y' = \sin y\,.$$

(e)

$$y' = ye^{-y}\,.$$

(f)

$$y' = y(4-y^2)\,.$$

(g)

$$y' = y^3 - 8y^2 + 12y\,.$$

(h)

$$y' = y^3 - 3y^2 - 2y + 4\,.$$

(i)

$$y' = y^4 - 8y^2 + 16\,.$$

(j)

$$y' = y^4 - 8y^3 + 16y^2\,.$$

(k)

$$y' = y^2 + 5y + 6\,.$$

(l)

$$y' = y^2 + 1\,.$$

(m)

$$y' = \frac{y^2-9}{y}\,.$$

3. Suppose that the following DE models the pressure within a container:

$$y' = y^4 - 7y^3 + 6y^2\,.$$

(a) Find the critical points, equilibrium solutions and classify them. Draw the phase portrait.
(b) If the pressure at $t = 0$ is 4, will it increase or decrease in the future? How much can it change on long term?

4. Look at the DE in Exercise 2. in Section 3.1.
(a) Find the critical points, equilibrium solutions and classify them. Draw the phase portrait.
(b) If at $t = 0$ the number of infected people is 15,000, will this number increase or decrease in the future? How much can it change on long term?

4.2. Existence and uniqueness of solutions for initial value problems

In this section we study the existence and uniqueness of solutions for IVPs. We will do this by analyzing the right hand side of first order DEs in normal form, but without solving the DEs in the way we did in Chapter 3. This is an important issue, because many DEs cannot be analytically solved and before we use the numerical methods from Section 4.4, we must be sure that the numerical methods lead to a valid approximate solution. We will see that disregarding this issue might lead to incorrect or incomplete answers.

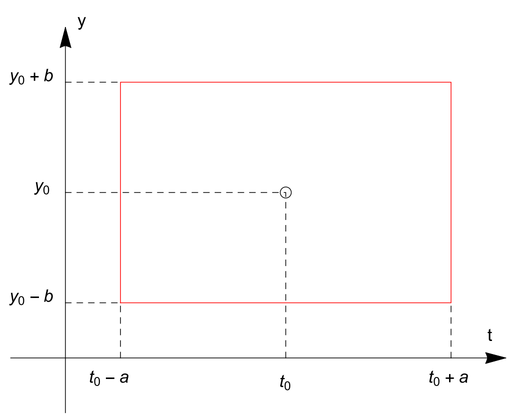
Consider the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \,, \end{cases} \tag{4.2.4}$$

where

$$(t,y) \in [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b] = R_{a,b} \,.$$

By this we assume that the function $f(t,y)$, as a function of two variables t and y, is defined on the rectangle $R_{a,b}$.



The question we can ask is under what conditions does the IVP have a solution curve through the point (t_0, y_0). The following two theorems give existence and uniqueness answers, based on the properties of $f(t,y)$ inside the rectangle $R_{a,b}$. We use the following numbers:

$$M = \text{Maximum of } \ |f(t,y)| \ \text{ when (t,y) belongs to } R_{a,b} \,,$$

and

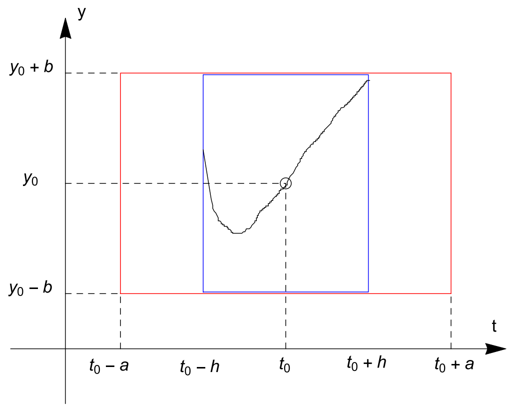
$$h = \min\{a, \frac{b}{M}\} \,.$$

We will apply the following two theorems.

Theorem 4.2.1 (Picard-Lindelöf Existence and Uniqueness Theorem). *If the function $f(t,y)$ and its partial derivative with respect to y, $\frac{\partial f}{\partial y}(t,y)$, are continuous on the rectangle $R_{a,b}$, then there exists a unique solution $y : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b]$ of the IVP (4.2.4).*

Theorem 4.2.2 (Peano's Existence Theorem). *If the function $f(t,y)$ is continuous in both the t and y variables on $R_{a,b}$, then the IVP (4.2.4) has at least one solution $y : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b]$.*

The following graph shows that, while we check the properties of $f(t,y)$ inside the red rectangle, we can assure the existence of a solution curve inside a smaller blue rectangle.



Our main goal is to apply, if possible, the Picard-Lindelöf Existence and Uniqueness Theorem.

Example 1. Consider the IVP:

$$\begin{cases} y' = -\frac{y}{t} \\ y(1) = 2\,. \end{cases}$$

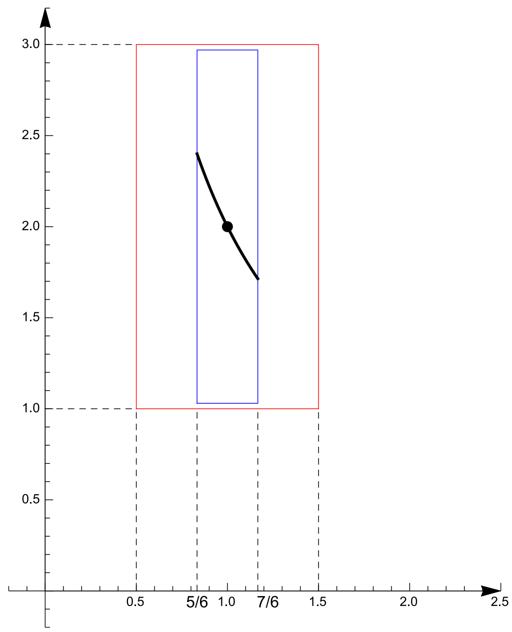
For this probem we have $t_0 = 1$, $y_0 = 2$ and

$$f(t,y) = -\frac{y}{t}\,, \quad \frac{\partial f}{\partial y}(t,y) = -\frac{1}{t}\,.$$

Both functions are continuous everywhere, except at the points for which $t = 0$. We will choose the number $a > 0$ in such a way to avoid $t = 0$. Any number $0 < a < 1$ is good for this purpose. With the choice of $a = 0.5$, the variable t belongs to the interval $[0.5, 1.5]$, hence it cannot be 0. For the choice of $b > 0$ we don't have any restrictions, so for simplicity let us use $b = 1$. This means that the variable y belongs to the interval $[1, 3]$. The rectangle $R_{a,b}$ has the form

$$R_{0.5,1} = [0.5, 1.5] \times [1, 3]\,.$$

To calculate M, we would need to use optimization methods for functions with two variables, which is part of Calculus 3 and it is not a prerequisite for this class. Hence, we will use simple logical arguments, like a fraction is the largest, when the numerator is the largest and the denominator is the smallest. Then $M = \frac{3}{0.5} = 6$ and $h = \min\{0.5, \frac{1}{6}\} = \frac{1}{6}$. Therefore, the Picard-Lindelöf Existence and Uniqueness Theorem guarantees the existence of a unique solution $y : \left[\frac{5}{6}, \frac{7}{6}\right] \to [1, 3]$.



Observation. As you can notice from the graph, the solution curve probably continues outside of the interval $[5/6, 7/6]$ and stays within the red rectangle. This means that the number h provided by the Picard-Lindelöf Existence and Uniqueness Theorem is not optimal. Let's have glimpse on how a more detailed analysis can extend the solution curve outside of the interval $[5/6, 7/6]$, but still within the rectangle $[0.5, 1.5] \times [1, 3]$. The IVP of this example can be rewritten as

$$y(t) = 2 + \int_1^t -\frac{y(s)}{s}\, ds\,.$$

If we want to check where does the solution curve exit the red rectangle, we have to evaluate $|y(t) - 2|$ and see when it reaches 1.

$$|y(t) - 2| = \left| \int_1^t -\frac{y(s)}{s}\, ds \right| \leq \int_1^t \frac{y(s)}{s}\, ds\,.$$

At this stage, in the proof of Picard-Lindelöf Existence and Uniqueness Theorem we calculate the maximum of $|f(t, y)| = \frac{y}{t}$ over the red rectangle, which means that we assume $0.5 \leq s \leq 1.5$ and $1 \leq y(s) \leq 3$, which gives

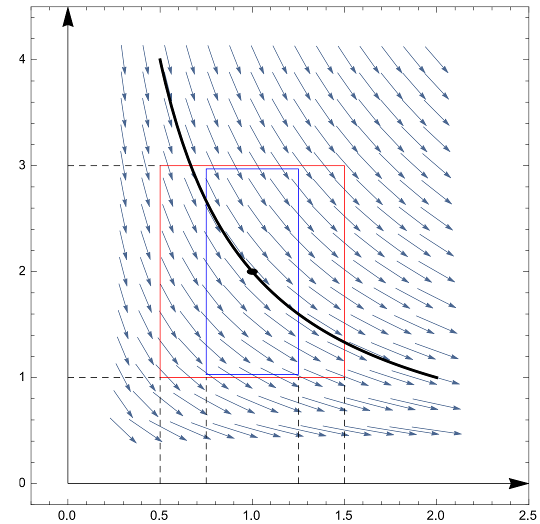
$$|y(t) - 2| \leq \frac{3}{0.5} \cdot h \leq 1\,,$$

and this leads to $h = \frac{1}{6}$. However, it is enough to consider the maximum of $\frac{y}{t}$ over a smaller rectangle $[1 - h, 1 + h] \times [1, 3]$, defined by a variable h, and this gives

$$|y(t) - 2| \leq \frac{3}{1 - h} \cdot h \leq 1\,,$$

which leads to $h = \frac{1}{4}$, which is a better result.

It is a good exercise to try finding the optimal h, without solving the DE. As an indication of what is happening, we can show a combination of the slope field from Section 4.1 and the graph from this section.



Example 2. Consider the IVP

$$\begin{cases} y' = 2t\sqrt[3]{y^2} \\ y(0) = 0\,. \end{cases}$$

For this problem $f(t,y) = 2t\sqrt[3]{y^2}$ and $\frac{\partial f}{\partial y}(t,y) = \frac{4t}{3\sqrt[3]{y}}$. Notice $t_0 = 0$ and $y_0 = 0$. The function $f(t,y)$ is continuous everywhere, however $\frac{\partial f}{\partial y}(t,y)$ is not continuous at the points where $y = 0$. In this problem $y_0 = 0$, and no matter how we choose $b > 0$, the interval $[-b, b]$ contains the 0. Therefore, we will be able to apply just Peano's Existence Theorem. Consider $a = 1$ and $b = 1$. With these choices, $-1 \leq t \leq 1$, $-1 \leq y \leq 1$, $R_{1,1} = [-1,1] \times [-1,1]$, $M = 2$ and $h = \min\{1, \frac{1}{2}\} = 0.5$.
Therefore, by Peano's Existence Theorem we have at least one solution $y : [-0.5, 0.5] \to [-1, 1]$.

Note. Let's see how good our answer is. In many cases the DE cannot be analytically solved, but in this problem it is with separable variables, so we can solve it. Indeed, we get two solutions, $y(t) = 0$ and $y(t) = \frac{1}{27}t^6$, both going through the initial point $(0,0)$.

Example 3. Consider the IVP

$$\begin{cases} y' = \frac{y^2}{t} \\ y(0) = 1\,. \end{cases}$$

In this problem $t_0 = 0$, $y_0 = 1$ and $f(t,y) = \frac{y^2}{t}$. The function $f(t,y)$ is not continuous (not even defined) where $t = 0$, and no matter how we choose $a > 0$, the interval $[-a, a]$ contains the 0. As the continuity of $f(t,y)$ is required for both theorems, we cannot apply any of them. The only answer available at this moment is that we don't have any conclusion about the existence and uniqueness of solutions to this (IVP).

Note. If we solve the DE from this exercise, we get the singular solution $y(t) \equiv 0$, which doesn't satisfy $y(0) = 1$, and the family of solutions $y(t) = \frac{-1}{\ln t + c}$. No member of this family is defined at $t = 0$.

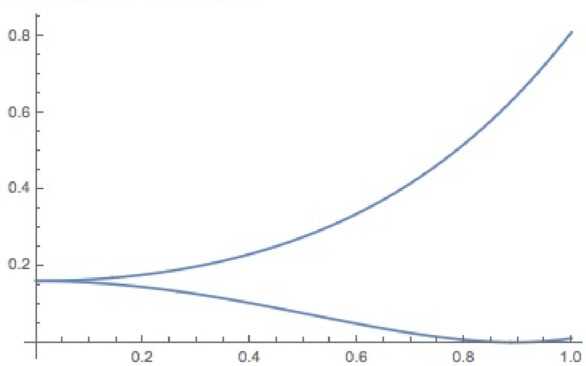
Example 4. This examples shows that we cannot blindly trust the answers given by computers. Consider the IVP

$$\begin{cases} y' = 2t\sqrt{y} \\ y(0) = 0.16\,. \end{cases}$$

The initial point is $(t_0, y_0) = (0, 0.16)$ and the function $f(t, y) = 2t\sqrt{y}$ and its partial derivative $\frac{\partial f}{\partial y}(t, y) = \frac{t}{\sqrt{y}}$ are continuous on the rectangle $R_{1,0.08} = [-1, 1] \times [0.08, 0.24]$, so by the Picard-Lindelöf Existence and Uniqueness Theorem we should have we have a unique solution going through the initial point $(0, 0.16)$. However, Mathematica gives two solutions.

```
In[30]:= sol1 = DSolve[{y'[t] == 2*t*Sqrt[y[t]], y[0] == 0.16}, y[t], t]
Out[30]= {{y[t] → 0.25 (0.64 - 1.6 t^2 + t^4)}, {y[t] → 0.25 (0.64 + 1.6 t^2 + t^4)}}

In[31]:= g[t_] := Evaluate[y[t] /. sol1]

In[34]:= Plot[g[t], {t, 0, 1}]
```

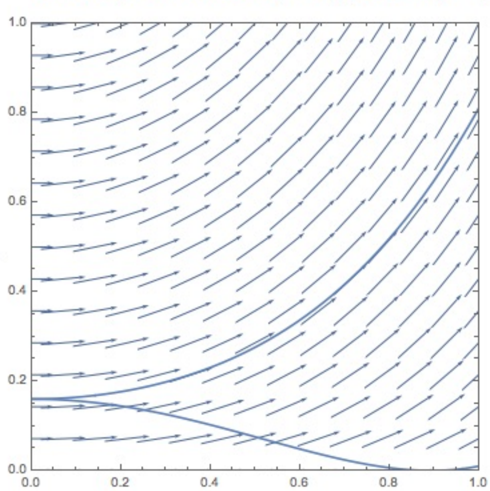
Out[34]=



If we add the slope field to the graph, we see that the decreasing curve $y = 0.25(0.64 - 1.6t^2 + t^4) = 0.25(t^2 - 0.8)^2$ doesn't fit and it is the result of a software mistake.

```
In[37]:= Show[VectorPlot[{Cos[ArcTan[2*t*Sqrt[y]]], Sin[ArcTan[2*t*Sqrt[y]]]}, {t, 0, 1}, {y, 0, 1},
           VectorStyle → Arrowheads[0.015], PlotRange → {{0, 1}, {0, 1}}], Plot[g[t], {t, 0, 1}]]
```

Out[37]=



Homework Exercises.

(1) Check the existence and uniqueness of solutions for the following IVPs. Sketch the rectangle $R_{a,b}$. In case of existence or existence and uniqueness of the solution draw an approximate solution curve through the initial point.

$$
\begin{aligned}
&(a) \quad (t-1)y' = y^2 + t\,, \quad y(0) = 1\,.\\
&(b) \quad (t-1)y' = y^2 + t\,, \quad y(1) = 0\,.\\
&(c) \quad y' = \sqrt{y^2 - 4}\,, \quad y(1) = 2\,.\\
&(d) \quad y' = \sqrt{y^2 - 4}\,, \quad y(1) = 3\,.\\
&(e) \quad y' = \sqrt[3]{t}y + y^2\,, \quad y(0) = 3\,.\\
&(f) \quad (t^2 + y^2)y' = y + 1\,, \quad y(1) = 1\,.\\
&(g) \quad y' = ty^2 + 3\,, \quad y(0) = 2\,.\\
&(h) \quad y' = \frac{ty}{t^2 - 1}\,, \quad y(0) = 1\,.\\
&(i) \quad t^2y' + ty = 1\,, \quad y(3) = 1\,.\\
&(j) \quad t^2y' + ty = 1\,, \quad y(0) = 1\,.\\
&(k) \quad ty^2y' = y^3 - t^3\,, \quad y(1) = 1\,.
\end{aligned}
$$

(2) Return to exercise 2 from Section 2.2. Do we have a unique solution for the IVPs? Why?

4.3. The method of successive approximations

This is a theoretical method, which is used to prove the existence and uniqueness theorem. Although, practically not as useful as the numerical methods from the next section, it offers great insight to the theory of initial value problems.
Consider the IVP

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \,, \end{cases} \tag{4.3.5}$$

and assume that we can use the PIcard-Lindelöf Existence and Uniqueness theorem to assure that we have a unique solution $y : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b]$. Integrate both sides of the DE from t to t_0:

$$\int_{t_0}^{t} y'(s)ds = \int_{t_0}^{t} f(s, y(s))\, ds \,.$$

By the Fundamental Theorem of Calculus we get that

$$y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))\, ds \,,$$

and hence any solution of the IVP (4.3.5) satisfies the equation

$$y(t) = \int_{t_0}^{t} f(s, y(s))\, ds + y_0 \,. \tag{4.3.6}$$

We will use an iteration, called the succesive approximation of the solution, for (4.3.6):

$$y_1(t) = \int_{t_0}^{t} f(s, y_0)\, ds + y_0$$

$$y_2(t) = \int_{t_0}^{t} f(s, y_1(s))\, ds + y_0$$

....

$$y_n(t) = \int_{t_0}^{t} f(s, y_{n-1}(s))\, ds + y_0 \tag{4.3.7}$$

....

As $n \to \infty$ the sequence of functions $y_n(t)$ converges uniformly to a function $y(t)$ on the interval $[t_0 - h, t_0 + h]$. Therefore, in the equation (4.3.7) we can let $n \to \infty$ and get that

$$y(t) = \int_{t_0}^{t} f(s, y(s))\, ds + y_0 \,,$$

which means that $y(t)$ is the unique solution of the IVP (4.3.5).

Example. Consider the IVP

$$\begin{cases} y' = y \\ y(0) = 1 \,, \end{cases}$$

The solution $y(t)$ satsfies the integral equation

$$y(t) = \int_{0}^{t} y(s)\, ds + 1 \,.$$

The successive approximation looks like:

$$y_1(t) = \int_0^t 1\, ds + 1 = t + 1$$

$$y_2(t) = \int_0^t (s+1)\, ds + 1 = \frac{t^2}{2} + t + 1$$

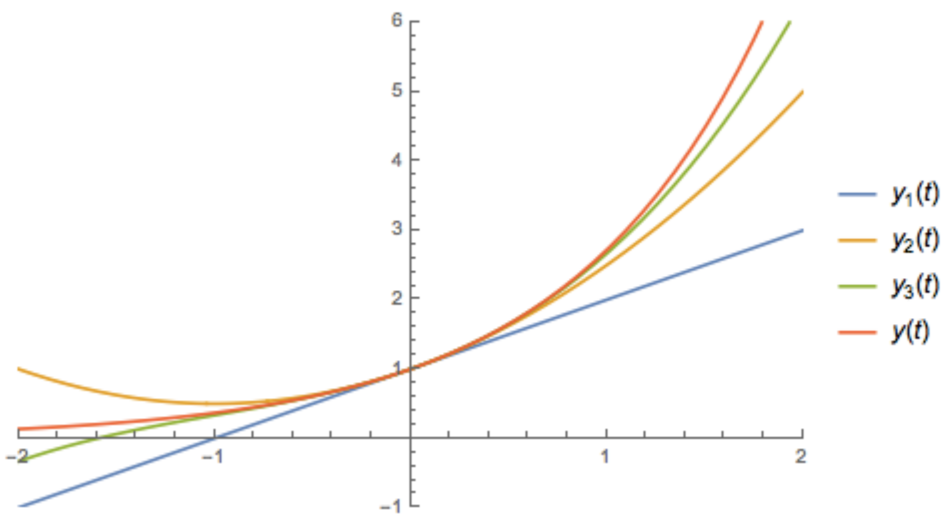
$$y_3(t) = \int_0^t \left(\frac{s^2}{2} + s + 1\right) ds + 1 = \frac{t^3}{6} + \frac{t^2}{2} + t + 1$$

....

$$y_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!}$$

....

$$y(t) = \lim_{n\to\infty} y_n(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t$$



Homework Exercises.

Calculate the first 3 terms of the method of successive approximations. Substitute $y_3(t)$ into the DE and verify how close $y_3(t)$ is to be a solution.
Optional: Try finding (not always easy or even possible) the formula for $y_n(t)$ and then calculate the solution as $y(t) = \lim_{n\to\infty} y_n(t)$.

$$
\begin{aligned}
&(1) \quad y' = -y\,,\ y(0) = 2\,. \\
&(2) \quad y' = 3y\,,\ y(0) = 1\,. \\
&(3) \quad y' = 2ty\,,\ y(0) = 1\,. \\
&(4) \quad y' = y - t\,,\ y(0) = 2\,. \\
&(5) \quad y' = \frac{t}{\sqrt{t^2+1}}\,,\ y(0) = 2\,. \\
&(6) \quad y' = y^2\,,\ y(0) = 1\,. \\
&(7) \quad y' + 2ty^2 = 0\,,\ y(0) = 1\,. \\
&(8) \quad y' = y + t\,,\ y(0) = 0\,. \\
&(9) \quad y' = ty^2 - 1\,,\ y(0) = 1\,. \\
&(10) \quad y' = \frac{ty}{\sqrt{t^2+1}}\,,\ y(0) = 2\,.
\end{aligned}
$$

4.4. Numerical methods for Differential equations

4.4.1. The Euler's method. Consider again the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \,. \end{cases}$$

Suppose that, as in the statement of the existence and uniqueness theorem, f and $\frac{\partial f}{\partial y}$ are continuous on $R_{a,b}$. Hence, we have a unique solution defined on $[t_0 - h, t_0 + h]$.

The following method, called Euler's method, provides the simplest numerical approximation of the solution. By numerical approximation we mean some algebraical calculations using $f(t, y)$, which is the right hand side of the DE.

Choose a small step $\varepsilon > 0$. We will determine approximate values of the solution at the following points:

$$\begin{aligned} &t_1 = t_0 + \varepsilon \,, \\ &t_2 = t_1 + \varepsilon = t_0 + 2\varepsilon \,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &t_n = t_0 + n\,\varepsilon \,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

For each t_n we define a number y_n which approximates the exact value of the solution $y(t_n)$. We write this approximation as $y_n \approx y(t_n)$.

Let us start with

$$y_1 = y_0 + f(t_0, y_0)\varepsilon \,.$$

By the fact that the slope of the solution curve at (t_0, y_0) is $f(t_0, y_0)$ we can use the linear approximation of functions by their first order Taylor polynomial to conclude that $y(t_1) \approx y_1$. Continue the process by setting

$$\begin{aligned} &y_2 = y_1 + f(t_1, y_1)\varepsilon \\ &y_3 = y_2 + f(t_2, y_2)\varepsilon \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &y_{n+1} = y_n + f(t_n, y_n)\varepsilon \,. \end{aligned}$$

The calculated points (t_0, y_0), (t_1, y_1), ... (t_n, y_n) can be connected by line segments to give a continuous curve, which is an approximation of the solution curve.

Example. Consider the IVP

$$\begin{cases} y' = 4t\sqrt{y} \\ y(0) = 0.16 \,, \end{cases} \quad . \tag{4.4.8}$$

We want to find an approximation of the solution on the interval $[0, 1]$. First, let us select a step size $\varepsilon = 0.25$. We use

$$f(t, y) = 4t\sqrt{y}$$

and

$$\boxed{t_0 = 0\,,\ y_0 = 0.16}\,.$$

Starting the first round of calculations, $t_1 = 0.25$ and $y_1 = 0.16 + (4 \cdot 0 \cdot \sqrt{0.16}) \cdot 0.25 = 0.16$.

$$\boxed{t_1 = 0.25\,,\ y_1 = 0.16}\,.$$

Continuing with the second round, $t_2 = 0.5$ and $y_2 = 0.16 + (4 \cdot 0.25 \cdot \sqrt{0.16}) \cdot 0.25 = 0.26$.

$$\boxed{t_2 = 0.5\,,\ y_2 = 0.26}\,.$$

In similar ways,

$$\boxed{t_3 = 0.75\,,\ y_3 = 0.5149}\,.$$

and

$$\boxed{t_4 = 1\,,\ y_4 = 1.053}\,.$$

In this way we found that,

$$y(0) = 0.16,\ y(0.25) \approx 0.16,\ y(0.5) \approx 0.26,\ y(0.75) \approx 0.5149,\ y(1) \approx 1.053\,.$$

We can use Mathematica to generate these numbers:

```
f[t_, y_] := 4 * t * Sqrt[y];
t0 = 0; y0 = 0.16; eps = 0.25; n = 4;
t = t0; y = y0;
Do[y = y + f[t, y] * eps;
 t = t + eps;
 Print[t, "...", y], {i, 1, n}]

0.25...0.16

0.5...0.26

0.75...0.514951

1....1.05315
```

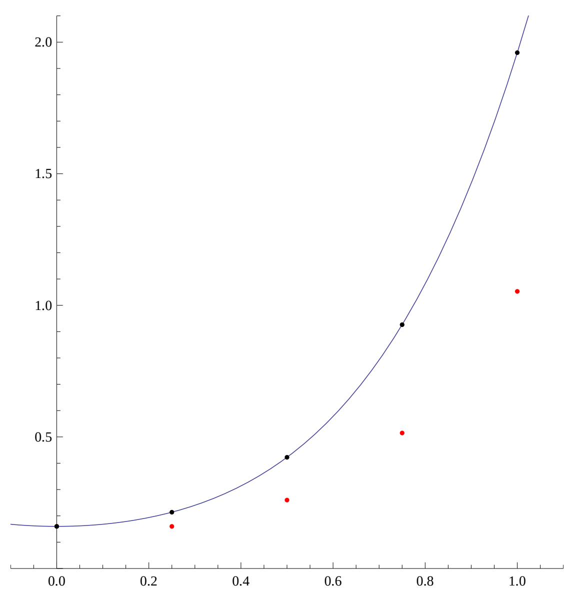
For comparison, let us calculate the exact values using the exact solution $y(t) = (t^2+0.4)^2$. Note that, in general, we don't know the exact solution.

$$y(0) = 0.16\,,\ \text{compared to}\ \ y_0 = 16$$
$$y(0.25) = (0.25^2 + 0.4)^2 = 0.2139\,,\ \text{compared to}\ \ y_1 = 0.16$$
$$y(0.5) = (0.5^2 + 0.4)^2 = 0.4225\,,\ \text{compared to}\ \ y_2 = 0.26$$
$$y(0.75) = (0.75^2 + 0.4)^2 = 0.9264\,,\ \text{compared to}\ \ y_3 = 0.5149$$
$$y(1) = (1 + 0.4)^2 = 1.96\,,\ \text{compared to}\ \ y_4 = 1.053$$

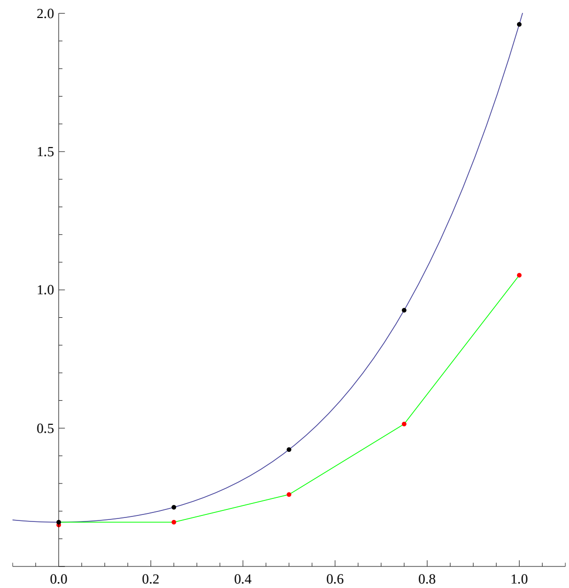
In order to see how the differences between the numerical approximations and exact values increase, we can add to the Mathematica code a third column with the exact values.

```
In[67]:= f[t_, y_] := 4 * t * Sqrt[y];
         t0 = 0; y0 = 0.16; eps = 0.25; n = 4;
         t = t0; y = y0;
         Do[y = y + f[t, y] * eps;
          t = t + eps;
          Print[t, "....", y, ".....", (t^2 + 0.4)^2], {i, 1, n}]

         0.25....0.16.....0.213906

         0.5....0.26.....0.4225

         0.75....0.514951.....0.926406

         1.....1.05315.....1.96
```

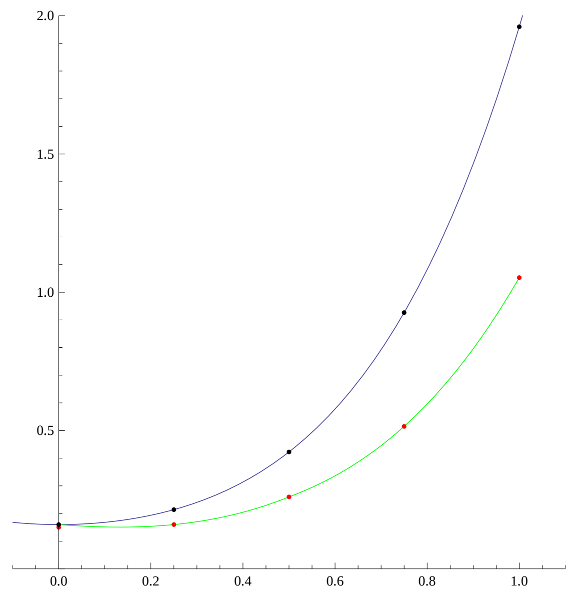
The numbers in the middle column are not very good approximations, which can be attributed to a large step size and a not very efficient approximation method. In the following graph the blue curve is the graph of the exact solution $y(t) = (t^2 + 0.4)^2$ and the red dots show the approximating values at the intermediate points.



An approximative solution curve can be given by connecting the points (t_i, y_i) by line segments. This method of connecting point by line segments, or other types of curves, is called interpolation.



Smoother interpolation curves are available, too:



As y_n is just an approximation of the exact value $y(t_n)$, we call the quantity

$$err_n = |y(t_n) - y_n| \,,$$

the error of the approximation at t_n. Let us try to estimate err_n.
The following calculations show the power of theoretical mathematics in finding the size of the error, without knowing the exact solution.
Suppose that both partial derivatives of $f(t, y)$ are continuous on the rectangle $R_{a,b}$. Then, the unique solution $y(t)$ of the IVP has a continuous second order derivative on $[t_0 - h,\, t_0 + h]$ and

$$y''(t) = \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t))\, y'(t) \,.$$

Hence, $|y''(t)|$ will have a finite maximum $M_2 \geq 0$ over the interval $[t_0 - h, t_0 + h]$. Using Taylor's theorem we get that

$$y(t_1) = y(t_0) + y'(t_0)\varepsilon + y''(t^*)\frac{\varepsilon^2}{2}\,,$$

for some $t_0 \leq t^* \leq t_1$. But, by Euler's method $y_1 = y(t_0) + y'(t_0)\varepsilon$, which gives

$$|y(t_1) - y_1| \leq \frac{M_2}{2}\varepsilon^2\,.$$

These calculation show that at each step we pick up a **local error** of order ε^2. But, we need $\frac{h}{\varepsilon}$ steps to cover the interval from t_0 to $t_0 + h$, so we can expect that the **global error** to be of order one less than the local error:

$$\frac{h}{\varepsilon}\frac{M}{2}\varepsilon^2 = C\,\varepsilon\,,$$

This means that

$$|y(t_n) - y_n| \leq C\,\varepsilon\,.$$

To improve the approximation of the solution we have two options: use smaller steps or improve the numerical method.

First, let us use a smaller step size $\varepsilon = 0.1$. We let Mathematica do the calculations and, as before, the second column contains the numerical approximations and the third column the exact values.

```
In[59]:= f[t_, y_] := 4 * t * Sqrt[y];
t0 = 0; y0 = 0.16; eps = 0.1; n = 10;
t = t0; y = y0;
Do[y = y + f[t, y] * eps;
 t = t + eps;
 Print[t, "....", y, ".....", (t^2 + 0.4) ^2], {i, 1, n}]

0.1....0.16.....0.1681

0.2....0.176.....0.1936

0.3....0.209562.....0.2401

0.4....0.264495.....0.3136

0.5....0.346782.....0.4225

0.6....0.464558.....0.5776

0.7....0.628139.....0.7921

0.8....0.850053.....1.0816

0.9....1.14509.....1.4641

1.....1.53032.....1.96
```

As we can see, $y_{10} = 1.53032$ is much closer to the exact value of $y(1) = 1.96$ than the earlier 1.053, which was calculated with a step size of 0.25.

4.4.2. The improved Euler (or Heun) method.

The previously introduced Euler method tends to underestimate the exact values in a case of a concave-up solution. To get a better approximation we will use an improved method, which is of a predictor-corrector type. This means that we approximate $y'(t_n)$ by averaging the slopes at the current and the following intermediate points.
To find y_{n+1}, we will calculate first an intermediate value y^*_{n+1}:

$$y^*_{n+1} = y_n + f(t_n, y_n) \cdot \varepsilon \,,$$

and then

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y^*_{n+1})}{2} \cdot \varepsilon \,.$$

For the same IVP (4.4.8) as before, with step size $\varepsilon = 0.25$, the calculated values are:

$$\boxed{t_0 = 0 \,,\ y_0 = 0.16} \,.$$

$$t_1 = 0.25 \,,\ y^*_1 = 0.16 + 4 \cdot 0 \cdot \sqrt{0.16} \cdot 0.25 = 0.16$$

$$y_1 = 0.16 + 0.25 \cdot \frac{4 \cdot 0 \cdot \sqrt{0.16} + 4 \cdot 0.25 \cdot \sqrt{0.16}}{2} = 0.21$$

$$\boxed{t_1 = 0.25 \,,\ y_1 = 0.21} \,.$$

$$t_2 = 0.5 \,,\ y^*_2 = 0.21 + 0.25 \cdot (4 \cdot 0.25 \cdot \sqrt{0.21}) = 0.3245$$

$$y_2 = 0.21 + 0.25 \cdot \frac{4 \cdot 0.25 \cdot \sqrt{0.21} + 4 \cdot 0.5 \cdot \sqrt{3245}}{2} = 0.4096$$

$$\boxed{t_2 = 0.5 \,,\ y_2 = 0.4096} \,.$$

$$t_3 = 0.75 \,,\ y^*_3 = 0.7296 \,,\ y_3 = 0.8899$$

$$\boxed{t_3 = 0.75 \,,\ y_3 = 0.8899} \,.$$

$$t_4 = 1 \,,\ y^*_4 = 1.5974 \,,\ y_4 = 1.8756$$

$$\boxed{t_4 = 1 \,,\ y_4 = 1.8756} \,.$$

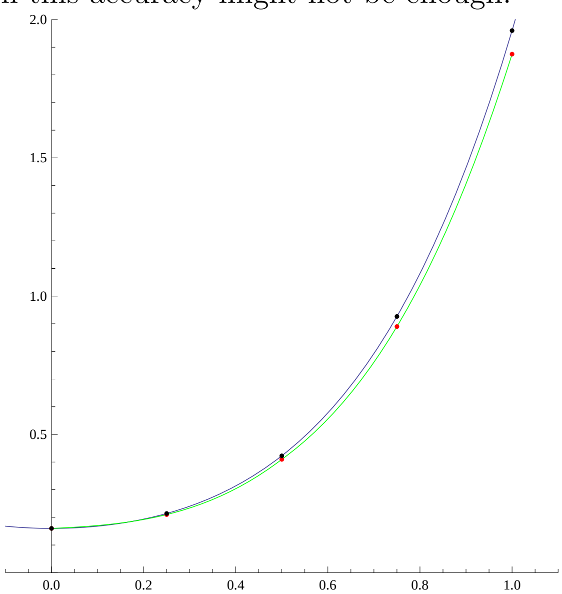
Therefore,

$$y(1) \approx 1.8756 \,.$$

Mathematica can be programmed in the following way. As before, the middle column contains the numerical approximations and the third column contains the exact values.

```
In[63]:= f[t_, y_] := 4 * t * Sqrt[y];
         t0 = 0; y0 = 0.16; eps = 0.25; n = 4;
         t = t0; y = y0;
         Do[y = y + 0.5 (f[t, y] + f[t + eps, y + f[t, y] * eps]) * eps;
          t = t + eps;
          Print[t, "....", y, ".....", (t^2 + 0.4)^2], {i, 1, n}]

         0.25....0.21.....0.213906

         0.5....0.409709.....0.4225

         0.75....0.890075.....0.926406

         1.....1.87586.....1.96
```

The local error for the improved Euler method is of order ε^3 and the global error is of order ε^2. The following graph shows how efficient this methods is. However, for some problems even this accuracy might not be enough.



4.4.3. The fourth order Runge-Kutta method.

This method uses a weighted average of four slopes at each (t_n, y_n). There are various versions of the Runge-Kutta method and the one we present here is the classical one with the average of four slopes. The general formula is the following.

$$
\begin{aligned}
s_1 &= f(t_n, y_n) \\
s_2 &= f\left(t_n + \frac{\varepsilon}{2}\,,\, y_n + \frac{s_1}{2}\,\varepsilon\right) \\
s_3 &= f\left(t_n + \frac{\varepsilon}{2}\,,\, y_n + \frac{s_2}{2}\,\varepsilon\right) \\
s_4 &= f\left(t_n + \varepsilon\,,\, y_n + s_3\,\varepsilon\right) \\
y_{n+1} &= y_n + \frac{s_1 + 2s_2 + 2s_3 + s_4}{6}\,\varepsilon
\end{aligned}
$$

For the IVP (4.4.8) studied earlier, let us use a step twice as large as for the Euler and Heun methods: $\varepsilon = 0.5$. Remember that $f(t, y) = 4t\sqrt{y}$.
Then for the first step we get the following results:

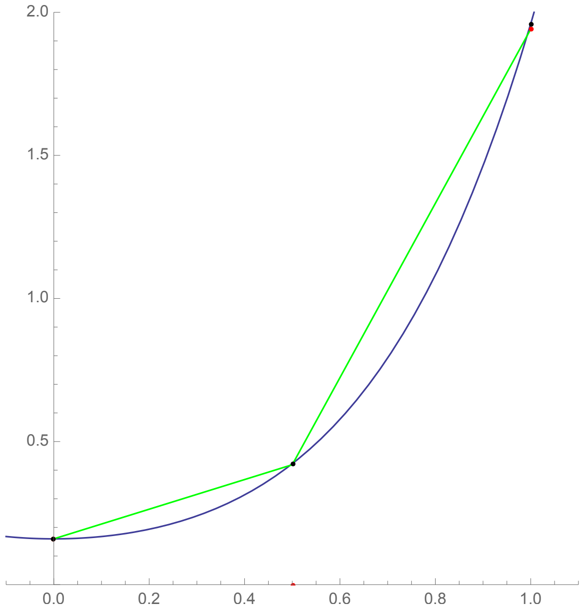
$$
\begin{aligned}
s_1 &= f(0, 0.16) = 0 \\
s_2 &= f(0.25, 0.16) = 0.4 \\
s_3 &= f\left(0.25, 0.16 + \frac{0.4}{2} \cdot 0.5\right) = 0.509902 \\
s_4 &= f\left(0.5, 0.16 + 0.509902 \cdot 0.5\right) = 1.28833 \\
y_1 &= 0.16 + \frac{0 + 2 \cdot 0.4 + 2 \cdot 0.509902 + 1.28833}{6}\,0.5 = 0.419011
\end{aligned}
$$

For the second step we get the following results:

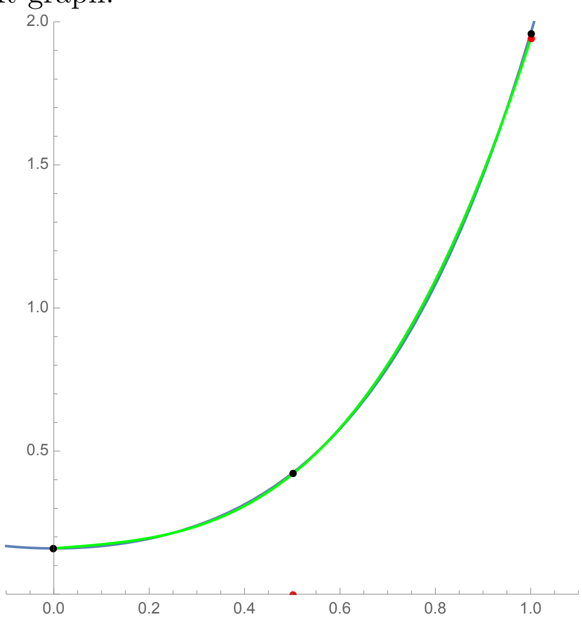
$$
\begin{aligned}
s_1 &= f(0.5, 0.419011) = 1.29452 \\
s_2 &= f\left(0.75, 0.419011 + \frac{1.29462}{2}\,0.5\right) = 2.58534 \\
s_3 &= f\left(0.75, 0.419011 + \frac{2.58534}{2}\,0.5\right) = 3.09647 \\
s_4 &= f\left(1, 0.419011 + 0.309647 \cdot 0.5\right) = 5.61034 \\
y_2 &= 0.419011 + \frac{1.29462 + 2 \cdot 2.58534 + 2 \cdot 0.3.09647 + 5.61034}{6}\,0.5 = 1.94138
\end{aligned}
$$

We can see that with just 2 steps the Runge-Kutta method gives better approximation of $y(1)$ than the Heun method with 4 step and the Euler method with 10 steps.
The local error of the Runge-Kutta method is of order ε^5, while the global error is of order ε^4.

Connecting the calculated points with line segments leads to the following graph.



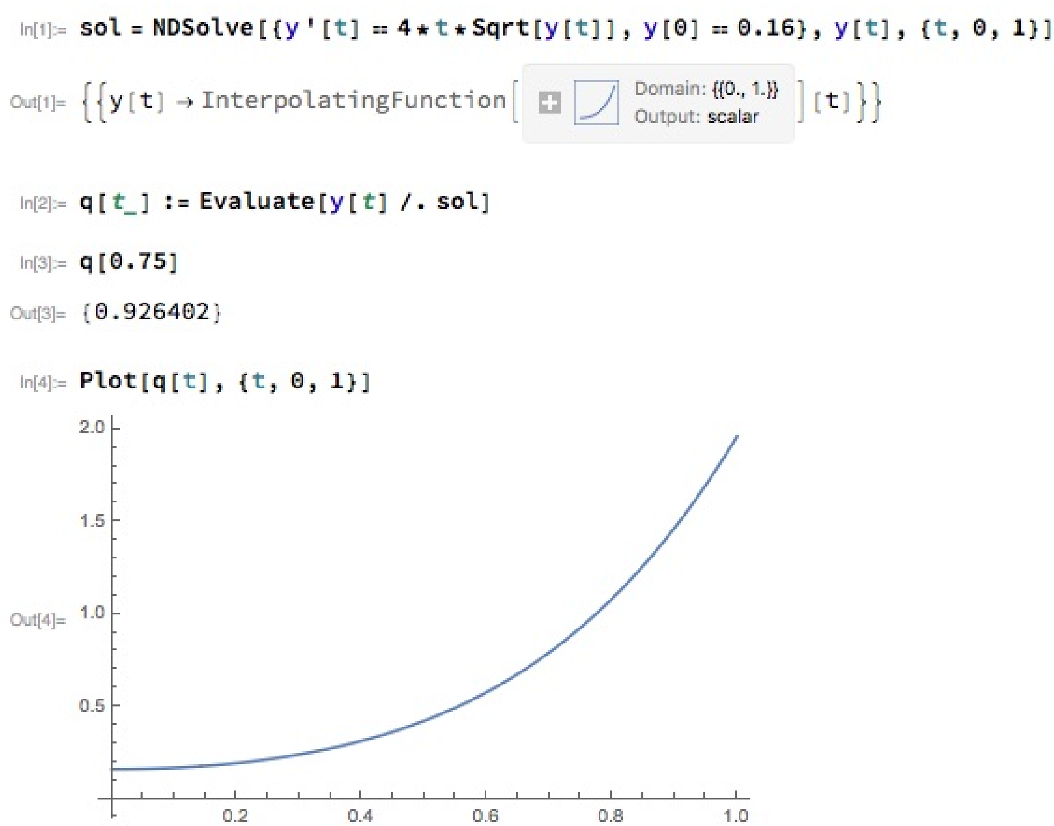
As you notice, even if the calculated points are on the exact solution curve, connecting them with line segments doesn't match the exact solution curve at other places. We can improve this by selecting a smaller step size, or use a smoother interpolation curve as in the next graph.



To further compare the approximation methods from this section, you can watch the following demonstration:
`http://demonstrations.wolfram.com/NumericalMethodsForDifferentialEquations/`

4.4.4. NDSolve command in Mathematica.

We can use the NDSolve command to get a numerical solution to an IVP. The expressions giving the solution looks as:
`NDSolve[{y’[t] == 4*t*Sqrt[y[t]], y[0] == 0.16}, y[t], {t, 0, 1}]`
If we want to get approximate values and graph of the solution then we assign a function to the numerical solution in the follwing way:
`sol = NDSolve[y’[t] == 4*t*Sqrt[y[t]], y[0] == 0.16, y[t], t, 0, 1]`
`q[t_] := Evaluate[y[t] /. sol]`
The function $q[t]$ is the approximate numerical solution of our problem. If we want to get the the approximate value of the solution for the input $t = 0.75$, then we just write `q[0.75]`, which gives 0.926402, an answer very close to the exact one 0.962406.
We can graph the solution with the command line
`Plot[q[t],{t,0,1}]` .



More examples for numerical approximations.

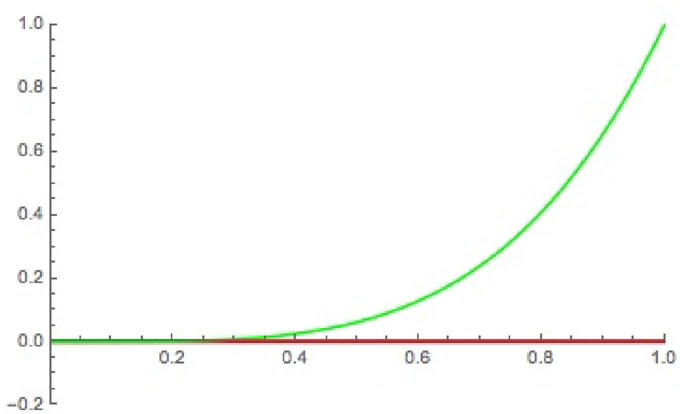
Example 1. This example shows what could happen if the IVP has multiple solutions. Consider the IVP

$$\begin{cases} y' = 4t\sqrt{y} \\ y(0) = 0\,, \end{cases} \quad . \tag{4.4.9}$$

The problem arises from the fact that there are two solutions. NDSolve gives the constant $y(t) \equiv 0$ function as a solution, while DSolve gives the function $y(t) = t^4$ as the solution. However, for more complicated DEs, DSolve might not provide an answer and we would loose information about other solutions.

```
In[5]:= sol1 = NDSolve[{y'[t] == 4 * t * Sqrt[y[t]], y[0] == 0}, y[t], {t, 0, 1}]

Out[5]= {{y[t] → InterpolatingFunction[ Domain: {{0., 1.}} Output: scalar ][t]}}

In[6]:= y1[t_] := Evaluate[y[t] /. sol1]

In[7]:= sol2 = DSolve[{y'[t] == 4 * t * Sqrt[y[t]], y[0] == 0}, y[t], t]

Out[7]= {{y[t] → t^4}}

In[8]:= y2[t_] := Evaluate[y[t] /. sol2]

In[12]:= Plot[{y1[t], y2[t]}, {t, 0, 1}, PlotStyle → {Red, Green},
          PlotRange → {{0, 1}, {-0.2, 1}}]
```

Out[12]=

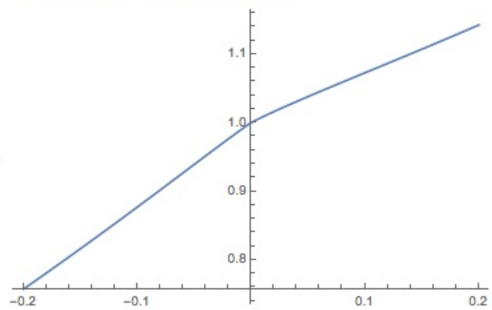


Example 2. For the following IVP analytic solutions are not possible and DSolve doesn't provide any answers. We can use only NDSolve, but we have to check that we can apply the Picard-Lindelöf Existence and Uniqueness Theorem before we start the computations. Consider the IVP

$$\begin{cases} y' = y^2 - \sqrt[3]{t} \\ y(0) = 1\,, \end{cases} \quad . \tag{4.4.10}$$

Using the Picard-Lindelöf Existence and Uniqueness Theorem for $a = 1$ and $b = 1$ leads to a unique solution $y : [-0.2, 0.2] \to [0, 2]$. NDSolve leads to the following graph.

```
In[14]:= sol3 = NDSolve[{y'[t] == (y[t])^2 - CubeRoot[t], y[0] == 1}, y[t], {t, -0.2, 0.2}]

Out[14]= {{y[t] → InterpolatingFunction[ Domain: {{-0.2, 0.2}} Output: scalar ][t]}}

In[15]:= y3[t_] := Evaluate[y[t] /. sol3]

In[16]:= Plot[y3[t], {t, -0.2, 0.2}]

Out[16]=
```

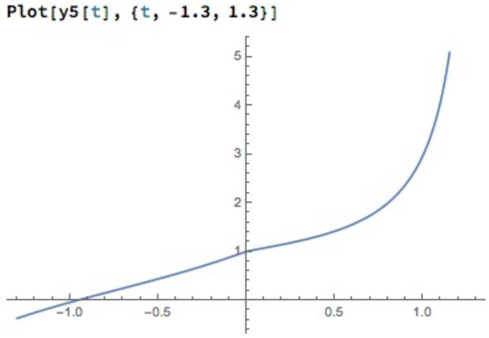


The graph shows no erratic behavior of the solution and therefore, probably, its domain can be extended. Trying to extend it from $[-0.2, 0.2]$ to $[-2, 2]$ leads to the following message.

```
In[20]:= sol4 = NDSolve[{y'[t] == (y[t])^2 - CubeRoot[t], y[0] == 1}, y[t], {t, -2, 2}]

NDSolve: At t == 1.3503053628761046`, step size is effectively zero; singularity or stiff system suspected.

Out[20]= {{y[t] → InterpolatingFunction[ Domain: {{-2., 1.35}} Output: scalar ][t]}}
```

The message shows that around $t = 1.35$ some problems arise. Let's try the domain $[-1.3, 1.3]$.

```
In[21]:= sol5 = NDSolve[{y'[t] == (y[t])^2 - CubeRoot[t], y[0] == 1}, y[t], {t, -1.3, 1.3}]

Out[21]= {{y[t] → InterpolatingFunction[ Domain: {{-1.3, 1.3}} Output: scalar ][t]}}

In[22]:= y5[t_] := Evaluate[y[t] /. sol5]

In[23]:= Plot[y5[t], {t, -1.3, 1.3}]

Out[23]=
```



The graph shows that at about $t = 1.35$ a possibly vertical asymptote shows up, hence the solution cannot be extended further to the right. However, it could be extended to the left.

Homework Exercises.

(1) Use each of the Euler, Heun and Runge-Kutta methods to approximate $y(t)$ on the interval $[1, 2]$ using a step size $\varepsilon = 0.5$, where $y(t)$ is the solution of the IVP

$$\begin{cases} y' = t + y \\ y(1) = 0\,. \end{cases}$$

(2) Use each of the Euler, Heun and Runge-Kutta methods to approximate $y(0.5)$ after two steps, where $y(t)$ is the solution of the IVP

$$\begin{cases} y' = -ty \\ y(0) = 1\,. \end{cases}$$

(3) Use each of the Euler, Heun and Runge-Kutta methods to approximate $y(1)$ after two steps, where $y(t)$ is the solution of the IVP

$$\begin{cases} y' = t - y \\ y(0) = 1\,. \end{cases}$$

(4) Use the Heun method to approximate $y(1)$ with step sizes $\varepsilon = 0.25$ and $\varepsilon = 0.1$, where $y(t)$ is the solution of the IVP

$$\begin{cases} y' = y^2 \\ y(0) = 2\,. \end{cases}$$

What are your conclusions?

(5) Use the Euler method to approximate $y(1.5)$ with step sizes $\varepsilon = 0.25$ and $\varepsilon = 0.1$, where $y(t)$ is the solution of the IVP

$$\begin{cases} y' = 3y^{2/3} \\ y(1) = 0\,. \end{cases}$$

What might go wrong and why?

(6) Use each of the Euler, Heun and Runge-Kutta methods to approximate $y(2.2)$ using a step size $\varepsilon = 0.4$, where $y(t)$ is the solution of the IVP

$$\begin{cases} y' = \frac{t}{2} + \frac{y}{4} \\ y(1) = 4\,. \end{cases}$$

Note: Check your answers for the Homework Exercises with Mathematica by using both the DSolve and the NDSolve.

CHAPTER 5

Higher order linear differential equations

5.1. General theory

A n$^{\text{th}}$-order linear DE has the form

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) = g(t)\,, \quad t \in I\,, \tag{5.1.1}$$

where the unknown function is $y(t)$ and the coefficients are the functions $a_k(t)$, $0 \le k \le n$.

Example. In the case of

$$(t^3-1)\, y^{(4)}(t) + \sqrt{t^2+4}\, y'''(t) - \sin t\, y'(t) + y(t) = e^t\,, \quad 1 < t < \infty\,,$$

$a_4(t) = t^3 - 1$, $a_3(t) = \sqrt{t^2+4}$, $a_2(t) = 0$, $a_1(t) = -\sin t$, $a_0(t) = 1$ and $g(t) = e^t$.

The general solution of a n$^{\text{th}}$-order linear DE has the form

$$y(t) = y_h(t) + y_p(t)\,,$$

where $y_h(t)$ is a n-parameter family of solutions of the linear and homogeneous DE

$$a_n(t)\, y^{(n)}(t) + a_{n-1}(t)\, y^{(n-1)}(t) + \cdots + a_1(t)\, y'(t) + a_0(t)\, y(t) = 0\,, \quad t \in I\,, \tag{5.1.2}$$

and $y_p(t)$ is a particular solution of the non-homogeneous DE (5.1.1). As a n-parameter family of solutions, $y_h(t)$ has to be determined as

$$y_h(t) = c_1 y_1(t) + \cdots + c_n y_n(t)\,,$$

where $y_1(t)\,, \cdots\; y_n(t)$ are solutions of the linear and homogeneous DE (5.1.2).
However, not every choice of n solutions is suitable. We must choose linearly independent solutions, which means that if

$$c_1 y_1(t) + \cdots + c_n y_n(t) = 0\,, \quad \text{for every } \; t \in I\,,$$

then each parameter must be 0:

$$c_1 = \cdots = c_n = 0\,.$$

To analytically check the linear independence of solutions, we must check the Wronskian determinant is not identically zero:

$$W(y_1(t)\,,\, y_2(t)\,, \cdots y_n(t)) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ y_1^{(n-1}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix} \neq 0\,,$$

for at least one $t \in I$.

Note: Determinants are calculated in the following way:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc\,,$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}\,.$$

Higher order determinants are calculated in a similar way by expanding them using the first row, and thus reducing the calculations to determinants of one size less.

Definition. The functions $y_1(t), \cdots, y_n(t)$ form a **Fundamental Set of Solutions** (shortly FSS) of the linear and homogeneous DE (5.1.2) if:

1. Each function is a solution.
2. They are linearly independent.

The following theorem gives us a method to check whether n functions form a FSS or not.

THEOREM 5.1.1. *If the functions $y_1(t), \cdots, y_n(t)$ are solutions of the linear and homogeneous DE (5.1.2) and $W(y_1(t), \cdots, y_n(t)) \neq 0$ for at least one $t \in I$, then they are linearly independent and form a FSS.*

Examples:
(1) Let us show that the functions $y_1(t) = t$ and $y_2(t) = t^3$ form a FSS for the DE

$$t^2\, y'' - 3t\, y' + 3\, y = 0\,, \quad t \in (0, +\infty)\,.$$

First, let us check that the two functions are solutions. By substituting $y_1(t) = t$ into the DE we get

$$t^2 \cdot 0 - 3t \cdot 1 + 3t = 0\,,$$

which leads to $0 = 0$. Repeat the process for $y_2(t) = t^3$, too.
Then

$$W(t, t^3) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 3t^3 - t^3 = 2t^2\,,$$

which is not zero for any (would be enough to check just for one) $t > 0$. Therefore, $y_1(t) = t$ and $y_2(t) = t^3$ form a FSS.
However, if we want to see whether $z_1(t) = t$ and $z_2 = 5t$ form a FSS, then we can check that they are solutions, but

$$W(t, 5t) = \begin{vmatrix} t & 5t \\ 1 & 5 \end{vmatrix} = 5t - 5t = 0\,,$$

which shows that they are not linearly independent. Therefore, they do not form a FSS.

Regarding the existence and uniqueness of solutions for IVPs corresponding to linear DEs we have the following theorem.

THEOREM 5.1.2. *Consider the IVP*

$$\begin{cases} a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t)y(t) = g(t)\,,\ t \in [\alpha, \beta] \\ \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}\,, \end{cases}$$

where $t_0 \in [\alpha, \beta]$ is a fixed point.
If the functions $a_n(t), \cdots, a_0(t), g(t)$ are continuous on the interval $[\alpha, \beta]$ and $a_n(t) \neq 0$ for any $\alpha \leq t \leq \beta$, then the IVP has a unique solution on the entire interval $[\alpha, \beta]$.

Homework Exercises.
(1) Determine whether the given functions form a FSS of the corresponding linear and homogeneous DE.

$$\begin{array}{ll} (a) & \cos 5t\,,\ \sin 5t\,, \quad y'' + 25y = 0\,,\ \ t \in \mathbb{R}. \\ (b) & e^{5t}\,,\ e^{-5t}\,, \quad y'' + 25y' = 0\,,\ \ t \in \mathbb{R}. \\ (c) & e^{5t}\,,\ e^{-5t}\,, \quad y'' - 25y = 0\,,\ \ t \in \mathbb{R}. \\ (d) & \dfrac{1}{t}\,,\ t\,,\ t^2\,, \quad t^3y''' + t^2y'' - 2ty' + 2y = 0\,,\ t > 0\,. \\ (e) & t^2 - t\,,\ t\,,\ t^2\,, \quad t^3y''' + t^2y'' - 2ty' + 2y = 0\,,\ t < 0\,. \\ (f) & e^{3t}\,,\ te^{3t}\,, \quad y'' - 6y' + 9y = 0\,,\ \ t \in \mathbb{R}. \\ (g) & 1\,,\ \cos 2t\,,\ \sin 2t\,, \quad y''' + 4y' = 0\,,\ \ t \in \mathbb{R}. \\ (h) & e^{-t}\,,\ e^{4t}\,, \quad y'' - 3y' - 4y = 0\,,\ \ t \in \mathbb{R}. \\ (i) & e^{t}\,,\ \cos t\,,\ \sin t\,, \quad y''' - y'' + y' - y = 0\,,\ \ t \in \mathbb{R}. \end{array}$$

(2) Determine the intervals on which IVPs corresponding to the given DEs have unique solutions:

$$\begin{array}{ll} (a) & (t^2 - 9)\,y''' + \sin t\; y'' - y = t. \\ (b) & \cos t\; y' + 3y = e^t. \\ (c) & y'' + \sqrt{1 - t^2}\; y' + \ln t\; y = 0. \\ (d) & y^{(4)} - y'' = t^3. \\ (e) & t^3\,y''' - 5t^2\,y'' + ty' - 5y = t + 2. \end{array}$$

5.2. Linear and homogeneous DEs with constant coefficients

The linear and homogeneous DEs with constant coefficients have the form

$$a_n\, y^{(n)}(t) + a_{n-1}\, y^{(n-1)}(t) + \cdots + a_1\, y'(t) + a_0\, y(t) = 0\,, \tag{5.2.1}$$

where the coefficients $a_n,\ a_{n-1}, \cdots, a_1, a_0$ are real numbers and $a_n \neq 0$.
We would like to find which functions of the form $y(t) = e^{rt}$ are solutions of the DE (5.2.1). Substituting $y(t) = e^{rt}$ in the DE (5.2.1) gives

$$\Big(a_n\, r^n + a_{n-1}\, r^{n-1} + \cdots + a_1\, r + a_0\Big)\, e^{rt} = 0\,.$$

Therefore, we have the following theorem:

THEOREM 5.2.1. *If r is a solution of the polynomial equation*

$$a_n\, r^n + a_{n-1}\, r^{n-1} + \cdots + a_1\, r + a_0 = 0\,, \tag{5.2.2}$$

then $y(t) = e^{rt}$ is a solution of the DE (5.2.1).

Equation (5.2.2) is called the **characteristic equation** of the DE (5.2.1). Every n^{th}-order polynomial equation has n real or complex solutions.

We will assign to each solution r of the characteristic equation (5.2.2) a solution of the DE (5.2.1). In this process we have to distinguish the following cases.

Simple real solution: If r is a simple real solution of (5.2.2), then we assign to it the function

$$e^{rt}\,.$$

Repeated real solutions: If r is a real solution repeated k times, then we assign to it k solutions:

$$e^{rt}, te^{rt}, \cdots, t^{k-1}e^{rt}\,.$$

Simple complex solution: If $r = a + ib$ is a complex solution of (5.2.2), then $a - ib$ is also a solutions, so we assign to r two solutions

$$e^{at}\cos(bt)\,,\ e^{at}\sin(bt)\,.$$

Repeated complex solutions: If $r = a + ib$ is a complex solution of (5.2.2) repeated k times, then we assign to it k pairs of solutions

$$e^{at}\cos(bt)\,,\ e^{at}\sin(bt)\,,\ te^{at}\cos(bt)\,,\ te^{at}\sin(bt)\,, \cdots t^{k-1}e^{at}\cos(bt)\,,\ t^{k-1}e^{at}\sin(bt)\,.$$

We finalize the theory of this section by the following theorem.

THEOREM 5.2.2. *If we assign to each solution of the characteristic equation a solution of the linear and homogeneous DE (5.2.1) in the ways shown above, then we get a fundamental set of solutions.*

Examples.

1. Solve the DE:

$$y'' - 9y = 0\,.$$

The characteristic equation $r^2 - 9 = 0$ has the solutions

$$r_1 = 3\,,\ r_2 = -3\,.$$

The functions assigned to them are

$$y_1(t) = e^{3t}\,,\ y_2(t) = e^{-3t}\,.$$

These two functions form a FSS, so the general solution has the form

$$y(t) = c_1 e^{3t} + c_2 e^{-3t}\,.$$

2. Solve the DE:

$$y'' + 9y = 0\,.$$

The characteristic equation $r^2 + 9 = 0$ has the solutions

$$r_1 = 3i\,,\ r_2 = -3i\,.$$

The functions assigned to them are

$$y_1(t) = \cos(3t)\,,\ y_2(t) = \sin(3t)\,.$$

These two functions form a FSS, so the general solution has the form

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t)\,.$$

3. Solve the DE:

$$y''' + 4y'' + 4y' = 0\,.$$

The characteristic equation $r^3 + 4r^2 + 4r = 0$ has the solutions

$$r_1 = 0\,,\ r_2 = r_3 = -2\,.$$

The functions assigned to them are

$$y_1(t) = 1\,,\ y_2(t) = e^{-2t}\,,\ y_3(t) = te^{-2t}$$

These three functions form a FSS, so the general solution has the form

$$y(t) = c_1 + c_2 e^{-2t} + c_3 te^{-2t}\,.$$

4. Solve the IVP:

$$y'' - 4y' + 13y = 0\,,\ y(0) = 0\,,\ y'(0) = 3\,.$$

The characteristic equation $r^2 - 4r + 13 = 0$ has the solutions

$$r_1 = 2 + 3i\,,\ r_2 = 2 - 3i\,.$$

The functions assigned to them are

$$y_1(t) = e^{2t}\cos(3t)\,,\; y_2(t) = e^{2t}\sin(3t)\,.$$

These two functions form a FSS, so the general solution of the DE has the form

$$y(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)\,.$$

Using the initial conditions we get $c_1 = 0$ and $c_2 = 1$, and, therefore, the unique solution of the IVP is

$$y(t) = e^{2t}\sin(3t)\,.$$

Homework Exercises. Solve the following DEs and IVPs:

1. $2y' - 5y = 0$.

2. $y'' + 4y' + 3y = 0$, $y(1) = 0$, $y'(1) = 2e^{-3}$.

3. $y''' + y = 0$.

4. $y^{(4)} - 16y = 0$.

5. $y''' + 5y'' = 0$.

6. $y''' - y'' + 4y' - 4y = 0$.

7. $y''' - 5y'' + 3y' + y = 0$.

8. $y'' - 8y' + 16y = 0$.

9. $y''' - y'' + y' - y = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$.

10. $y^{(4)} - 5y'' + 4y = 0$.

11. $y^{(4)} + 5y'' + 4y = 0$.

12. $y^{(4)} - 50y'' + 625y = 0$.

13. $y''' + y'' + y' = 0$.

14. $y'' - 6y' + 13y = 0$.

15. $y''' - 3y'' + 4y' - 2y = 0$.

5.3. Linear and non-homogeneous DEs with constant coefficients

The previous section provided methods to find y_h, so we are left to find a particular solution y_p. Two methods will be presented.

5.3.1. Variation of parameters for second order linear equations.

Variation of parameters can be used for any linear DE, as long as we know a FSS of the homogeneous equation. Here we will present it just for second order linear DEs.

Consider the DE

$$a_2\, y''(t) + a_1\, y'(t) + a_0\, y(t) = g(t)\,,$$

where a_2, a_1, a_0 are real numbers, $a_2 \neq 0$ and $g(t)$ is not the constant zero function.
Let us assume that we already obtained the solution of the homogeneous DE

$$a_2\, y'' + a_1\, y' + a_0\, y = 0\,,$$

and it has the form

$$y_h(t) = c_1\, y_1(t) + c_2\, y_2(t)\,.$$

The variation of parameters method means that we are looking for the particular solution in the form

$$y_p(t) = c_1(t)\, y_1(t) + c_2(t)\, y_2(t)\,,$$

where the $c_1(t)$ and $c_2(t)$ are unknown functions left to be determined. By requesting that $y_p(t)$ be a solution of the non-homogeneous DE, we get the system

$$\begin{cases} c_1'(t)\, y_1(t) + c_2'(t)\, y_2(t) &= 0 \\ c_1'(t)\, y_1'(t) + c_2'(t)\, y_2'(t) &= \frac{g(t)}{a_2}\,. \end{cases}$$

Solving this system gives $c_1'(t)$ and $c_2'(t)$. By integrating them we get $c_1(t)$ and $c_2(t)$, and from here we find $y_p(t)$. The final solution is given by $y(t) = y_h(t) + y_p(t)$.

Example: Solve the DE

$$y'' - 3y' + 2y = e^{5t}\,.$$

Step 1. The homogeneous equation

$$y'' - 3y' + 2y = 0$$

has the characteristic equation

$$r^2 - 3r + 2 = 0\,,$$

which has the solutions $r_1 = 1$ and $r_2 = 2$. The FSS assigned to them is formed by the functions $y_1(t) = e^t$ and $y_2(t) = e^{2t}$. Therefore,

$$y_h(t) = c_1\, e^t + c_2\, e^{2t}\,.$$

Step 2. We search the particular solution in the form

$$y_p(t) = c_1(t)\, e^t + c_2(t)\, e^{2t}\,,$$

and this leads to the system

$$\begin{cases} c_1'(t)\ e^t + c_2'(t)\ e^{2t} & = 0 \\ c_1'(t)\ e^t + c_2'(t)\ 2e^{2t} & = e^{5t}\,. \end{cases}$$

Subtracting the first equation from the second gives

$$c_2'(t)e^{2t} = e^{5t}\,,$$

and hence $c_2'(t) = e^{3t}$ and $c_2(t) = \frac{1}{3}e^{3t}$. Substituting e^{3t} for $c_2'(t)$ in the first equation gives $c_1'(t) = -e^{4t}$ and hence $c_1(t) = -\frac{1}{4}e^{4t}$.
Therefore,

$$y_p(t) = -\frac{1}{4}e^{4t}e^t + \frac{1}{3}e^{3t}e^{2t} = \frac{1}{12}e^{5t}\,.$$

Step 3. The complete solution is

$$y(t) = c_1\ e^t + c_2\ e^{2t} + \frac{1}{12}e^{5t}\,.$$

5.3.2. The undetermined coefficients method and the superposition principle.

The undetermined coefficients method applies if the function $g(t)$ on the right hand side of the DE has one of the following forms:

1. If $g(t) = P(t)e^{\alpha t}$, where $P(t)$ is a polynomial of degree m and α is not a solution of the characteristic equation, then we search $y_p(t)$ in the following form

$$y_p(t) = (b_m t^m + \cdots + b_1 t + b_0)e^{\alpha t}\,,$$

where the unknown coefficients $b_m, \cdots, b_1, b_0$ are determined by substituting $y_p(t)$ in the non-homogeneous DE.

2. If $g(t) = P(t)e^{\alpha t}$, where $P(t)$ is a polynomial of degree m and α is a solution of the characteristic equation repeated k-times, then we search $y_p(t)$ in the following form

$$y_p(t) = t^k(b_m t^m + \cdots + b_1 t + b_0)e^{\alpha t}\,,$$

where the unknown coefficients $b_m, \cdots, b_1, b_0$ are determined by substituting $y_p(t)$ in the non-homogeneous DE.

3. If $g(t) = P(t)e^{\alpha t}\cos(\beta t) + Q(t)e^{\alpha t}\sin(\beta t)$, where $P(t)$ and $Q(t)$ are a polynomials of degree at most m and $\alpha + i\beta$ is not a solution of the characteristic equation, then we search $y_p(t)$ in the following form

$$y_p(t) = (b_m t^m + \cdots + b_1 t + b_0)e^{\alpha t}\cos(\beta t) + (d_m t^m + \cdots d_1 t + d_0)e^{\alpha t}\sin(\beta t)\,,$$

where the unknown coefficients $b_m, \cdots, b_1, b_0$, $d_m, \cdots, d_1, d_0$ are determined by substituting $y_p(t)$ in the non-homogeneous DE.

4. If $g(t) = P(t)e^{\alpha t}\cos(\beta t) + Q(t)e^{\alpha t}\sin(\beta t)$, where $P(t)$ and $Q(t)$ are a polynomials of degree at most m and $\alpha + i\beta$ is a solution of the characteristic equation repeated k-times, then we search $y_p(t)$ in the following form

$$y_p(t) = t^k(b_m t^m + \cdots + b_1 t + b_0)e^{\alpha t}\cos(\beta t) + t^k(d_m t^m + \cdots d_1 t + d_0)e^{\alpha t}\sin(\beta t)\,,$$

where the unknown coefficients $b_m, \cdots, b_1, b_0,\ d_m, \cdots, d_1, d_0$ are determined by substituting $y_p(t)$ in the non-homogeneous DE.

In the following examples we focus just on finding y_p and ask the reader to complete the details of finding y_h.

Example 1. Solve the DE

$$y'' - y' - 2y = 2t + 3\,.$$

Step 1. Using the method from Section 5.2 we get $y_h(t) = c_1 e^{2t} + c_2 e^{-t}$.
Step 2. In this exercise $g(t) = (2t+3)e^{0\,t}$ and $\alpha = 0$, which is not a solution of the characteristic equation $r^2 - r - 2 = 0$. So, we search for $y_p(t)$ in the form

$$y_p(t) = (b_1 t + b_0)e^{0\,t} = b_1 t + b_0\,.$$

Substituting $y_p(t)$ into the DE leads to

$$-b_1 - 2b_1 t - 2b_0 = 2t + 3\,,$$

which can be rearranged as

$$-2b_1 t - 2b_0 - b_1 = 2t + 3\,.$$

The two sides must be identically the same, so we have $-2b_1 = 2$ and $-2b_0 - b_1 = 3$, which gives $b_1 = -1$ and $b_0 = -1$, and hence $y_p(t) = -t - 1$.
Step 3. The final form of the solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - t - 1\,.$$

Example 2. Solve the DE

$$y'' - y' = 2t + 3\,.$$

Step 1. Using the method from Section 5.2 we obtain $y_h(t) = c_1 + c_2 e^{t}$.
Step 2. In this exercise $g(t) = (2t+3)e^{0\,t}$ and $\alpha = 0$, which is a simple ($k = 1$) solution of the characteristic equation $r^2 - r = 0$. So, we search for $y_p(t)$ in the form

$$y_p(t) = t(b_1 t + b_0)e^{0\,t} = b_1 t^2 + b_0 t\,.$$

Substituting $y_p(t)$ into the DE leads to

$$2b_1 - 2b_1 t - b_0 = 2t + 3\,,$$

which can be rearranged as

$$-2b_1 t + 2b_1 - b_0 = 2t + 3\,.$$

The two sides must be identically the same, so we have $-2b_1 = 2$ and $2b_1 - b_0 = 3$, which gives $b_1 = -1$ and $b_0 = -5$ and hence $y_p(t) = -t^2 - 5t$.
Step 3. The final form of the solution is

$$y(t) = c_1 + c_2 e^{t} - t^2 - 5t\,.$$

Example 3. Solve the DE
$$y''' + y'' - y' - y = 4\cos t\,.$$
Step 1. Using the method from Section 5.2 we obtain $y_h(t) = c_1e^t + c_2e^{-t} + c_3te^{-t}$.
Step 2. In this exercise, $g(t) = 4e^{0t}\cos t$ and $\alpha + i\beta = i$, which is not a solution of the the characteristic equation. So, we search $y_p(t)$ in the form
$$y_p(t) = a\cos t + b\sin t\,.$$
By substituting $y_p(t)$ into the DE and grouping the similar terms we get that
$$(2a - 2b)\sin t + (-2a - 2b)\cos t = 4\cos t\,,$$
which leads to the system
$$\begin{cases} 2a - 2b &= 0 \\ -2a - 2b &= 4\,. \end{cases}$$
This gives $a = b = -1$ and therefore $y_p(t) = -\cos t - \sin t$.
Step 3. The final form of the solution is
$$y(t) = c_1e^t + c_2e^{-t} + c_3te^{-t} - \cos t - \sin t\,.$$

The superposition principle:
This method works just in the case of linear DEs. If the right hand side is the sum of k functions,
$$g(t) = g_1(t) + \cdots + g_k(t)\,,$$
then we search the particular solution as
$$y_p(t) = y_{p1}(t) + \cdots + y_{pk}(t)\,,$$
where each function is a particular solution of the corresponding term of the right hand side.

Example 4.. Consider the linear DE:
$$y'' + 4y = te^t - 24e^{2t}\,.$$
Step 1. Solving the homogeneous equation gives $y_h(t) = c_1\cos(2t) + c_2\sin(2t)$.
Step 2. We search the particular solution in the form $y_p(t) = (at+b)e^t + de^{2t}$. By substituting $y_p(t)$ into the DE we get
$$(5at + 5b + 2a)e^t + 8de^{2t} = te^t - 24e^{2t}\,,$$
which gives the system
$$\begin{cases} 5a &= 1 \\ 5b + 2a &= 0 \\ 8d &= -24\,. \end{cases}$$
Hence, $a = \frac{1}{5}$, $b = -\frac{2}{25}$, $d = -3$ and $y_p(t) = (\frac{1}{5}t - \frac{2}{25})e^t - 3e^{2t}$.
Step 3. The final form of the solution is
$$y(t) = c_1\cos(2t) + c_2\sin(2t) + (\frac{1}{5}t - \frac{2}{25})e^t - 3e^{2t}\,.$$

5.3.3. Use Mathematica to solve higher order DEs.

```
In[1]:= DSolve[y''[t] + 4*y[t] == t*Exp[t] - 24*Exp[2*t], y[t], t]

Out[1]= {{y[t] → C[1] Cos[2 t] + C[2] Sin[2 t] - 1/25 e^t (2 + 75 e^t - 5 t) (Cos[2 t]^2 + Sin[2 t]^2)}}

In[2]:= FullSimplify[DSolve[y''[t] + 4*y[t] == t*Exp[t] - 24*Exp[2*t], y[t], t]]

Out[2]= {{y[t] → -1/25 e^t (2 + 75 e^t - 5 t) + C[1] Cos[2 t] + C[2] Sin[2 t]}}
```

The FullSimplify command can be really useful in simplifying the solutions to a form similar to what we get without the use of computers.

Homework exercises.

1. Solve the following DEs and IVPs:

1. $y' - 3y = 6$.
2. $y'' - 4y' + 3y = e^t \sin(2t)$, $y(0) = 0$, $y'(0) = 1$.
3. $y''' + y = t + e^{-2t}$.
4. $y^{(4)} - 16y = t^2 + t$.
5. $y''' + 5y'' = t + 3$.
6. $y''' - y'' + 4y' - 4y = \sin(2t)$.
7. $4y'' + 5y' + y = \frac{1}{e^t}$.
8. $y'' + y = \tan t$.
9. $y'' + y = \cos t$.
10. $y'' + y = te^{-2t}$.
11. $y'' + 9y = \frac{1}{\sin(3t)}$.
12. $y'' + 3y' + 2y = \frac{1}{1+e^t}$.
13. $y'' + 3y' + 2y = \frac{1+e^t}{e^t}$.

14. $y'' + 3y' + 2y = t^2$.

15. $y^{(4)} - 5y'' + 4y = e^t$.

16. $y^{(4)} - 5y'' + 4y = e^{3t}$.

17. $y^{(4)} + 5y'' + 4y = \cos(2t) - \sin(2t)$.

18. $y^{(4)} - 50y'' + 625y = 125$.

19. $y'' + 2y' + y = \frac{1}{te^t}$.

2. The vertical displacement from its natural length of a spring-mass system is described by

$$y''(t) + 2y'(t) + 10y(t) = 0\,,$$

where the time t is measured in seconds.
Describe the position of the mass after 20 seconds if the initial position is 0 and initial velocity is $-1\ m/s$.

3. Find the charge $q(t)$ on the capacitor in a series RLC circuit which is modeled by the IVP

$$\frac{1}{8}q'' + 5q' + 500q = 0V\,,\ q(0) = 0\,C\,,\ q'(0) = 20\,A\,.$$

Find the charge after 3 seconds.

4. Find the charge $q(t)$ on the capacitor in a series RLC circuit which is modeled by the IVP

$$\frac{5}{3}q'' + 10q' + 30q = 110V\,,\ q(0) = 0\,C\,,\ q'(0) = 2\,A\,.$$

What is the charge after 1 second? What is the long term behavior of $q(t)$?

5. (a) Find the charge $q(t)$ on the capacitor in a series RLC circuit which is modeled by the IVP

$$\frac{1}{10}q'' + 2q' + 100q = \cos(10t) + \sin(10t)V\,,\ q(0) = 0\,C\,,\ q'(0) = 0\,A\,.$$

(b) Use DSolve to find $q(t)$, plot it and estimate the maximum charge during the first second.

6. Consider the problem of a free falling object with mass M. Assume that only gravity and air resistance act upon the object.

(a) As a first model, let us suppose that the air resistance is proportional to the velocity $v(t)$ of the object. Newton's second law of motion gives the DE

$$Mv'(t) = Mg - kv(t)\,,\ t \geq 0\,.$$

More exactly, this is a first order linear DE with constant coefficients:

$$Mv'(t) + kv(t) = Mg\,,\ t \geq 0\,.$$

Suppose that 2 objects with mass $M_1 = 10$ kg and $M_2 = 20$kg are released from an altitude of 3000 meters with initial vertical velocity 0. Suppose that the constant $k = 0.5$ for both objects. Answer the following questions:
(i) Calculate the velocities $v_1(t)$ and $v_2(t)$ of the two objects.
(ii) What are their terminal (highest) velocities?
(iii) Which object is falling faster?
(iv) What are their speeds after 5 seconds?

(b) (Optional) The role of this exercise is to show that another mathematical model might lead to a much more difficult DE. In certain cases, the air resistance can be modeled as

$$F_{air} = C \cdot 0.5 \cdot \rho \cdot v(t)^2 \cdot A\,,$$

where C is the drag coefficient, ρ is the air density, and A is the reference area of the object. C and A are constants, but the air density depends on air temperature and pressure which vary with altitude. A simple function modeling air density is the following

$$\rho(y) = 1.2 - 0.00011y\,,$$

where y is the elevation above see level. This function is obtained by supposing that the air is dry, the temperature at see level is $20°C$ and is dropping at a rate of $6°C$ per 1000 meters. If an object with mass of 10kg is released at 3000 meters and we denote by $s(t) = 3000 - y(t)$ the distance the object dropped until time t, then we get the a DE of the form

$$10s''(t) + C \cdot A \cdot (0.87 + 0.00011s(t)) \cdot (s'(t))^2 = 100\,,$$

which is a second order non-linear differential equation.
Use DSolve and NDSolve to estimate the altitude and velocity of the object after 10 seconds.

5.4. The Cauchy-Euler DE

The Cauchy-Euler DE has the form

$$a_n \cdot t^n \cdot y^{(n)}(t) + a_{n-1} \cdot t^{n-1} \cdot y^{(n-1)}(t) + \cdots + a_1 \cdot t \cdot y'(t) + a_0 \cdot y(t) = g(t)\,, \qquad (5.4.1)$$

which has to be solved for $t < 0$ or $t > 0$. This is a linear DE with non-constant coefficients and we will reduce it to a linear DE with constant coefficients. In order to achieve this, we use the substitutions

$$t = e^x \quad \text{or} \quad x = \ln t\,, \quad \text{if} \quad t > 0\,,$$

and

$$t = -e^x \quad \text{or} \quad x = \ln(-t)\,, \quad \text{if} \quad t < 0\,.$$

Let us consider the $t > 0$ case.
We have to substitute the derivatives in t with derivatives in x. Using the chain rule we get that

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} = \frac{dy}{dx}\,\frac{1}{t} = \frac{dy}{dx}\,e^{-x}\,.$$

Furthermore,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\,\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\,e^{-x}\right)\,e^{-x} = \left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\,e^{-2x}\,,$$

and

$$\frac{d^3y}{dt^3} = \frac{d}{dt}\,\left(\frac{d^2y}{dt^2}\right) = \frac{d}{dx}\left(\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\,e^{-2x}\right)\,e^{-x} = \left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)\,e^{-3x}\,.$$

If we continue in this way, we can express any derivative in t in terms of derivatives in x and by substituting them into the equation (5.4.1), we obtain a linear differential equation with constant coefficients.

Example Solve the following DE:

$$t^3y''' + 5t^2y'' + 7ty' + 8y = 2\ln t\,, \quad t > 0\,.$$

Let us use the substitution $t = e^x$ and get

$$e^{3x}\,\left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right)\,e^{-3x} + 5e^{2x}\,\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)\,e^{-2x} + 7e^x\,\frac{dy}{dx}\,e^{-x} + 8y = 2x$$

The exponential functions are canceling each other, so we get

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 8y = 2x\,.$$

Solving this DE according to the methods from the previous sections gives

$$y(x) = c_1e^{-2x} + c_2\cos(2x) + c_3\sin(2x) + \frac{1}{4}x - \frac{1}{8}\,.$$

We get the final form of the solution by substituting $x = \ln t$ in the previous line:

$$y(t) = c_1t^{-2} + c_2\cos(2\ln t) + c_3\sin(2\ln t) + \frac{1}{4}\ln t - \frac{1}{8}\,.$$

Homework Exercises.

Solve the following DEs and IVPs.

1. $t^2y'' - ty' + y = \sin(\ln t)\,, \quad t > 0.$

2. $t^3y''' - 6y = 2t + 3\,,\ t > 0\,.$

3. $t^2y'' + ty' + y = 0\,,\ y(1) = 1\,,\ y'(1) = 2\,.$

4. $t^2y'' + ty' - y = \frac{1}{t}\,,\ t > 0\,.$

5. $t^3y''' - 6ty' + 12y = t^2\,,\ t > 0\,.$

6. $t^2y'' - ty' + 5y = 2\ln t + t\,,\ t > 0\,.$

7. $t^3y' - 3t^2y = 1\,,\ t > 0\,.$

8. $t^3y''' - 6ty' + 12y = t^2\,,\ t < 0\,.$

CHAPTER 6

Solving linear differential equations with the Laplace transform

6.1. Definition and properties of the Laplace transform

The Laplace transform changes a linear DE into an algebraical equation, which can be solved by algebraical methods. Finally, the algebraical solution is transformed back into a solution of the original DE.
As an addition to the methods presented in the previous chapter, the Laplace transform will help us to solve linear DEs with discontinuous right hand sides.

DEFINITION 6.1.1. *We say that a function* $y : [0, +\infty) \to \mathbb{R}$ *is* **piecewise continuous on** $[0, \infty)$ *if* $\lim_{t\to 0^+} y(t)$ *exists and* $y(t)$ *is continuous on every interval of finite length* $[0, b]$*, except maybe a finite number of points, where the function has jump discontinuities.*

DEFINITION 6.1.2. *We say the function* $y : [0, +\infty) \to \mathbb{R}$ *is of* **exponential order** $c \geq 0$ *if there are positive constants* M *and* T *such that*

$$|y(t)| \leq Me^{ct}\,, \text{ for all } t \geq T\,.$$

DEFINITION 6.1.3. *Consider a function* $y : [0, +\infty) \to \mathbb{R}$*, which is piecewise continuous on* $[0, +\infty)$ *and is of exponential order* c*. The* **Laplace transform** *of the function* $y(t)$ *is defined as*

$$\boxed{\mathcal{L}[y(t)](s) = \int_0^\infty e^{-st} y(t)\, dt\,, \quad s > c\,.} \tag{6.1.1}$$

Properties of the Laplace transform:

Existence: The Laplace transform is an improper integral, which could converge or diverge depending on the value of s.
However, if the function $y(t)$ is piecewise continuous on $[0, +\infty)$ and of exponential order c, then the improper integral converges for $s > c$, so $\mathcal{L}[y(t)](s)$ exists and is finite.

Linearity: Suppose that $\lambda \in \mathbb{R}$ and the functions $y(t)$ and $z(t)$ are piecewise continuous on $[0, +\infty)$ and of exponential order c. Then for all $s > c$ we have:

$$\boxed{\begin{aligned} \mathcal{L}[y(t) + z(t)](s) &= \mathcal{L}[y(t)](s) + \mathcal{L}[z(t)](s) \\ \mathcal{L}[\lambda y(t)](s) &= \lambda \mathcal{L}[y(t)](s)\,. \end{aligned}}$$

The linearity of the Laplace transform makes it compatible with linear differential equations.

Laplace transforms of elementary functions:

$$(1)\quad \boxed{\mathcal{L}[1](s) = \frac{1}{s}\,,\ s > 0.}$$

$$(2)\quad \boxed{\mathcal{L}[t^n](s) = \frac{n!}{s^{n+1}}\,,\ s > 0\,,\ n \in \mathbb{N}.}$$

$$(3)\quad \boxed{\mathcal{L}[e^{at}](s) = \frac{1}{s-a}\,,\ s > a.}$$

$$(4)\quad \boxed{\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2+b^2}\,,\ s > 0.}$$

$$(5)\quad \boxed{\mathcal{L}[\cos(bt)](s) = \frac{s}{s^2+b^2}\,,\ s > 0.}$$

$$(6)\quad \mathcal{L}[\sinh(bt)](s) = \frac{b}{s^2-b^2}\,,\ s > |b|.$$

$$(7)\quad \mathcal{L}[\cosh(bt)](s) = \frac{s}{s^2-b^2}\,,\ s > |b|.$$

Let us prove these these formulas.

(1) If $s > 0$ then

$$\mathcal{L}[1](s) = \int_0^\infty e^{-ts}\,dt = -\frac{e^{-ts}}{s}\Big|_0^\infty = \frac{1}{s}\,.$$

(2)

$$\mathcal{L}[t](s) = \int_0^\infty e^{-ts}\,t\,dt = -\frac{te^{-ts}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-ts}\,dt = \frac{1}{s^2}\,.$$

$$\mathcal{L}[t^2](s) = \int_0^\infty e^{-ts}\,t^2\,dt = -\frac{t^2e^{-ts}}{s}\Big|_0^\infty + \frac{2}{s}\int_0^\infty e^{-ts}\,t\,dt = \frac{2}{s^3}\,.$$

By mathematical induction, if $n \geq 2$,

$$\mathcal{L}[t^n](s) = \int_0^\infty e^{-ts}\,t^n\,dt = -\frac{t^ne^{-ts}}{s}\Big|_0^\infty + \frac{n}{s}\mathcal{L}[t^{n-1}](s) = \frac{n!}{s^{n+1}}\,.$$

(3) If $s > a$, then

$$\mathcal{L}[e^{at}](s) = \int_0^\infty e^{-ts}\,e^{at}dt = \int_0^\infty e^{-t(s-a)}\,dt = -\frac{e^{-t(s-a)}}{s-a}\Big|_0^\infty = \frac{1}{s-a}\,.$$

(4) If $s > 0$, then

$$\begin{aligned}
\mathcal{L}[\sin(bt)](s) &= \int_0^\infty e^{-ts} \sin(bt)\, dt \\
&= -\frac{e^{-ts} \sin(bt)}{s}\Big|_0^\infty + \frac{b}{s} \int_0^\infty e^{-ts} \cos(bt)\, dt \\
&= -\frac{be^{-ts} \cos(bt)}{s^2}\Big|_0^\infty - \frac{b^2}{s^2} \int_0^\infty e^{-ts} \sin(bt)\, dt \\
&= \frac{b}{s^2} - \frac{b^2}{s^2} \mathcal{L}[\sin(bt)](s)\,.
\end{aligned}$$

Therefore,

$$\mathcal{L}[\sin(bt)](s) = \frac{b}{s^2 + b^2}\,.$$

(5) It is similar to (4).

(6) If $s > |b|$, then

$$\begin{aligned}
\mathcal{L}[\sinh(bt)](s) &= \mathcal{L}\left[\frac{e^{bt} - e^{-bt}}{2}\right](s) = \frac{1}{2}\left(\mathcal{L}[e^{bt}](s) - \mathcal{L}[e^{-bt}](s)\right) \\
&= \frac{1}{2}\left(\frac{1}{s-b} - \frac{1}{s+b}\right) = \frac{b}{s^2 - b^2}\,.
\end{aligned}$$

(7) If $s > |b|$, then

$$\begin{aligned}
\mathcal{L}[\cosh(bt)](s) &= \mathcal{L}\left[\frac{e^{bt} + e^{-bt}}{2}\right](s) = \frac{1}{2}\left(\mathcal{L}[e^{bt}](s) + \mathcal{L}[e^{-bt}](s)\right) \\
&= \frac{1}{2}\left(\frac{1}{s-b} + \frac{1}{s+b}\right) = \frac{s}{s^2 - b^2}\,.
\end{aligned}$$

Homework Exercises.

1. Which of the following functions are of exponential order c? Find c, if the answer is yes.

(a)

$$y(t) = 5t^2 + 2t + 1\,.$$

(b)

$$y(t) = \sin(3t)\,.$$

(c)

$$y(t) = 4\,e^{2t}\,.$$

(d)

$$y(t) = e^{t^2}\,.$$

(e)

$$y(t) = \begin{cases} 2 & \text{if} \quad t = 3 \\ \frac{1}{t-3} & \text{if} \quad t \neq 3 . \end{cases}$$

(f)

$$y(t) = \cos t \, e^{3t} .$$

(g)

$$y(t) = e^{-5t} .$$

2. Which of the following functions are piecewise continuous on $[0, +\infty)$?
(a)

$$y(t) = t^2 \, e^t .$$

(b)

$$y(t) = \begin{cases} 0 & \text{if} \quad t = 4 \\ \frac{1}{t-4} & \text{if} \quad t \neq 4 . \end{cases}$$

(c)

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < 5 \\ \frac{1}{t-4} & \text{if} \quad t \geq 5 . \end{cases}$$

(d)

$$y(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \sin \frac{1}{t} & \text{if} \quad t > 0 . \end{cases}$$

(e)

$$y(t) = \lfloor t \rfloor , \quad \text{the integer part of t} .$$

3. Which of the functions from exercises 1 and 2 are both of exponential order c and piecewise continuous on $[0, +\infty)$?

4. Find the Laplace transforms of the following functions and give the interval on which the Laplace transforms are defined:

(a)

$$y(t) = 2t + 3 .$$

(b)

$$y(t) = t^2 + 2t + 1 .$$

(c)

$$y(t) = (\cos t + \sin t)^2 .$$

(d)

$$y(t) = t^2 + e^{4t} .$$

(e)

$$y(t) = \left(1 + e^{3t}\right)^2 .$$

(f)

$$y(t) = \sinh^2 t\,.$$

(g)

$$y(t) = \frac{t^2 + 5t + 6}{t + 2}\,.$$

(h)

$$y(t) = \sin(5t) + \cos(5t)\,.$$

(i)

$$y(t) = e^{-2t} + 3e^{2t}\,.$$

(j)

$$y(t) = \cos^2(t)\,.$$

(h)

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 1 \\ t & \text{if} \quad t \ge 1\,. \end{cases}$$

6.2. Further properties of the Laplace transform. Transforms of the Heaviside function and the Dirac Delta function

6.2.1. Translation on the s-axis.

If the function $y(t)$ is piecewise continuous on $[0, +\infty)$ and of exponential order c, then

$$\boxed{\mathcal{L}[e^{at}\, y(t)](s) = \mathcal{L}[y(t)]\,(s-a)\,, \quad \text{if } \; s > a + c\,.} \tag{6.2.2}$$

We can prove this formula in the following way:

$$\begin{aligned}\mathcal{L}[e^{at}\, y(t)](s) &= \int_0^\infty e^{-ts}\, e^{at}\, y(t)\, dt = \int_0^\infty e^{-t(s-a)}\, y(t)\, dt \\ &= \mathcal{L}[y(t)](s-a)\,.\end{aligned}$$

Examples:

(1)

$$\mathcal{L}[e^{2t}\, t^3](s) = \mathcal{L}[t^3]\,(s-2) = \left.\frac{3!}{s^4}\right|_{s\to s-2} = \frac{6}{(s-2)^4}\,, \quad \text{if } \; s > 2\,.$$

(2)

$$\mathcal{L}[e^{-t}\, \cos(2t)](s) = \mathcal{L}[\cos(2t)]\,(s+1) = \left.\frac{s}{s^2+4}\right|_{s\to s+1} = \frac{s+1}{(s+1)^2+4}\,, \quad \text{if } \; s > 0\,.$$

6.2.2. Derivatives of the Laplace transform.

For simplcity of notations, the Laplace transform of a function denoted by a lower case letter will be denoted by the same upper case letter. For example:

$$\mathcal{L}[y(t)]\,(s) = Y(s)\,.$$

To find a formula for $Y^{(n)}(s)$ we start with $Y'(s)$ and give some explanations.

$$\begin{aligned}Y'(s) = \frac{d}{ds}\int_0^\infty e^{-ts}\, y(t)\, dt &= \int_0^\infty \frac{d}{ds} e^{-ts}\, y(t)\, dt \\ &= \int_0^\infty (-t) e^{-ts}\, y(t)\, dt = -\mathcal{L}[ty(t)](s)\,.\end{aligned}$$

Hence,

$$\mathcal{L}[ty(t)](s) = -Y'(s)\,,$$

and continuing this process, by mathematical induction, we get that for any $n \in \mathbb{N}$ we have

$$\boxed{\mathcal{L}[t^n\, y(t)] = (-1)^n\, Y^{(n)}(s)\,.} \tag{6.2.3}$$

As you can see, the process of calculating the derivatives of $Y(s)$ involves differentiating under the integral sign, which requires the use of uniform convergence of the improper integrals

$\int_0^\infty e^{-ts}\, y(t)\, dt$ in s for $s \geq c + \varepsilon$, where $\varepsilon > 0$ symbolizes any small positive number.

Examples.
(1)

$$\mathcal{L}[t^2](s) = \mathcal{L}[t^2 \cdot 1](s) = \frac{d^2}{ds^2}\left(\frac{1}{s}\right) = \frac{2}{s^3}\,.$$

(2)

$$\mathcal{L}[t^3 e^{2t}](s) = -\frac{d^3}{ds^3}\left(\frac{1}{s-2}\right) = \frac{6}{(s-2)^3}\,.$$

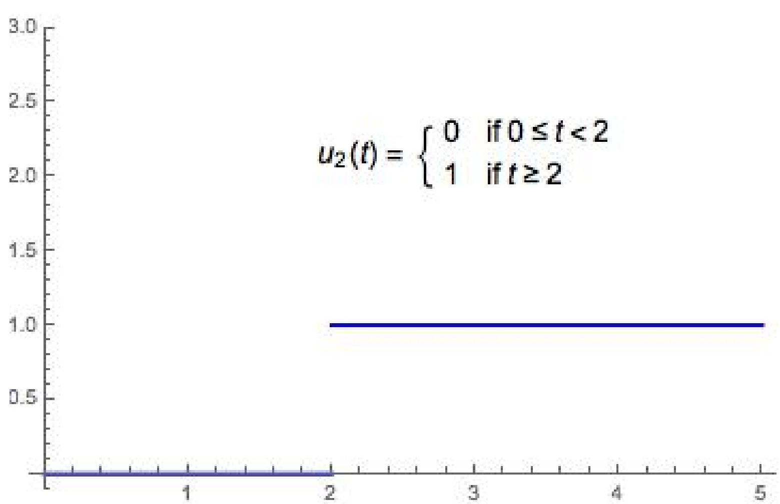
(3)

$$\mathcal{L}[t \sin t](s) = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}\,.$$

6.2.3. The Laplace transform of the unit step function and of piecewise continuous functions.

The unit step function is frequently used to model the turning "off" and "on" of external forces and it is defined by:

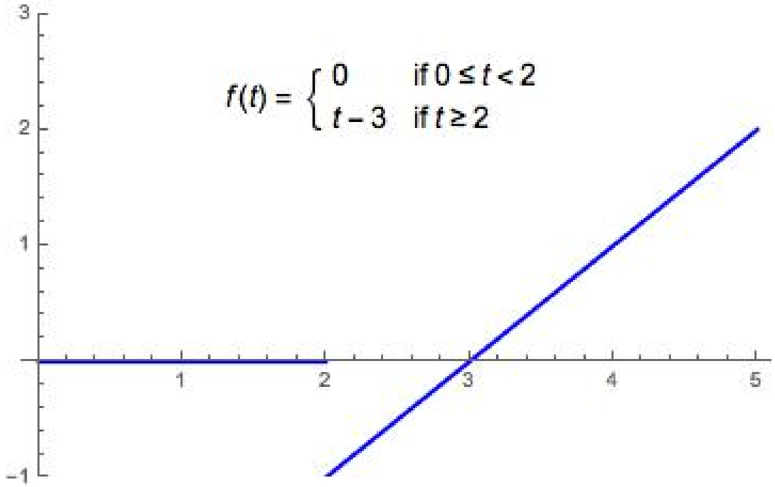
$$u_a(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t < a \\ 1 & \text{if} \quad t \geq a\,. \end{cases}$$



With the aid of the unit step function we can rewrite the piecewise continuous functions in a form suitable for the Laplace transform. Let's see two examples:

Consider

$$f(t) = \begin{cases} 0\,, & \text{if} \quad 0 \leq t < 2 \\ t-3\,, & \text{if} \quad t \geq 2\,. \end{cases}$$

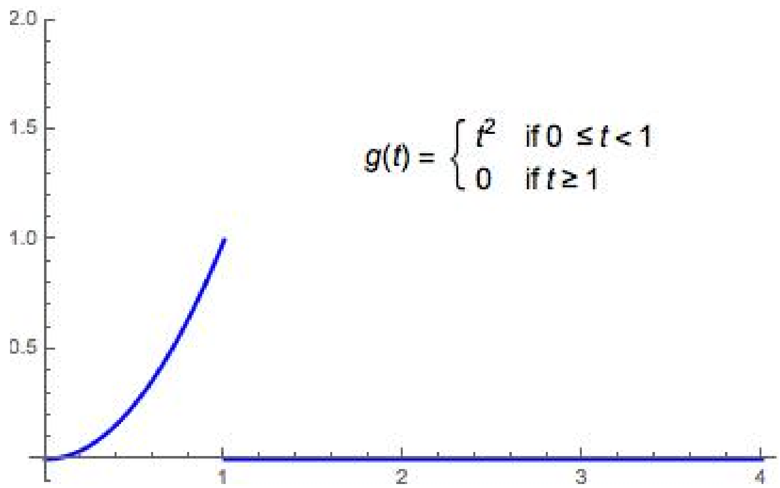


We can write

$$f(t) = (t-3)\, u_2(t)\,.$$

Consider now

$$g(t) = \begin{cases} t^2\,, & \text{if} \quad 0 \le t < 1 \\ 0\,, & \text{if} \quad t \ge 1\,. \end{cases}$$



Then,

$$g(t) = t^2 - t^2\, u_1(t)\,.$$

For another piecewise continuous function consider

$$f(t) = \begin{cases} 0\,, & \text{if} \quad 0 \le t < 1 \\ 3\,, & \text{if} \quad 1 \le t < 2 \\ 0\,, & \text{if} \quad t \ge 2 \end{cases}$$

we have

$$f(t) = 3u_1(t) - 3u_2(t)\,.$$

Let's continue by calculating the Laplace transforms of these piecewise continuous functions.

$$\mathcal{L}[u_a(t)]\,(s) = \int_a^\infty e^{-ts}\,dt = -\frac{e^{-ts}}{s}\bigg|_a^\infty = \frac{e^{-as}}{s}\,.$$

In general,

$$\mathcal{L}[f(t-a)\,u_a(t)]\,(s) = \int_a^\infty e^{-\tau s}\,f(\tau - a)\,d\tau$$

$$t = \tau - a\,,\quad dt = d\tau$$

$$= \int_0^\infty e^{-(t+a)s}\,f(t)\,dt = \int_0^\infty e^{-as}\,e^{-ts}\,f(t)\,dt = e^{-as}\int_0^\infty e^{-ts}\,f(t)\,dt$$
$$= e^{-as}\,\mathcal{L}[f(t)](s)\,.$$

Therefore,

$$\boxed{\mathcal{L}[f(t-a)\,u_a(t)]\,(s) = e^{-as}\,\mathcal{L}[f(t)](s)\,.} \qquad (6.2.4)$$

Examples.

(1)

$$\mathcal{L}[(t-2)u_2(t)]\,(s) = e^{-2s}\,\mathcal{L}[t]\,(s) = \frac{e^{-2s}}{s^2}\,.$$

(2)

$$\mathcal{L}[e^{t-3}u_3(t)]\,(s) = e^{-3s}\,\mathcal{L}[e^t]\,(s) = \frac{e^{-3s}}{s-1}\,.$$

(3)

$$\mathcal{L}[(t-3)\,u_2(t)]\,(s) = \mathcal{L}[(t-2)\,u_2(t) - u_2(t)]\,(s)$$
$$= \mathcal{L}[(t-2)\,u_2(t)]\,(s) - \mathcal{L}[u_2(t)]\,(s)$$
$$= e^{-2s}\,\mathcal{L}[t]\,(s) - \frac{e^{-2s}}{s} = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} = \frac{1-s}{s^2}\,e^{-2s}\,.$$

(4)

$$\begin{aligned}
&\mathcal{L}[(t^2+1)\; u_1(t)]\;(s) \\
&= \mathcal{L}[(t-1)^2 + 2(t-1) + 2)\; u_1(t)]\;(s) \\
&= \mathcal{L}[(t-1)^2\, u_1(t)]\;(s) + 2\mathcal{L}[(t-1)\, u_1(t)]\;(s) + 2\mathcal{L}[u_1(t)]\;(s) \\
&= e^{-s}\;\mathcal{L}[t^2]\;(s) + 2e^{-s}\;\mathcal{L}[t]\;(s) + 2\frac{e^{-s}}{s} \\
&= \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right)\; e^{-s}\;.
\end{aligned}$$

6.2.4. The Dirac Delta function. The Dirac Delta function describes forces of large magnitude acting only for a very short time. Actually it is not a function, it is a distribution, or generalized function, but a description of the distributions theory is beyond the level of this course. Hence, we will just define the Dirac function in an elementary way and give its Laplace transform.

For $a \geq 0$ define

$$\delta_a(t) = \begin{cases} +\infty\,, & \text{if} \quad t = a \\ 0\,, & \text{if} \quad t \neq a\,, \end{cases}$$

and formally require

$$\int_{-\infty}^{\infty} \delta_a(t)\, dt = 1\,.$$

We will use the notation $\delta(t)$ instead of $\delta_0(t)$.

The Laplace transform of the Dirac Delta function is given by

$$\boxed{\mathcal{L}[\delta_a(t)]\;(s) = e^{-as}\,.} \tag{6.2.5}$$

Hence,

$$\boxed{\mathcal{L}[\delta(t)]\;(s) = 1\,.} \tag{6.2.6}$$

Homework Exercises.

Find the Laplace transforms of the following functions:

(1) $e^t \sin t$

(2) $e^{-t} \cos t$

(3) $e^{3t}t^2$

(4) $t\cosh t$

(5) $t^2 - 3te^t$

(6) t^2e^{3t}

(7) $e^{-2t}\sin 4t + 3t$

(8) $(t-3)u_3(t)$

(9) $(t-3)^2\ u_3(t)$

(10) $\sin(t-\pi)\ u_\pi(t)$

(11) $\sin^2 t$

(12) $\cos^2(3t)$

(13) $10\delta_3(t)$

(14) $\sin t + \delta_\pi(t)$

(15) $(t-2)^3\ u_2(t) + \delta_2(t)$

6.3. The inverse Laplace transform

By Lerch's theorem, if two piecewise continuous functions have the same Laplace transform, then they can differ just at the discontinuity points. By assuming that at discontinuity points we consider the right hand side limit as the value of the function at that point, we find that the Laplace transform is a one-to-one transformation. Therefore, we can define its inverse transformation, which reverses the effect of the Laplace transform.

Definition 6.3.1. *If $Y(s) = \mathcal{L}[y(t)]\,(s)$ then define*

$$\boxed{\mathcal{L}^{-1}\,[Y(s)]\,(t) = y(t)\,.}$$

Note: The inverse Laplace transform is linear, which means that

$$\mathcal{L}^{-1}[Y(s)+Z(s)]\,(t) = \mathcal{L}^{-1}[Y(s)]\,(t) + \mathcal{L}^{-1}[Z(s)]\,(t) = y(t)+z(t)\,,$$

and

$$\mathcal{L}^{-1}[aY(s)]\,(t) = a\,\mathcal{L}^{-1}[Y(s)]\,(t) = a\,y(t)\,.$$

Examples:

(1)

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right]\,(t) = 1\,.$$

(2)

$$\mathcal{L}^{-1}\left[\frac{1}{s^3}\right]\,(t) = \frac{1}{2}t^2\,.$$

(3)

$$\mathcal{L}^{-1}\left[\frac{1}{s-5}\right]\,(t) = e^{5t}\,.$$

(4)

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]\,(t) = \cos(2t)\,.$$

(5)

$$\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+4}\right]\,(t) = e^{-3t}\,\cos(2t)\,.$$

(6)

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{s}{(s+3)^2+4}\right](t) = \mathcal{L}^{-1}\left[\frac{s+3-3}{(s+3)^2+4}\right](t) \\
&= \mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+4} - \frac{3}{(s+3)^2+4}\right](t) \\
&= \mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+4}\right](t) - \frac{3}{2}\,\mathcal{L}^{-1}\left[\frac{2}{(s+3)^2+4}\right](t) \\
&= e^{-3t}\ \cos(2t) - \frac{3}{2}\,e^{-3t}\ \sin(2t)\,.
\end{aligned}
$$

(7)

$$\mathcal{L}^{-1}\left[\frac{1}{(s-5)^2}\right](t) = te^{5t}\,.$$

(8)

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s}\right](t) = u_2(t)\,.$$

(9)

$$\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s-5}\right](t) = e^{5(t-2)}\ u_2(t)\,.$$

(10)

$$\mathcal{L}^{-1}\left[1\right](t) = \delta(t)\,.$$

(11)

$$\mathcal{L}^{-1}\left[e^{-3s}\right](t) = \delta_3(t)\,.$$

For the next exercise we have to use partial fraction decomposition.

(12)

$$
\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{2s^2+s+2}{s^3+s^2+2s+2}\right](t) = \mathcal{L}^{-1}\left[\frac{2s^2+s+2}{(s+1)(s^2+2)}\right](t) \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{s}{s^2+2}\right](t) \\
&= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) + \mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right](t) \\
&= e^{-t} + \cos(\sqrt{2}t)\,.
\end{aligned}
$$

6.3.1. Calculate the Laplace transform and inverse Laplace transform using Mathematica.

To calculate the Laplace transform we can use the following commands:

```
LaplaceTransform[Sin[t], t, s]
```

For the inverse Laplace transform we can use:

```
InverseLaplaceTransform[1/(1 + s), s, t]
```

Homework Exercises.

Find the inverse Laplace transforms of the following functions:

(1)
$$Y(s) = s^{-5}.$$

(2)
$$Y(s) = \frac{(s-3)^2}{s^5}.$$

(3)
$$Y(s) = \left(\frac{3}{s} + \frac{1}{s^2}\right)^2.$$

(4)
$$Y(s) = \frac{3}{s-2}.$$

(5)
$$Y(s) = \frac{1}{2s+1}.$$

(6)
$$Y(s) = \frac{5}{s^2+36}.$$

(7)
$$Y(s) = \frac{-3s}{s^2+1}.$$

(8)
$$Y(s) = \frac{s}{4s^2+1}.$$

(9)
$$Y(s) = \frac{2s+4}{s^2+9}.$$

(10)
$$Y(s) = \frac{1}{s^2+2s+10}.$$

(11)

$$Y(s) = \frac{1}{s^2 + 3s - 10}\,.$$

(12)

$$Y(s) = \frac{1}{s^4 + 5s^2 + 6}\,.$$

(13)

$$Y(s) = \frac{3}{(s-2)^4}\,.$$

(14)

$$Y(s) = \frac{s}{(s+1)^2}\,.$$

(15)

$$Y(s) = \frac{1}{s(s+1)^2}\,.$$

(16)

$$Y(s) = \frac{e^{-s}}{s^2}\,.$$

(17)

$$Y(s) = \frac{e^{-2s}}{s^2 + s}\,.$$

(18)

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 4}\,.$$

(19)

$$Y(s) = \frac{s\ e^{-s\pi/4}}{s^2 + 4}\,.$$

(20)

$$Y(s) = \frac{s}{(s^2 + 9)^2}\,.$$

6.4. Solving IVPs of linear DEs with the Laplace transform

Laplace transforms of the derivatives.

If $y(t), y'(t), \cdots, y^{(n-1)}(t)$ are continuous on $[0, +\infty)$, are of exponential order c and $y^{(n)}(t)$ is piecewise continuous on $[0, +\infty)$, then

$$\boxed{\mathcal{L}[y^{(n)}(t)](s) = s^n \mathcal{L}[y(t)](s) - s^{n-1}\, y(0) - s^{n-2}\, y'(0) - \cdots - s\, y^{(n-2)}(0) - y^{(n-1)}(0)\,.}$$

To see how this works, let us start calculating $\mathcal{L}[y'(t)](s)$ using integration by parts. For simplicity, let us work with these improper integrals as with the usual definite integrals, but we should not forget that this is possible, because our assumptions make these improper integrals convergent. Also, by the exponential order c of the function $y(t)$ we know that for $s > c$ we have $\lim_{t\to\infty} e^{-st} y(t) = 0$.

$$\begin{aligned}\mathcal{L}[y'(t)](s) &= \int_0^\infty e^{-ts}\, y'(t)\, dt \\ &= e^{-ts} y(t)\Big|_0^\infty + s \int_0^\infty e^{-ts}\, y(t)\, dt \\ &= -y(0) + s\, \mathcal{L}[y(t)](s)\,.\end{aligned}$$

We can continue to evaluate higher order derivatives in the following way:

$$\begin{aligned}\mathcal{L}[y''(t)](s) &= \int_0^\infty e^{-ts}\, y''(t)\, dt \\ &= e^{-ts} y'(t)\Big|_0^\infty + s \int_0^\infty e^{-ts}\, y'(t)\, dt \\ &= -y'(0) - s y(0) + s^2\, \mathcal{L}[y(t)](s)\,.\end{aligned}$$

You should pay close attention to Example 1. This is an easy exercise, but the more complicated ones follow exactly the same steps.

Example 1.

Solve the IVP

$$y' - 2y = 6\,, \quad y(0) = 1\,.$$

This is a linear DE with constant coefficients, so we can apply the Laplace transform to both sides of the equation:

$$\mathcal{L}[y'(t) - 2y(t)]\ (s) = \mathcal{L}[6]\ (s)\,.$$

By the linearity of the Laplace transform we get

$$\mathcal{L}[y'(t)]\ (s) - 2\mathcal{L}[y(t)]\ (s) = \mathcal{L}[6]\ (s)\,.$$

We use the notation $Y(s) = \mathcal{L}[y(t)]\ (s)$ and by the formula for the transformation of derivatives we get

$$sY(s) - 1 - 2Y(s) = \frac{6}{s}\,.$$

Solving this equation in $Y(s)$ gives:

$$Y(s) = \frac{s+6}{s(s-2)}\,.$$

The partial fraction decomposition of the right hand side gives:

$$Y(s) = \frac{4}{s-2} - \frac{3}{s}\,.$$

Now we apply the inverse Laplace transform to both sides:

$$\mathcal{L}^{-1}[Y(s)]\ (t) = \mathcal{L}^{-1}\left[\frac{4}{s-2} - \frac{3}{s}\right]\ (t)\,.$$

The linearity of the inverse transform gives:

$$y(t) = 4\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]\ (t) - 3\mathcal{L}^{-1}\left[\frac{1}{s}\right]\ (t)\,.$$

Hence, we get the final form of the solution

$$y(t) = 4e^{-2t} - 3\,.$$

Example 2.

Solve the IVP

$$y'' - 2y' + 2y = 0\,,\ y(0) = 1\,,\ y'(0) = 2\,.$$

Using the Laplace transform we get that

$$s^2Y(s) - s - 2 - 2\Big(sY(s) - 1\Big) + 2Y(s) = 0\,.$$

Solving this equation in $Y(s)$ gives

$$Y(s) = \frac{s}{s^2 - 2s + 2}\,.$$

The denominator cannot be factored, so we have to complete the square and then find the inverse Laplace transform.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{s}{s^2-2s+2}\right]\ (t) = \mathcal{L}^{-1}\left[\frac{s}{(s-1)^2+1}\right]\ (t) \\ &= \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right]\ (t) + \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right]\ (t) \\ &= e^t\cos t + e^t \sin t \end{aligned}$$

Example 3.

Let's see what is happening when we solve the same DE, but without the initial conditions. The DE is:

$$y'' - 2y' + 2y = 0\,.$$

For the unspecified initial conditions we use undetermined numbers $y(0) = a$ and $y'(0) = b$. Applying the Laplace transform to the DE leads to

$$s^2Y(s) - as - b - 2\Big(sY(s) - a\Big) + 2Y(s) = 0\,.$$

Solving this equation in $Y(s)$ gives

$$Y(s) = \frac{as + b - 2a}{s^2 - 2s + 2}\,.$$

Therefore,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{as+b-2a}{s^2-2s+2}\right](t) \\ &= \mathcal{L}^{-1}\left[\frac{as-a}{(s-1)^2+1}\right](t) + \mathcal{L}^{-1}\left[\frac{b-a}{(s-1)^2+1}\right](t) \\ &= ae^t\cos t + (b-a)e^t\sin t \end{aligned}$$

Renaming $a = c_1$ and $b - a = c_2$ we get the general solution

$$y(t) = c_1\, e^t \cos t + c_2\, e^t \sin t\,.$$

Example 4.

Solve the IVP $y'' - 4y' + 4y = t^3e^{2t}\,,\ y(0) = 6\,,\ y'(0) = -2.$
Using the Laplace transform we get that

$$s^2Y(s) - 6s + 2 - 4sY(s) + 24 + 4Y(s) = \frac{6}{(s-2)^4}\,.$$

Therefore,

$$(s^2 - 4s + 4)Y(s) = 6s - 26 + \frac{6}{(s-2)^4}\,.$$

Hence,

$$\begin{aligned} Y(s) &= \frac{6s-26}{(s-2)^2} + \frac{6}{(s-2)^6} \\ &= \frac{6(s-2)}{(s-2)^2} - \frac{14}{(s-2)^2} + \frac{6}{(s-2)^6} \\ &= \frac{6}{s-2} - \frac{14}{(s-2)^2} + \frac{6}{(s-2)^6}\,. \end{aligned}$$

By the inverse Laplace transform we get that

$$y(t) = 6e^{2t} - 14te^{2t} + \frac{1}{20}t^5 e^{2t}\,.$$

Example 5.

Solve the DE

$$y'' + y = f(t) = \begin{cases} 0\,, & \text{if} \quad 0 \le t < 1 \\ 3\,, & \text{if} \quad t \ge 1 \end{cases}$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

We can write $f(t) = 3u_1(t)$ and then apply the Laplace transform to the differential equation. We get

$$s^2 Y(s) - 1 + Y(s) = 3\,\frac{e^{-s}}{s}\,,$$

which leads to

$$Y(s) = 3e^{-s}\frac{1}{s(s^2+1)} + \frac{1}{s^2+1}\,.$$

Partial fraction decomposition gives

$$\begin{aligned} Y(s) &= 3e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{s^2+1} \\ &= 3e^{-s}\,\frac{1}{s} - 3e^{-s}\,\frac{s}{s^2+1} + \frac{1}{s^2+1}\,. \end{aligned}$$

The inverse Laplace transform provides now the answer

$$y(t) = 3u_1(t) - 3\cos(t-1)\,u_1(t) + \sin t\,.$$

Example 6.

Solve the IVP

$$y'' + 3y' + 2y = e^t + \delta_5(t)\,,\ y(0) = 1\,,\ y'(0) = 0\,.$$

Applying the Laplace transform gives

$$s^2 Y(s) - s + 3Y(s) - 3 + 2Y(s) = \frac{1}{s-1} + e^{-5s}\,.$$

Hence,

$$Y(s) = \frac{1}{(s-1)(s+1)(s+2)} + \frac{e^{-5s}}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}\,.$$

The partial fraction decompositions give

$$\begin{aligned} Y(s) = &\frac{1/6}{s-1} - \frac{1/2}{s+1} + \frac{1/3}{s+2} \\ &+ \frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s+2} \\ &+ \frac{2}{s+1} - \frac{1}{s+2} \end{aligned}$$

Hence,

$$Y(s) = \frac{1/6}{s-1} + \frac{3/2}{s+1} - \frac{2/3}{s+2} + \frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s+2}\,.$$

The inverse Laplace transform gives

$$y(s) = \frac{1}{6}e^{t} + \frac{3}{2}e^{-t} - \frac{2}{3}e^{-2t} + e^{-(t-5)}u_5(t) - e^{-2(t-5)}u_5(t)\,.$$

6.4.1. Solving differential equations using Mathematica and the Laplace transform.

Let us solve the following IVP:

$$y'' + 3y' + 2y = e^{2t}\,\cos t\,,\ y(0) = 1\,,\ y'(0) = -1\,.$$

First, let's give a name to the DE:

```
diffeq = y''[t] + 3*y'[t] + 2*y[t] == Exp[2*t]*Cos[t]
```

Then, we transform this DE with the Laplace transform:

```
transeq = LaplaceTransform[diffeq, t, s] /.  {y[0] -> 1, y'[0] -> -1,
LaplaceTransform[y[t], t, s] -> Y}
```

Now, we solve the transformed equation

```
sol = Solve[transeq, Y]  ,
```

and name the solution as

```
Z = Y /.  sol
```

The inverse Laplace transform gives now the final solution. The FullSimplify is needed to change the complex exponential form into a real expression.
The solution function can be defined with

```
DEsol[t_]:  = FullSimplify[InverseLaplaceTransform[Z, s, t]]  .
```

The whole process looks like:

```
In[38]:= diffeq = y''[t] + 3*y'[t] + 2*y[t] == Exp[2*t]*Cos[t]
```

Out[38]= $2\,y[t] + 3\,y'[t] + y''[t] == e^{2t}\,\mathrm{Cos}[t]$

```
In[39]:= transeq = LaplaceTransform[diffeq, t, s] /. {y[0] → 1, y'[0] → -1, LaplaceTransform[y[t], t, s] → Y}
```

Out[39]= $1 - s + 2\,Y + s^2\,Y + 3\,(-1 + s\,Y) == \frac{-2 + s}{1 + (-2 + s)^2}$

```
In[40]:= sol = Solve[transeq, Y]
```

Out[40]= $\left\{\left\{Y \to \frac{8 - 2\,s - 2\,s^2 + s^3}{(5 - 4\,s + s^2)\,(2 + 3\,s + s^2)}\right\}\right\}$

```
In[41]:= Z = Y /. sol
```

Out[41]= $\left\{\frac{8 - 2\,s - 2\,s^2 + s^3}{(5 - 4\,s + s^2)\,(2 + 3\,s + s^2)}\right\}$

```
InverseLaplaceTransform[Z, s, t]
```

$\left\{\frac{4\,e^{-2t}}{17} + \frac{7\,e^{-t}}{10} + \left(\frac{1}{340} - \frac{i}{340}\right) e^{(2-i)\,t}\,\left((2 + 9\,i) + (9 + 2\,i)\,e^{2\,i\,t}\right)\right\}$

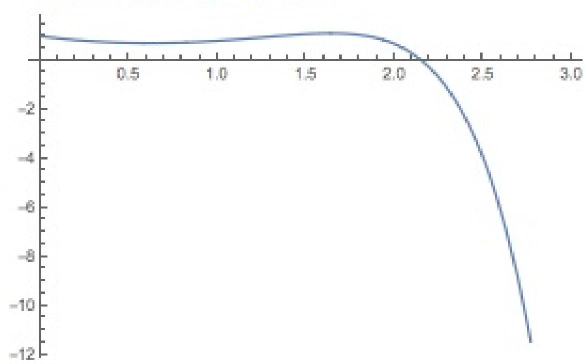
```
In[42]:= FullSimplify[InverseLaplaceTransform[Z, s, t]]
```

Out[42]= $\left\{\frac{1}{170}\,e^{-2t}\,\left(40 + 119\,e^{t} + e^{4t}\,(11\,\mathrm{Cos}[t] + 7\,\mathrm{Sin}[t])\right)\right\}$

```
In[43]:= DEsol[t_] := FullSimplify[InverseLaplaceTransform[Z, s, t]]
```

```
In[44]:= Plot[DEsol[t], {t, 0, 3}]
```

Out[44]=



Homework Exercises:

1. Use the Laplace transform to solve the following IVPs:

1.
$$y' - 2y = 6\,,\ y(0) = 1\,.$$

2.
$$y'' + 5y' + 6y = 0\,,\ y(0) = 0\,,\ y'(0) = 2\,.$$

3.
$$y'' + y = \cos(2t)\,,\ y(0) = 1\,,\ y'(0) = 0\,.$$

4.
$$y'' - 6y' + 8y = 0\,,\ y(0) = 0\,,\ y'(0) = -3\,.$$

5.
$$y'' + 4y = e^{-t}\,,\ y(0) = 0\,,\ y'(0) = 0\,.$$

6.
$$y'' + 3y' + 2y = e^{t} + e^{-t}\,,\ y(0) = 0\,,\ y'(0) = 0\,.$$

7.
$$y'' - 2y' + 2y = 0\,,\ y(0) = 1\,,\ y'(0) = 2\,.$$

8.
$$y'' - y = \begin{cases} 0\,, & \text{if} \quad t < 1 \\ t\,, & \text{if} \quad t \geq 1\,, \end{cases}\,,\ y(0) = 0\,,\ y'(0) = 1\,,$$

9.
$$y''' + 3y'' + 9y' - 13y = 0\,,\ y(0) = 0\,,\ y'(0) = 2\,,\ y''(0) = 10\,.$$

10.
$$y''' + 2y'' - y' - 2y = \sin(3t)\,,\ y(0) = 0\,,\ y'(0) = 0\,,\ y''(0) = 1\,.$$

11.
$$y'' - 8y' + 16y = t^2 e^{4t}\,,\ y(0) = 1\,,\ y'(0) = 0\,.$$

12.
$$y'' - 5y' + 6y = u_1(t)\,,\ y(0) = 0\,,\ y'(0) = 1\,.$$

13.
$$y'' + 9y = \begin{cases} 0\,, & \text{if} \quad t < \pi \\ \sin t\,, & \text{if} \quad t \geq \pi\,, \end{cases}\,,\ y(0) = 1\,,\ y'(0) = 0\,,$$

14.
$$y'' - 3y' + 2y = \delta(t)\,,\ y(0) = 0\,,\ y'(0) = 1\,.$$

15.
$$y' + 5y = \delta_1(t)\,,\ y(0) = 2\,.$$

16.
$$y'' - 2y' = e^{t} + \delta_3(t)\,,\ y(0) = 0\,,\ y'(0) = 0\,.$$

2. The vertical displacement from its natural length of a spring-mass system is described by

$$y''(t) + 3y'(t) + 2y(t) = 5\delta_3(t)\,,$$

where the time t is measured in seconds and the right hand side models a sharp downward blow on the mass of magnitude 5 at $t = 3$ seconds.

Describe the position of the mass after 10 seconds, if the mass is released 0.2m above the equilibrium position.

3. The charge $q(t)$ on the capacitor in an series RLC circuit is given by the DE

$$\frac{1}{8}q'' + 5q' + 500q = E(t)\,,\ q(0) = 0\,C\,,\ q'(0) = 20\,A\,,$$

where

$$E(t) = \begin{cases} 0, & \text{if} \quad 0 \leq t < 10\pi \\ 100(\sin(50t) + \cos(50t)), & \text{if} \quad t \geq 10\pi \end{cases} .$$

Find $q(40)$.

6.5. Solving systems of first order linear differential equations with the Laplace transform

With the Laplace transform we can solve systems of linear differential equations with constant coefficients, too, No extra preparation is needed.
Let's solve the following system of differential equations:

$$\begin{cases} y'(t) &= 2y(t) + 3z(t) \\ z'(t) &= 2y(t) + z(t) \end{cases} \tag{6.5.7}$$

with the initial conditions $y(0) = 1$, $z(0) = 4$.

We apply the Laplace transform to the DEs and for simplicity we write Y and Z instead of $Y(s)$ and $Z(s)$.

$$\begin{cases} sY - 1 &= 2Y + 3Z \\ sZ - 4 &= 2Y + Z\,. \end{cases}$$

By rearranging the terms we get that

$$\begin{cases} (s-2)Y - 3Z &= 1 \\ -2Y + (s-1)Z &= 4\,. \end{cases} \tag{6.5.8}$$

We can eliminate Z by multiplying the first equation by $(s-1)$, the second equation by 3 and then adding them. In this way we get that

$$Y(s) = \frac{s+11}{(s-4)(s+1)}\,.$$

The partial fraction decomposition leads to

$$Y(s) = \frac{3}{s-4} - \frac{2}{s+1}\,,$$

and therefore, by the inverse Laplace transform, we obtain that

$$y(t) = 3e^{4t} - 2e^{-t}\,.$$

Solving the first equation of the system (6.5.8) in Z leads to

$$\begin{aligned} Z =& \frac{(s-2)Y}{3} - \frac{1}{3} = \frac{s-2}{s-4} - \frac{2}{3}\,\frac{s-2}{s+1} - \frac{1}{3} \\ =& 1 + \frac{2}{s-4} - \frac{2}{3}\left(1 - \frac{3}{s+1}\right) - \frac{1}{3} \\ =& \frac{2}{s-4} + \frac{2}{s+1}\,. \end{aligned}$$

So, by the inverse Laplace transform we get that

$$z(t) = 2e^{4t} + 2e^{-t}\,.$$

Therefore, system (6.5.7) has the following pair of solutions

$$\begin{aligned} y(t) &= 3e^{4t} - 2e^{-t} \\ z(t) &= 2e^{4t} + 2e^{-t} \end{aligned}$$

6.5.1. Using Mathematica to solve systems of DEs.

```
In[1]:= DSolve[{y'[t] == 2*y[t] + 3*z[t], z'[t] == 2*y[t] + z[t], y[0] == 1, z[0] == 4},
   {y[t], z[t]}, t]
Out[1]= {{y[t] → e^-t (-2 + 3 e^5 t), z[t] → 2 e^-t (1 + e^5 t)}}
```

Homework Exercises.

1. Solve the following IVPs associated to systems of DEs.

1.
$$\begin{cases} y'(t) &= y(t) - 2z(t) \\ z'(t) &= y(t) + 4z(t) \end{cases}, \quad y(0) = 3\,, \; z(0) = -1\,.$$

2.
$$\begin{cases} y'(t) &= y(t) + z(t) \\ z'(t) &= -y(t) + z(t) \end{cases}, \quad y(0) = 2\,, \; z(0) = 3\,.$$

3.
$$\begin{cases} y'(t) &= y(t) + z(t) \\ z'(t) &= 4y(t) + z(t) \end{cases}, \quad y(0) = 6\,, \; z(0) = 0\,.$$

4.
$$\begin{cases} y'(t) &= 3y(t) - z(t) \\ z'(t) &= 4y(t) - z(t) \end{cases}, \quad y(0) = 0\,, \; z(0) = 1\,.$$

5.
$$\begin{cases} y'(t) &= y(t) + z(t) + 2 \\ z'(t) &= -2y(t) - z(t) - 1 \end{cases}, \quad y(0) = 1\,, \; z(0) = -1\,.$$

6.
$$\begin{cases} y'(t) &= -y(t) + 2z(t) + e^t \\ z'(t) &= -y(t) + z(t) - e^t \end{cases}, \quad y(0) = 0\,, \; z(0) = 0\,.$$

7.
$$\begin{cases} y'(t) &= 3y(t) + 2z(t) + \sin t \\ z'(t) &= -2y(t) - z(t) \end{cases}, \quad y(0) = 0\,, \; z(0) = 1\,.$$

8.

$$\begin{cases} y'(t) &= 2y(t) - z(t) + e^t \\ z'(t) &= 3y(t) - 2z(t) + 4t \end{cases}, \quad y(0) = 1\,,\ z(0) = 2\,.$$

9.

$$\begin{cases} y'(t) &= 2z(t) + 2 \\ z'(t) &= y(t) + 3z(t) + e^{-t} \end{cases}, \quad y(0) = 0\,,\ z(0) = 0\,.$$

10.

$$\begin{cases} y'(t) &= 2y(t) + 2z(t) + e^t \\ z'(t) &= y(t) + 3z(t) + 4t \end{cases}.$$

2. Suppose that we have two tanks with salt water. Fresh water flows into the first tank and is stirred with the existing salt water. The mixture flows into the second tank and after well-stirred, part of the outflow flows back into the first tank. We denote by $y(t)$ and $z(t)$ the amount of salt in the two tanks. Knowing the rate of flows, measured in gallon/hour, suppose that we obtained the following system of DEs:

$$\begin{cases} y'(t) &= -y(t) + 4z(t) \\ z'(t) &= y(t) - z(t) \end{cases}.$$

Find out the salt present in the two tanks after 3 hours, if the initial amounts were $y(0) = 100$ and $z(0) = 200$ pounds.

CHAPTER 7

Appendix: Mathematica files

Derivatives and plots with Mathematica

Define the function:

In[1]:=
```
f[t_] := t / (t^2 - 1)
```

Two options to calculate the derivative:

In[2]:=
```
f'[t]
```

Out[2]= $-\frac{2\,t^2}{\left(-1+t^2\right)^2}+\frac{1}{-1+t^2}$

In[3]:=
```
D[f[t], t]
```

Out[3]= $-\frac{2\,t^2}{\left(-1+t^2\right)^2}+\frac{1}{-1+t^2}$

Simplify the expression if needed:

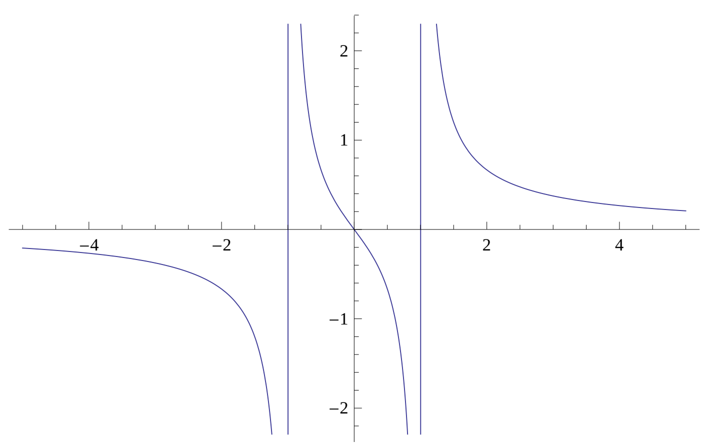
In[4]:=
```
FullSimplify[f'[t]]
```

Out[4]= $-\frac{1+t^2}{\left(-1+t^2\right)^2}$

Graph the function f(t) on the interval [-5,5]:

In[5]:=
```
Plot[f[t], {t, -5, 5}]
```

Out[5]=



Integration with *Mathematica*

Calculate the indefinite integral $\int (t^2 + 1) \sin(t) \, dt$:

In[1]:=
```
Integrate[(t^2 + 1) * Sin[t], t]
```

Out[1]=
```
-Cos[t] - (-2 + t^2) Cos[t] + 2 t Sin[t]
```

Simplify the expression

In[2]:=
```
FullSimplify[Integrate[(t^2 + 1) * Sin[t], t]]
```

Out[2]=
```
Cos[t] - t^2 Cos[t] + 2 t Sin[t]
```

You have to realize that this is the same as $(1-t^2)$ cos(t) + 2t sin(t).
The answer we expect to get for the indefinite integral is $(1-t^2)$ cos(t) + 2t sin(t) + c.

Calculate a definite integral $\int_0^{\pi} (t^2 + 1) \sin(t) \, dt$:

In[3]:=
```
Integrate[(t^2 + 1) * Sin[t], {t, 0, Pi}]
```

Out[3]= $-2 + \pi^2$

If we want a decimal number answer than we can use

In[4]:=
```
NIntegrate[(t^2 + 1) * Sin[t], {t, 0, Pi}]
```

Out[4]=
```
7.8696
```

Analytical solutions of Differential Equations

We will use “DSolve” to get an analytical solution to the DE y’(t) = 2ty(t).

In[1]:=
```
DSolve[y'[t]==2*t*y[t],y[t],t]
```

Out[1]= $\{\{y[t] \to e^{t^2} C[1]\}\}$

The answer corresponds to the one patrameter family of solutions y(t) = c e^{t^2}.

Let’s solve noe the IVP y’(t)=2ty(t), y(1)=2.

In[2]:=
```
DSolve[{y'[t] == 2 * t * y[t], y[1] == 2}, y[t], t]
```

Out[2]= $\{\{y[t] \to 2\, e^{-1+t^2}\}\}$

The answer corresponds to the solution y(t) = 2 $e^{-1} e^{t^2} = \frac{2}{e} e^{t^2}$.

If we want to plot the solution, first we have to define the solution as a function:

In[3]:=
```
sol = DSolve[{y'[t] == 2 * t * y[t], y[1] == 2}, y[t], t]
```

Out[3]= $\{\{y[t] \to 2\, e^{-1+t^2}\}\}$

In[4]:=
```
z[t_] := Evaluate[y[t] /. sol]
```

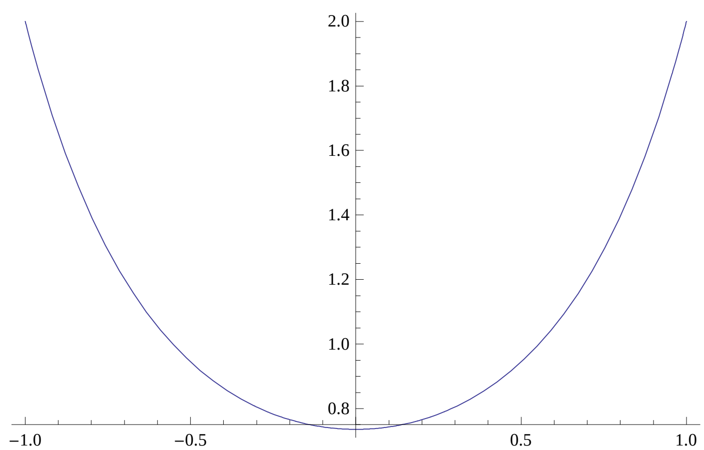
Now, z(t) is the solution function and we can use it for evaluation and graphing:

In[6]:=
```
z[0.1]
```

Out[6]= {0.743153}

In[7]:=
```
Plot[z[t], {t, -1, 1}]
```

Out[7]=

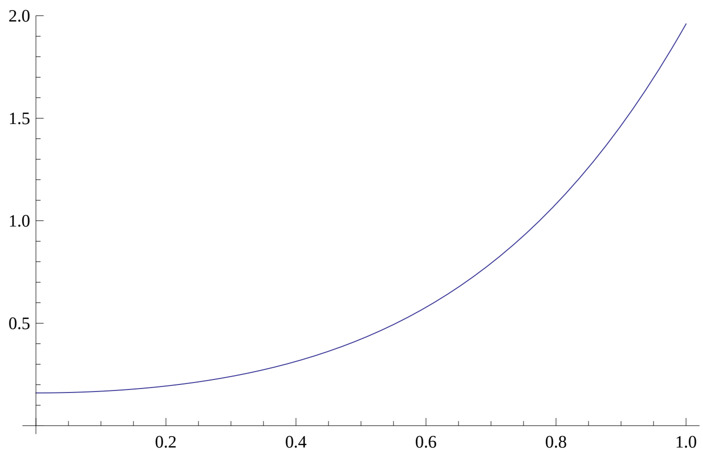


Numerical Solutions of Differential Equations with Mathematica

We will solve numerically the IVP $y'(t)=4t\sqrt{y(t)}$, y(0)=0.16.

```
In[2]:= sol = NDSolve[{y'[t] == 4 * t * Sqrt[y[t]], y[0] == 0.16}, y[t], {t, 0, 1}]

Out[2]= {{y[t] → InterpolatingFunction[{{0., 1.}}, <>][t]}}

In[3]:= q[t_] := Evaluate[y[t] /. sol]

In[4]:= q[0.75]

Out[4]= {0.926402}

In[5]:= Plot[q[t], {t, 0, 1}]
```

Out[5]=



The Laplace transform and inverse Laplace transform of functions.

In[23]:=
```
LaplaceTransform[Sin[t], t, s]
```

Out[23]= $\frac{1}{1+s^2}$

In[25]:=
```
LaplaceTransform[t^2 * Exp[3 * t] + t, t, s]
```

Out[25]= $\frac{2}{(-3+s)^3}+\frac{1}{s^2}$

In[27]:=
```
InverseLaplaceTransform[1 / (1 + s), s, t]
```

Out[27]= e^{-t}

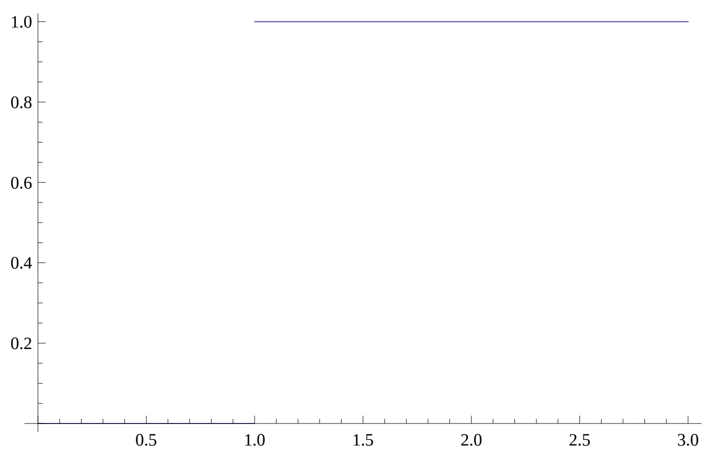
In[28]:=
```
InverseLaplaceTransform[Exp[-s] / s, s, t]
```

Out[28]= HeavisideTheta[-1 + t]

HeavisideTheta is the unit step function, so HeavisideTheta[-1+t]=$u_1(t)$.

In[29]:=
```
Plot[HeavisideTheta[-1 + t], {t, 0, 3}]
```

Out[29]=



In[30]:=
```
InverseLaplaceTransform[1, s, t]
```

Out[30]= DiracDelta[t]

In[31]:=
```
InverseLaplaceTransform[Exp[-3 * s], s, t]
```

Out[31]= DiracDelta[-3 + t]

With our notations DiracDelta[-3+t]=$u_3(t)$.

Solving IVPs with the Laplace transform.

Let's solve the IVP y''+3y'+2y=e^{2t}+cos(t), y(0)=1, y'(0)=-1.

```
In[8]:= diffeq = y''[t] + 3*y'[t] + 2*y[t] == Exp[2*t]*Cos[t]
```

Out[8]= $2\,y[t] + 3\,y'[t] + y''[t] == e^{2t}\,\text{Cos}[t]$

```
In[9]:= transeq = LaplaceTransform[diffeq, t, s] /.
          {y[0] → 1, y'[0] → -1, LaplaceTransform[y[t], t, s] → Y}
```

Out[9]= $1 - s + 2\,Y + s^2\,Y + 3\,(-1 + s\,Y) == \frac{-2 + s}{1 + (-2 + s)^2}$

```
In[10]:= sol = Solve[transeq, Y]
```

Out[10]= $\left\{\left\{Y \to \frac{8 - 2\,s - 2\,s^2 + s^3}{(5 - 4\,s + s^2)\,(2 + 3\,s + s^2)}\right\}\right\}$

```
In[11]:= Z = Y /. sol
```

Out[11]= $\left\{\frac{8 - 2\,s - 2\,s^2 + s^3}{(5 - 4\,s + s^2)\,(2 + 3\,s + s^2)}\right\}$

```
In[12]:= InverseLaplaceTransform[Z, s, t]
```

Out[12]= $\left\{\frac{4\,e^{-2t}}{17} + \frac{7\,e^{-t}}{10} + \left(\frac{1}{340} - \frac{i}{340}\right) e^{(2-i)\,t}\,\left((2 + 9\,i) + (9 + 2\,i)\,e^{2\,i\,t}\right)\right\}$

```
In[13]:= FullSimplify[InverseLaplaceTransform[Z, s, t]]
```

Out[13]= $\left\{\frac{1}{170}\,e^{-2t}\,\left(40 + 119\,e^{t} + e^{4t}\,(11\,\text{Cos}[t] + 7\,\text{Sin}[t])\right)\right\}$

```
In[16]:= DEsol[t_] := FullSimplify[InverseLaplaceTransform[Z, s, t]]
```

```
In[18]:= N[DEsol[1]]
```

Out[18]= {0.803708}