CHAPTER 2 : RELATIONS AND FUNCTIONS-I

In our daily life, we come across many patterns that characterise relations such as brother and sister, father and son, teacher and student etc. In mathematics also, we come across many relations such as number m is greater than number n, line ℓ is perpendicular to line m etc. the concept of relation has been established in mathematical form. The word "function" was introduced by leibnitz in 1694. Function is defiend as a special type of relation. In the present chapter we shall discuss cartesion product of sets, relation between two sets , conditions for a relation to be a function, different types of functions and their properties.

OBJECTIVES

After studying this lesson, you will be able to :

- define cartesion product of two sets.
- define. relation, function and cite examples there of
- find domain and range of a function
- draw graph of functions.
- define and cite examples of even and odd turnotions.
- determine whether a function is odd or even or neither
- define and cite examples of functions like $|x|, [x]$ the greatest integer functions, polynomial functions, logarithmic and exponential functions
- to Find sum. difference, product and quotient of real functions.

EXPECTED BACKGROUND KNOWLEDGE

- concept of ordered pairs.

2.1 CARTESIAN PRODUCT OF TWO SETS

Consider two sets A and B where

$A=\{1, 2\}$, $B=\{3, 4, 5\}$.

Set of all ordered pairs of elements of A and B is $\{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

This set is denoted by $A \times B$ and is called the cartesian product of sets A and B.

i.e. $A\times B = \{(1, 3), (1, 4),(1, 5),(2, 3),(2, 4),(2, 5)\}$

Cartesian product of sets B and A is denoted by $B\times A$.

In the present example, it is given by

$B\times A = \{(3, 1),(3, 2),(4, 1),(4, 2),(5, 1),(5, 2)\}$, Clearly $A\times B \neq B\times A$.

In the set builder form :

$A\times B = \{(a,b) : a \in A \text{ and } b \in B\}$ and $B\times A = \{(b,a) : b \in B \text{ and } a \in A\}$

Note : If $A = \phi$ or $B = \phi$ or $A, B = \phi$

then $A \times B = B \times A = \phi$.

Example 2.1

(1) Let $A=\{a,b,c\}$, $B=\{d,e\}$, $C=\{a,d\}$.

Find (i) $A\times B$ (ii) $B\times A$ (iii) $A\times(B \cup C)$ (iv) $(A \cap C) \times B$

(v) $(A \cap B) \times C$ (vi) $A \times (B - C)$.

Solution : (i) $A\times B = \{(a, d),(a, e), (b, d), (b, e), (c, d), (c, e)\}$.

(ii) $B\times A = \{(d, a),(d, b), (d, c), (e, a) (e, b),(e, c)\}$.

(iii) $A = \{a, b, c\}$, $B \cup C = \{a,d,e\}$.

$\therefore$ $A \times (B \cup C) = \{(a, a),(a, d),(a, e),(b, a),(b, d),(b, e), (c, a),(c, d),(c, e)$.

(iv) $A \cap C = \{a\}$, $B=\{d, e\}$. $\therefore$ $(A \cap C)\times B = \{(a, d), (a, e)\}$

(v) $A \cap B = \phi$, $C=\{a,d\}$, $\therefore$ $A \cap B \times C = \phi$

(vi) $A = \{a,b,c\}$, $B - C = \{e\}$. $\therefore$ $A \times (B - C) = \{(a, e), (b, e), (c, e)\}$.

2.1.1 Number of elements in the Cartesian product of two finite sets

Let A and B be two non-empty sets. We know that $A \times B = \{(a, b); a \in A \text{ and } b \in B\}$

Then number of elements in Cartesian product of two finite sets A and B

i.e. $n(A \times B) = n(A).\ n(B)$

Example 2.2 Suppose $A = \{1,2,3\}$ and $B=\{x, y\}$, show that $n(A \times B) = n(A) \times n(B)$

Solution : Here $n(A) = 3$, $n(B) = 2$

$\therefore$ $A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$

$n(A \times B) = n(A) \times n(B)$, $= 3 \times 2 = 6$

Example 2.3 If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$

Solution : We know that $n(A \times B) = n(A) \times n(B)$

$$n(A \times B) = 5 \times 4 = 20$$

2.1.2 Cartesian product of the set of real numbers R with itself upto R × R × R

Ordered triplet $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$

Here (a, b, c) is called an ordered triplet.

The Cartesian product R × R represents the set $R \times R = \{(x, y) : x, y \in R\}$ which represents the coordinates of all the points in two dimensional plane and the Cartesian product R × R × R represent the set $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ which represents the coordinates of all the points in three dimenstional space.

Example 2.4 If $A = \{1, 2\}$, form the set $A \times A \times A$.

Solution : $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2)$

$(2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

2.2 RELATIONS

Consider the following example :

A={Mohan, Sohan, David, Karim} and B={Rita, Marry, Fatima}

Suppose Rita has two brothers Mohan and Sohan, Marry has one brother David, and Fatima has one brother Karim. If we define a relation R " is a brother of" between the elements of A and B then clearly.

Mohan R Rita, Sohan R Rita, David R Marry, Karim R Fatima.

After omiting R between two names these can be written in the form of ordered pairs as :

(Mohan, Rita), (Sohan, Rita), (David, Marry), (Karima, Fatima).

The above information can also be written in the form of a set R of ordered pairs as

R= {(Mohan, Rita), (Sohan, Rita), (David, Marry), Karim, Fatima}

Clearly $R \subseteq A \times B$, i.e. $R = \{(a, b) : a \in A, b \in B \text{ and } aRb\}$

If A and B are two sets then a relation R from A toB is a sub set of A×B.

If (i) $R = \phi$, R is called a void relation.

(ii) R=A×B, R is called a universal relation.

(iii) If R is a relation defined from A to A, it is called a relation defined on A.

(iv) $R = \{(a, a) \ \forall \ a \in A\}$, is called the identity relation.

2.2.1 Domain and Range of a Relation

If R is a relation between two sets then the set of first elements (components) of all the ordered pairs of R is called Domain and set of 2nd elements of all the ordered pairs of R is called range, of the given relation. In the previous example. Domain = {Mohan, Sohan, David, Karim}, Range = {Rita, Marry, Fatima}

Example 2.5 Given that $A = \{2, 4, 5, 6, 7\}$, $B = \{2, 3\}$.

R is a relation from A to B defined by

$$R = \{(a, b) : a \in A,\ b \in B \text{ and a is divisible by b}\}$$

find (i) R in the roster form (ii) Domain of R (iii) Range of R (iv) Repersent R diagramatically.

Solution : (i) $R = \{(2, 2), (4, 2), (6, 2), (6, 3)\}$

(ii) Domain of R = {2, 4, 6} (iii) Range of R = {2, 3}

(iv)

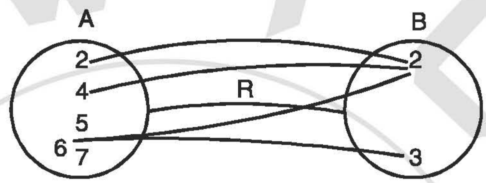


Fig. 2.1

Example 2.6 If R is a relation 'is greater than' from A to B, where

A= {1, 2, 3, 4, 5} and B = {1,2,6}.

Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.

Solution :

(i) $R = \{(2,1), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2)\}$

(ii) Domain of R = {2, 3, 4, 5} (iii) Range of R = {1, 2}

2.2.2 Co-domain of a Relation

If R is a relation from A to B, then B is called codomain of R.

For example, let A = {1, 3, 4, 5, 7} and B = {2, 4, 6, 8} and R be the relation 'is one less than' from A to B, then R = {(1, 2), (3, 4), (5, 6), (7, 8)}

so codomain of R = {2, 4, 6, 8}

Example 2.7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$ and write down the domain, range and codomain of R.

Solution : R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

Domain of A = {1, 2, 3, 4, 5}

Range of R = {2, 3, 4, 5, 6} and Codomain of R = {1, 2, 3, 4, 5, 6}

2.3 DEFINITION OF A FUNCTION

Consider the relation f : $\{(a,1), (b,2), (c,3), (d,5)\}$ from set A = $\{a,b,c,d\}$ to set B = $\{1,2,3,4\}$.

In this relation we see that each element of A has a unique image in B. This relation f from set A to B where every element of A has a unique image in B is defined as a function from A to B. So we observe that ***in a function no two ordered pairs have the same first element.***

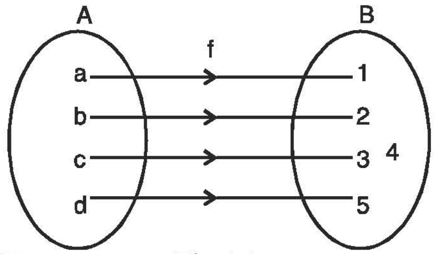


Fig. 2.2

We also see that $\exists$ an element $\in$ B, i.e., 4 which does not have its preimage in A. Thus here:

(i) the set B will be termed as co-domain and (ii) the set $\{1, 2, 3, 5\}$ is called the range. From the above we can conclude that ***range is a subset of co-domain.***

Symbolically, this function can be written as

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

2.3.1 Real valued function of a real variable

A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, then it is called a real function.

> Let R be the set of all real numbers and X, Y be two non-empty subsets of R, then a rule 'f' which associates to each $x \in$ X, a unique element y of Y is called a real valued function of the real variable or simply a real function and we write it as f : X $\to$ Y
>
> A real function 'f' is a rule which associates to each possible real number x, a unique real number $f(x)$.

Example 2.8 Which of the following relations are functions from A to B. Write their domain and range. If it is not a function give reason ?

(a) $\{(1,-2),(3,7),(4,-6),(8,1)\}$, $A=\{1,3,4,8\}$, $B=\{-2,7,-6,1,2\}$

(b) $\{(1,0),(1-1),(2,3),(4,10)\}$, $A=\{1,2,4\}$, $B=\{0,-1,3,10\}$

(c) $\{(a,b),(b,c),(c,b),(d,c)\}$, $A=\{a,b,c,d,e\}$ $B=\{b,c\}$

(d) $\{(2,4),(3,9),(4,16),(5,25),(6,36\}$, $A=\{2,3,4,5,6\}$, $B=\{4,9,16,25,36\}$

(e) $\{(1,-1),(2,-2),(3,-3),(4,-4),(5,-5)\}$, $A=\{0,1,2,3,4,5\}$,
$B=\{-1,-2,-3,-4,-5\}$

(f) $\left\{\left(\sin\frac{\pi}{6},\frac{1}{2}\right),\left(\cos\frac{\pi}{6},\frac{\sqrt{3}}{2}\right),\left(\tan\frac{\pi}{6},\frac{1}{\sqrt{3}}\right),\left(\cot\frac{\pi}{6},\sqrt{3}\right)\right\}$,

$$A=\left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}\quad B=\left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3},1\right\}$$

(g) $\{(a, b), (a, 2), (b, 3), (b, 4)\}$, $A=\{a, b\}$, $B=\{b, 2, 3, 4\}$.

Solution :

(a) It is a function. Domain $=\{1, 3, 4, 8\}$, Range $=\{-2, 7, -6, 1\}$

(b) It is not a function. Because Ist two ordered pairs have same first elements.

(c) It is not a function. Because Domain $=\{a, b, c, d\} \neq A$, Range $=\{b, c\}$

(d) It is a function. Domain $=\{2, 3, 4, 5, 6\}$, Range $=\{4, 9, 16, 25, 36\}$

(e) It is not a function .

Because Domain $=\{1, 2, 3, 4, 5\} \neq A$, Range $=\{-1, -2, -3, -4, -5\}$

(f) It is a function .

Domain $=\left\{\sin\frac{\pi}{6},\cos\frac{\pi}{6},\tan\frac{\pi}{6},\cot\frac{\pi}{6}\right\}$, Range $=\left\{\frac{1}{2},\frac{\sqrt{3}}{2},\frac{1}{\sqrt{3}},\sqrt{3}\right\}$

(g) It is not a function. Because first two ordered pairs have same first component and last two ordered pairs have also same first component.

Example 2.9 State whether each of the following relations represent a function or not.

(a)

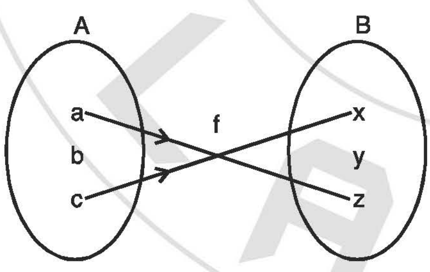


Fig. 2.3

(b)

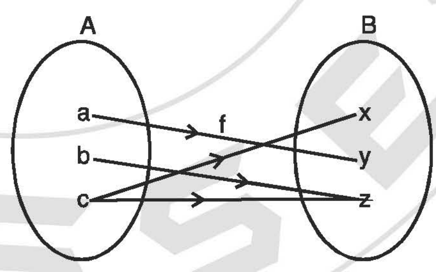


Fig. 2.4

(c)

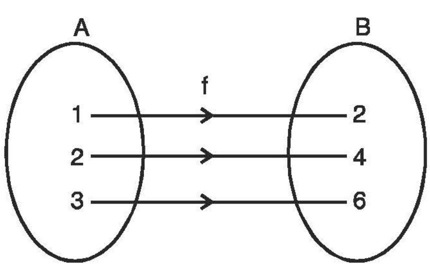


Fig. 2.5

(d)

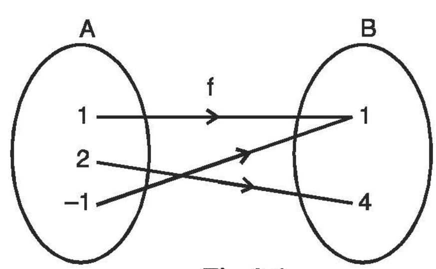


Fig. 2.6

Solution :

(a) f is not a function because the element b of A does not have an image in B.

(b) f is not a function because the element c of A does not have a unique image in B.

(c) f is a function because every element of A has a unique image in B.

(d) f is a function because every element in A has a unique image in B.

Example 2.10 Which of the following relations from $R \rightarrow$. R are functions?

(a) $y = 3x + 2$ (b) $y < x + 3$ (c) $y = 2x^2 + 1$

Solution : (a) $y = 3x + 2$ Here corresponding to every element $x \in R, \exists$ a unique element $y \in R$. $\therefore$ It is a function.

(b) $y < x + 3$.

For any real value of x we get more than one real value of y. $\therefore$ It is not a function.

(c) $y = 2x^2 + 1$

For any real value of x, we will get a unique real value of y. $\therefore$ It is a function.

Example 2.11 Let N be the set of natural numbers. Define a real valued function $f : N \rightarrow N$ by $f(x) = 2x + 1$. Using the definition find $f(1), f(2), f(3), f(4)$

Solution : $f(x) = 2x + 1$

$f(1) = 2 \times 1 + 1 = 2 + 1 = 3,\ f(2) = 2 \times 2 + 1 = 4 + 1 = 5$

$f(3) = 2 \times 3 + 1 = 6 + 1 = 7,\ f(4) = 2 \times 4 + 1 = 6 + 1 = 9$

2.3.2 Some More Examples on Domain and Range

Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 2.12 Find the domain of each of the following functions :

(a) $y = \dfrac{1}{x}$ (b) $y = \dfrac{1}{x - 2}$ (c) $y = \dfrac{1}{(x + 2)(x - 3)}$

Solution : The function $y = \dfrac{1}{x}$ can be described by the following set of ordered pairs.

$$\left\{ \ldots\ldots, \left(-2, -\frac{1}{2}\right), (-1, -1), (1, 1)\left(2, \frac{1}{2}\right), \ldots. \right\}$$

Here we can see that x can take all real values except 0 because the corresponding image, i.e., $\dfrac{1}{0}$ is not defined. $\therefore$ Domain $= R - \{0\}$ [Set of all real numbers except 0]

Note : Here range $= R - \{ 0 \}$

(b) x can take all real values except 2 because the corresponding image, i.e., $\frac{1}{(2-2)}$ does not exist. $\therefore$ Domain $= R - \{2\}$

(c) Value of y does not exist for $x = -2$ and $x = 3$ $\therefore$ Domain $= R - \{-2, 3\}$

Example 2.13 Find domain of each of the following functions :

(a) $y = +\sqrt{x-2}$ (b) $y = +\sqrt{(2-x)(4+x)}$

Solution :(a) Consider the function $y = +\sqrt{x-2}$

In order to have real values of y, we must have $(x-2) \geq 0$ i.e. $x \geq 2$

$\therefore$ Domain of the function will be all real numbers ≥ 2.

(b) $y = +\sqrt{(2-x)(4+x)}$

In order to have real values of y, we must have $(2-x)(4+x) \geq 0$

We can achieve this in the following two cases :

Case I : $(2-x) \geq 0$ and $(4+x) \geq 0$ $\Rightarrow$ $x \leq 2$ and $x \geq -4$

$\therefore$ Domain consists of all real values of x such that $-4 \leq x \leq 2$

Case II : $2 - x \leq 0$ and $4 + x \leq 0$ $\Rightarrow$ $2 \leq x$ and $x \leq -4$.

But, x cannot take any real value which is greater than or equal to 2 and less than or equal to -4.

$\therefore$ From both the cases, we have Domain = $-4 \leq x \leq 2 \; \forall \; x \in R$

Example 2.14 For the function

$f(x) = y = 2x + 1$, find the range when domain $= \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution : For the given values of x, we have

$f(-3) = 2(-3) + 1 = -5$, $f(-2) = 2(-2) + 1 = -3$

$f(-1) = 2(-1) + 1 = -1$, $f(0) = 2(0) + 1 = 1$

$f(1) = 2(1) + 1 = 3$, $f(2) = 2(2) + 1 = 5$, $f(3) = 2(3) + 1 = 7$

The given function can also be written as a set of ordered pairs.

i.e., $\{(-3,-5),(-2,-3),(-1,-1),(0,1)(1,3),(2,5)(3,7)\}$

$\therefore$ Range $= \{-5, -3, -1, 1, 3, 5, 7\}$

Example 2.15 If $f(x) = x + 3$, $0 \leq x \leq 4$, find its range.

Solution : Here $0 \leq x \leq 4$ or $0 + 3 \leq x + 3 \leq 4 + 3$ or $3 \leq f(x) \leq 7$

$\therefore$ Range $= \{f(x) : 3 \leq f(x) \leq 7\}$

Example 2. 16 If $f(x) = x^2, \quad -3 \le x \le 3$, find its range.

Solution : Given $-3 \le x \le 3$ or $0 \le x^2 \le 9$ or $0 \le f(x) \le 9$

$\therefore$ Range $= \{f(x) : 0 \le f(x) \le 9\}$

2.4 GRAPHICAL REPRESENTATION OF FUNCTIONS

Since any function can be represented by ordered pairs, therefore, a graphical representation of the function is always possible. For example, consider $y = x^2$.

x	0	1	–1	2	–2	3	–3	4	–4
y	0	1	1	4	4	9	9	16	16

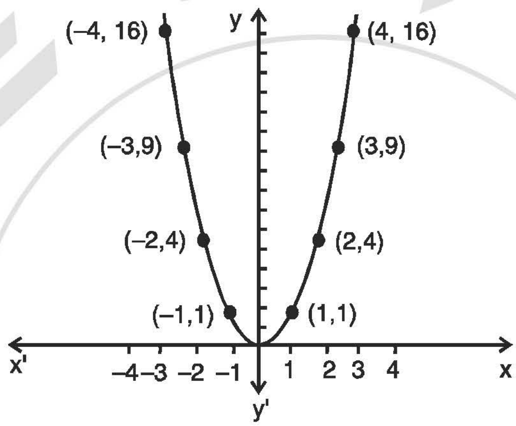


Fig. 2.16

Does this represent a function?

Yes, this represent a function because corresponding to each value of x $\exists$ a unique value of y.

Now consider the equation $x^2 + y^2 = 25$

x	0	0	3	3	4	4	5	–5	–3	–3	–4	–4
y	5	–5	4	–4	3	–3	0	0	4	–4	3	–3

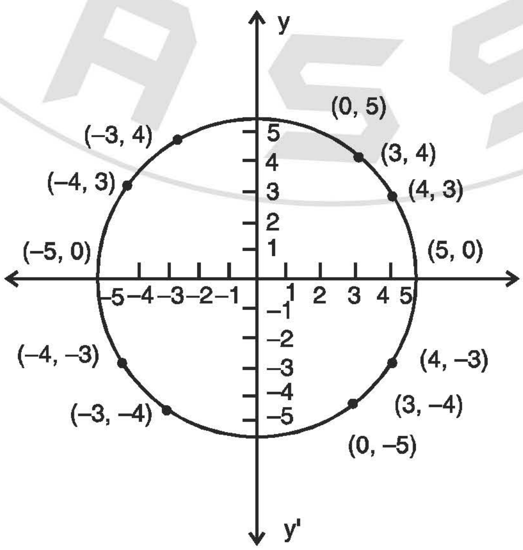


Fig. 2.17

This graph represents a circle. Does it represent a function ?

No, this does not represent a function because corresponding to the same value of x, there does not exist a unique value of y.

2.5 SOME SPECIAL FUNCTIONS

2.5.1 Monotonic Function

Let $F : A \to B$ be a function then F is said to be monotonic on an interval (a,b) if it is either increasing or decreasing on that interval.

For function to be increasing on an interval (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2) \quad \forall \quad x_1 \; x_2 \in (a, b)$$

and for function to be decreasing on (a,b)

$$x_1 < x_2 \Rightarrow F(x_1) > F(x_2) \quad \forall \quad x_1 \; x_2 \in (a, b)$$

A function may not be monotonic on the whole domain but it can be on different intervals of the domain.

Consider the function $F : R \to R$ defined by $f(x) = x^2$.

Now $\quad \forall \; x_1, x_2 \in [0, \infty], \; x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $[0, \infty]$.

($\because$ It is only increasing function on this interval)

But $\quad \forall \; x_1, x_2 \in (-\infty, 0), \; x_1 < x_2 \Rightarrow F(x_1) > F(x_2)$

$\Rightarrow$ F is a ***Monotonic Function*** on $(-\infty, 0)$

($\because$ It is only a decreasing function on this interval)

Therefore if we talk of the whole domain given function is not monotonic on R but it is monotonic on $(-\infty, 0)$ and $(0, \infty)$. Again consider the function $F : R \to R$ defined by $f(x) = x^3$.

Clearly $\forall \; x_1 \; x_2 \in$ domain $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$

$\therefore$ Given function is ***monotonic*** on R i.e. on the whole domain.

2.5.2 Even Function

A function is said to be an even function if for each x of domain $F(-x) = F(x)$

For example, each of the following is an ***even function***.

(i) If $F(x) = x^2$ then $F(-x) = (-x)^2 = x^2 = F(x)$

(ii) If $F(x) = \cos x$ then $F(-x) = \cos(-x) = \cos x = F(x)$

(iii) If $F(x) = |x|$ then $F(-x) = |-x| = |x| = F(x)$

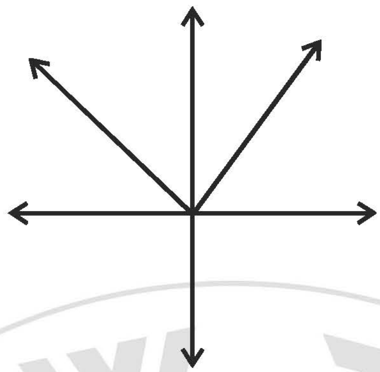


Fig. 2.25

The graph of this even function (modulus function) is shown in the figure above.

Observation Graph is symmetrical about y-axis.

2.5.3 Odd Function

A function is said to be an odd function if for each x

$$f(-x) = -f(x)$$

For example,

(i) If $f(x) = x^3$

then $f(-x) = (-x)^3 = -x^3 = -f(x)$

(ii) If $f(x) = \sin x$

then $f(-x) = \sin(-x) = -\sin x = -f(x)$

Graph of the odd function y = x is given in Fig.2.26

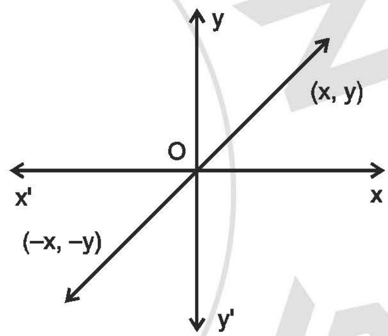


Fig. 2.26

Observation Graph is symmetrical about origin.

2.5.4 Greatest Integer Function (Step Function)

$f(x) = [x]$ which is the greatest integer less than or equal to x.

$f(x)$ is called Greatest Integer Function or Step Function. Its graph is in the form of steps, as shown in Fig. 2.27.

Let us draw the graph of $y = [x], x \in R$

$[x] = 1, \quad 1 \le x < 2$

$[x] = 2, \quad 2 \le x < 3$

$[x] = 3, \quad 3 \le x < 4$

$[x] = 0, \quad 0 \le x < 1$

$[x] = -1, \quad -1 \le x < -0$

$[x] = -2, \quad -2 \le x < -1$

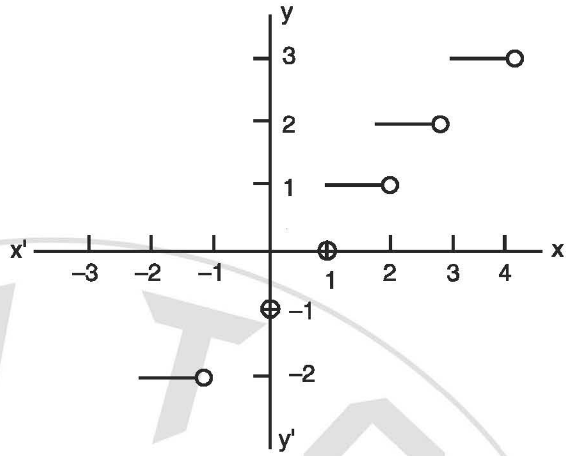


Fig. 2.27

- Domain of the step function is the set of real numbers.
- Range of the step function is the set of integers.

2.5.5 Polynomial Function

Any function defined in the form of a polynomial is called a polynomial function.

For example, (i) $f(x) = 3x^2 - 4x - 2$, (ii) $f(x) = x^3 - 5x^2 - x + 5$, (iii) $f(x) = 3$ are all polynomial functions.

Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function.

2.5.6 Rational Function

Function of the type $f(x) = \dfrac{g(x)}{h(x)}$, where $h(x) \ne 0$ and $g(x)$ and $h(x)$ are polynomial functions are called rational functions.

For example, $f(x) = \dfrac{x^2 - 4}{x + 1}, x \ne -1$ is a rational function.

2.5.7 Reciprocal Function: Functions of the type $y = \dfrac{1}{x}, x \ne 0$ is called a reciprocal function.

2.5.8 Exponential Function A swiss mathematician Leonhard Euler introduced a number e in the form of an infinite series. In fact

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots\ldots + \frac{1}{\lfloor n} + \ldots\ldots \qquad \ldots\ldots(1)$$

It is well known that the sum of this infinite series tends to a finite limit (i.e., this series is convergent) and hence it is a positive real number denoted by e. This number e is a transcendental irrational number and its value lies between 2 an 3.

Now consider the infinite series $1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+.....+\frac{x^n}{\lfloor n}+.....$

It can be shown that the sum of this infinite series also tends to a finite limit, which we denote by

e^x. Thus, $e^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+.....+\frac{x^n}{\lfloor n}+.....$ ......(2)

This is called the **Exponential Theorem** and the infinite series is called the **exponential series**. We easily see that we would get (1) by putting x = 1 in (2).

The function $f(x)=e^x$, where x is any real number is called an **Exponential Function**.

The graph of the exponential function $y=e^x$ is obtained by considering the following important facts :

(i) As x increases, the y values are increasing very rapidly, whereas as x decreases, the y values are getting closer and closer to zero.

(ii) There is no x-intercept, since $e^x \neq 0$ for any value of x.

(iii) The y intercept is 1, since $e^0=1$ and $e \neq 0$.

(iv) The specific points given in the table will serve as guidelines to sketch the graph of e^x (Fig. 2.28).

x	−3	−2	−1	0	1	2	3
$y=e^x$	0.04	0.13	0.36	1.00	2.71	7.38	20.08

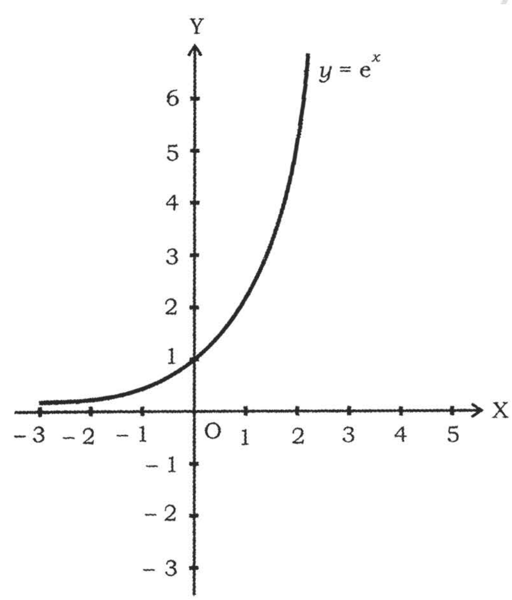


Fig. 2.28

If we take the base different from e, say a, we would get exponential function

$$f(x) = a^x \text{ ,provbided } a > 0, \quad a \neq 1.$$

For example, we may take a = 2 or a = 3 and get the graphs of the functions

$y = 2^x$ (See Fig. 2.29)

and $y = 3^x$ (See Fig. 2.30)

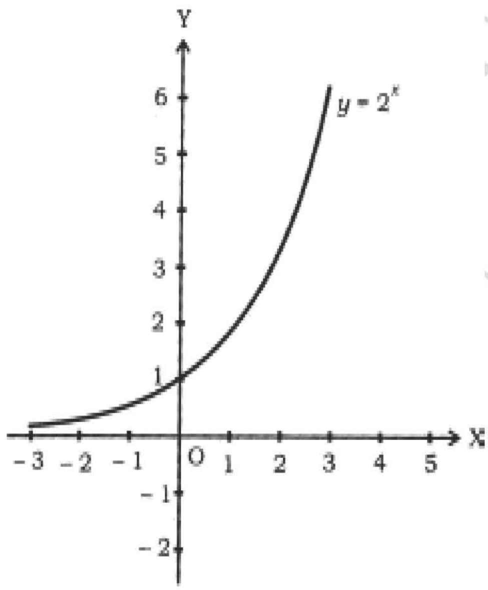


Fig. 2.29

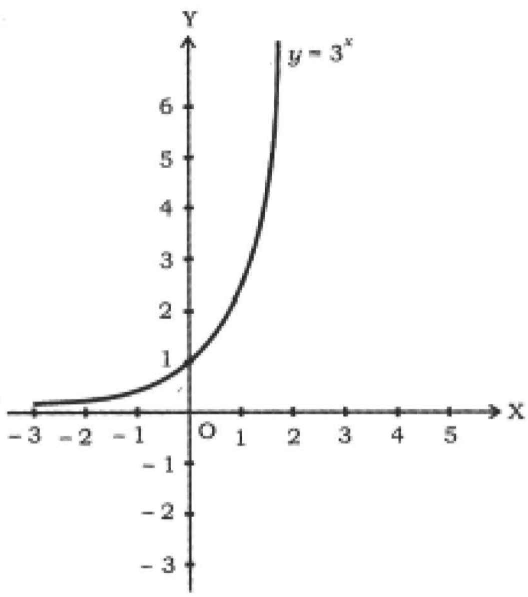


Fig. 2.30

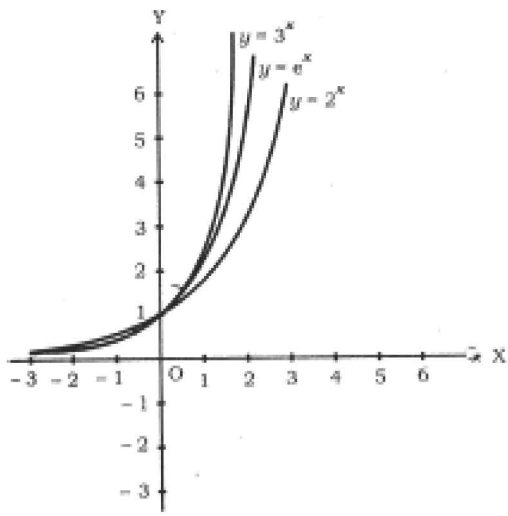


Fig. 2.31

2.5.9 Logarithmic Functions

Now Consider the function $y = e^x$ .....(3)

We write it equivalently as $x = \log_e y$ Thus, $y = \log_e x$ .....(4)

is the inverse function of $y = e^x$

The base of the logarithm is not written if it is e and so $\log_e x$ is usually written as log x.

As $y = e^x$ and $y = \log x$ are inverse functions, their graphs are also symmetric w.r.t. the line $y = x$. The graph of the function $y = \log x$ can be obtained from that of $y = e^x$ by reflecting it in the line y = x.

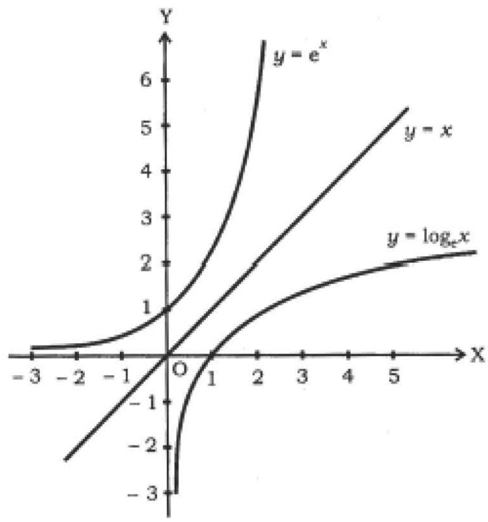


Fig. 2.32

Note

(i) The learner may recall the laws of indices which you have already studied in the Secondary Mathematics : If a > 0, and m and n are any rational numbers, then

$$a^m \cdot a^n = a^{m+n},\ a^m \div a^n = a^{m-n},\ (a^m)^n = a^{mn},\ a^0 = 1$$

(ii) The corresponding laws of logarithms are

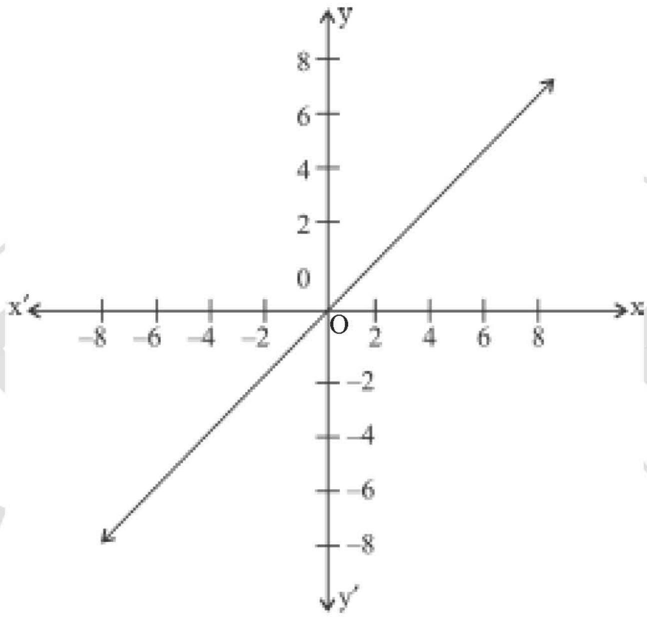
$$\log_a(mn) = \log_a m + \log_a n,\ \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a(m^n) = n\log_a m,\ \log_b m = \frac{\log_a m}{\log_a b} \text{ or } \log_b m = \log_a m \log_b a$$

Here a, b > 0, $a \neq 1$, $b \neq 1$.

2.5.10 Identity Function

Let R be the set of real numbers. Define the real valued function f:R→R by $y = f(x) = x$ for each $x \in$R. Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line. It passes through the origin.

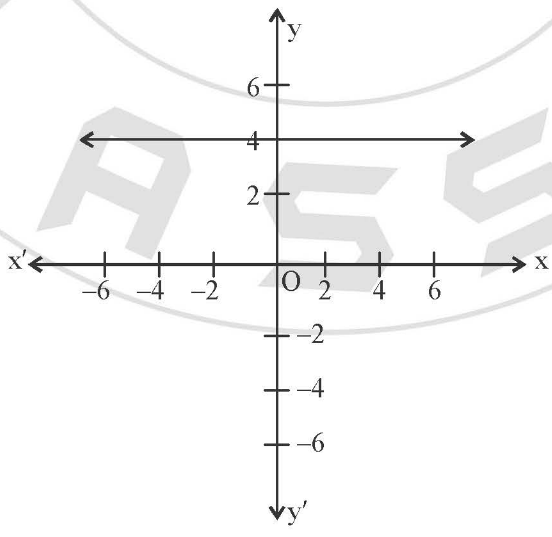


$f(x) = x$ **Fig. 2.33**

2.5.11 Constant Function

Define the function f : R → R by $y = f(x) = c$, $x \in$R where c is a constant and each $x \in$R. Here domain of f is R and its range is $\{c\}$.

The graph is a line parallel to x-axis. For example, $f(x) = 4$ for each $x \in$R, then its graph will be shown as



$f(x) = 4$ **Fig. 2.34**

2.5.12 Signum Function

The function $f : R \to R$ defined by $f(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$ is called a signum function.

The domain of the signum function is R and the range is the set $\{-1, 0, 1\}$.

The graph of the signum function is given as under :

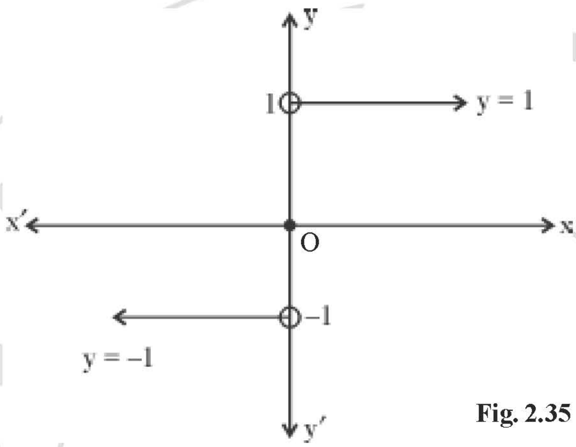


Fig. 2.35

2.6 Sum, difference, product and quotient of functions

(i) Addition of two real functions :

Let $f : X \to R$ and $g : X \to R$ be any two functions, where $X \subset R$. Then, we define $(f + g) : X \to R$ by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$

(ii) **Subtraction of a real function from another :**

Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$. Then, we define $(f - g) : X \to R$ by $(f - g)x = f(x) - g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

then $(f - g)(x) = f(x) - g(x) = x^2 - (2x + 1) = x^2 - 2x - 1$

(iii) **Multiplication of two real functions :**

The product of two real functions $f : X \to R$ and $g : X \to R$ is a function $f g : X \to R$ defined by $(f g)(x) = f(x) \,.\, g(x)$, for all $x \in X$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $f g(x) = f(x) \,.\, g(x) = x^2 . (2x + 1) = 2x^3 + x^2$

(iv) **Quotient of two real functions :**

Let f and g be two real functions defined from $X \to R$ where $X \subset R$. The real quotient of f by g denoted by $\frac{f}{g}$ is a function defined by

$$\left(\frac{f}{g}\right)x = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0,\ x \in X$$

Let $f(x) = x^2$, $g(x) = 2x + 1$

Then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq \frac{-1}{2}$

Example 2. 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f + g)(x)$, $(f - g)(x)$, $(f g)(x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution : We have $f(x) = \sqrt{x}, g(x) = x$

Then $(f + g)(x) = f(x) + g(x) = \sqrt{x} + x$

$(f - g)(x) = f(x) - g(x) = \sqrt{x} - x$

$(f g)(x) = f(x) \, . \, g(x) = \sqrt{x} \, . \, x = x^{3/2}$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = x^{\frac{-1}{2}}, x \neq 0$

LET US SUM UP

- Cartesian product of two sets A and B is the set of all ordered pairs of the elements of A and B. It is denoted by $\times$ i.e $\times$ =$\{(a,b):\ \in\ \ \in\ \}$

- Relation is a sub set of A×B where *A* and *B* are sets.

  i.e. $R \subseteq A \times B = \{(a, b) : a \in A \text{ and } b \in B \text{ and } aRb\}$

- Function is a special type of relation.

- Functrions $f : A \to B$ is a rule of correspondence from *A* to *B* such that to every element of *A* $\exists$ a unique element in *B*.

- Functions can be described as a set of ordered pairs.

- Let *f* be a function from *A* to *B*.

  Domain : Set of all first elements of ordered pairs belonging to *f*.

  Range : Set of all second elements of ordered pairs belonging to *f*.

- Functions can be written in the form of equations such as $y = f(x)$

  where x is independent variable, y is dependent variable.

  Domain : Set of independent variable.

  Range : Set of dependent variable.

  Every equation does not represent a function.

- **Vertical line test :** To check whether a graph is a function or not, we draw a line parallel to y-axis. If this line cuts the graph in more than one point, we say that graph does not represent a function.

- A function is said to be monotonic on an interval if it is either increasing or decreasing on that interval.

- A function is called even function if $f(x) = f(-x)$, and odd function if $f(-x) = -f(x)$, $x, -x \in D_f$

- $f, g : X \to R$ and $X \subset R$, then

  $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$

  $(f \,.\, g)x = f(x) \,.\, g(x)$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided $g(x) \neq 0$.

- A real function has the set of real number or one of its subsets both as its domain and as its range.