CHAPTER 4 : PRINCIPLE OF MATHEMATICAL INDUCTION

In your daily life, you must be using various kinds of reasoning depending on the situation you are faced with. For instance, if you are told that your friend just has a child, you would know that it is either a girl or a boy. In this case, you would be applying general principles to a particular case. This form of reasoning is an example of **deductive** logic.

Now let us consider another situation. When you look around, you find students who study regularly, do well in examinations, you may formulate the general rule (rightly or wrongly) that "any one who studies regularly will do well in examinations". In this case, you would be formulating a general principle (or rule) based on several particular instances. Such reasoning is inductive, a process of reasoning by which general rules are discovered by the observation and consideration of several individual cases. Such reasoning is used in all the sciences, as well as in Mathematics.

Mathematical induction is a more precise form of this process. This precision is required because a statement is accepted to be true mathematically only if it can be shown to be true for each and every case that it refers to.

In the present chapter, first of all we shall introduce you with a statement and then we shall introduce the concept of principale of Mathematical induction, which we shall be using in proving some statements.

OBJECTIVES

After studying this lesson, you will be able to:

- To check whether the given sentence is a statement or not.
- state the Principle of Mathematical Induction;
- verify the truth or otherwise of the statement P(n) for n = 1;
- verify P(k+1) is true, assuming that P(k) is true;
- use principle of mathematical induction to establish the truth or otherwise of mathematical statements;

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.

10.1 WHAT IS A STATEMENT ?

In your daily interactions, you must have made several assertions in the form of sentences. Of these assertions, the ones that are **either true** or **false** are called **statement** or **propositions**. For instance,

"I am 20 years old" and "If $x = 3$, then $x^2 = 9$" are statements, but 'When will you leave ?' And 'How wonderful!' are not statements.

Notice that a statement has to be a definite assertion which can be true or false, but not both. For example, 'x - 5 = 7' is not a statement, because we don't know what x, is. If $x = 12$, it is true, but if $x = 5$, 'it is not true. Therefore, '$x - 5 = 7$' is not accepted by mathematicians as a statement.

But both '$x - 5 = 7 \Rightarrow x = 12$' and $x - 5 = 7$ for any real number x' are statements, the first one true and the second one false.

Example 10.1 Which of the following sentences is a statement ?

(i) India has never had a woman President. , (ii) 5 is an even number.

(iii) $x^n > 1$, (iv) $(a + b)^2 = a^2 + 2ab + b^2$

Solution : (i) and (ii) are statements, (i) being true and (ii) being false. (iii) is not a statement, since we can not determine whether it is true or false, unless we know the range of values that x and n can take.

Now look at (iv). At first glance , you may say that it is not a statement, for the very same reasons that (iii) is not. But look at (iv) carefully. It is true for any value of a and b. It is an identity. Therefore, in this case, even though we have not specified the range of values for a and b, (iv) is a statement.

Some statements, like the one given below are about natural numbers in general. Let us look at the statement :

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

This involves a general natural number n. Let us call this statement P (n) [P stands for proposition].

Then P (1) would be $1 = \frac{1(1+1)}{2}$

Similarly, P (2) would be the statement , $1 + 2 = \frac{2(2+1)}{2}$ and so on.

Let us look at some examples to help you get used to this notation.

Example 10.2 If P (n) denotes $2^n > n-1$, write P (1), P (k) and P $(k+1)$, where $k \in N$.

Solution : Replacing n by 1, k and $k + 1$, respectively in P (n), we get

$P(1): 2^1 > 2 - 1$, i.e., $2 > 1$, $P(k): 2^k > k - 1$

$P(k+1): 2^{k+1} > (k+1) - 1$, i.e., $2^{k+1} > k$

Example 10.3 If P (n) is the statement, '$1 + 4 + 7 + (3n - 2) = \dfrac{n(3n-1)}{2}$

write $P(1)$, $P(k)$ and $P(k+1)$.

Solution : To write $P(1)$, the terms on the left hand side (LHS) of $P(n)$ continue till $3 \times 1 - 2$, i.e., 1. So, P (1) will have only one term in its LHS, i.e., the first term.

Also, the right hand side (RHS)of $P(1) = \dfrac{1 \times (3 \times 1 - 1)}{2} = 1$, Therefore, $P(1)$ is $1 = 1$.

Replacing n by 2, we get

$$P(2): 1 + 4 = \frac{2 \times (3 \times 2 - 1)}{2}, \text{ i.e., } 5 = 5.$$

Replacing n by k and $k + 1$, respectively, we get

$$P(k): 1 + 4 + 7 + ... + (3k - 2) = \frac{k(3k-1)}{2}$$

$$P(k+1): 1 + 4 + 7 + ... + (3k - 2) + [3(k+1) - 2] = \frac{(k+1)[3(k+1)-1]}{2}$$

$$\text{i.e., } 1 + 4 + 7 + ... + (3k + 1) = \frac{(k+1)[(3k+2)}{2}$$

10.2 The Principle of Mathematical Induction:

Let $P(n)$ be a statement involving a natural number n. If

(i) it is true for $n = 1$, i.e., $P(1)$ is true; and

(ii) assuming $k \geq 1$ and $P(k)$ to be true, it can be proved that $P(k+1)$ is true; then $P(n)$ must be true for every natural number n.

***Note** that condition (ii) above **does not** say that P(k) **is** true. It says that **whenever** P(k) is true, then P(k + 1) is true'.*

Let us see, for example, how the principle of mathematical induction allows us to conclude that $P(n)$ is true for $n = 11$.

By (i) $P(1)$ is true. As $P(1)$ is true, we can put $k = 1$ in (ii), So $P(1 + 1)$, i.e., $P(2)$ is true. As $P(2)$ is true, we can put $k = 2$ in (ii) and conclude that $P(2 + 1)$, i.e., $P(3)$ is true. Now put $k = 3$ in (ii), so we get that $P(4)$ is true. It is now clear that if we continue like this, we shall get that $P(11)$ is true.

It is also clear that in the above argument, 11 does not play any special role. We can prove that $P(137)$ is true in the same way. Indeed, it is clear that $P(n)$ is true for all $n > 1$.

Example 10.4 Prove that, $1+2+3+\cdots+n=\frac{n}{2}(n+1)$, where n is a natural number.

Solution: We have , $P(n) : 1 + 2 + 3 + ... + n = \frac{n}{2}(n + 1)$

Therefore, $P(1)$ is '$1 = \frac{1}{2}(1+1)$', which is true,. Therefore, $P(1)$ is true.

Let us now see, is $P(k + 1)$ true whenever $P(k)$ is true.

Let us, therefore, assume that $P(k)$ is true, i.e., $1 + 2 + 3 \ldots + k = \frac{k}{2}(k+1)$ ....(i)

Now, $P(k + 1)$ is $1 + 2 + 3 + \ldots + k + (k + 1) = \frac{(k+1)(k+2)}{2}$

It will be true, if we can show that LHS = RHS

The LHS of $P(k + 1) = (1 + 2 + 3 \ldots + k) + (k + 1) = \frac{k}{2}(k+1) + (k + 1)$ ....[From (i)]

$$= (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2} = \text{RHS of } P(k+1)$$

So, $P(k + 1)$ is true, if we assume that $P(k)$ is true.

Since $P(1)$ is also true, both the conditions of the principle of mathematical induction are fulfilled, we conclude that the given statement is true for every natural number n.

As you can see, we have proved the result in three steps – the **basic step** [i.e., checking (i)], the **Induction step** [i.e., checking (ii)], and hence arriving at the end result.

Example 10.5 For every natural number n, prove that $\left(x^{2n-1}+y^{2n-1}\right)$ is divisible by $(x+y)$, where $x, y \in N$.

Solution: Let us see if we can apply the principle of induction here. Let us call $P(n)$ the statement '$\left(x^{2n-1}+y^{2n-1}\right)$is divisible by $(x+y)$',

Then $P(1)$ is '$\left(x^{2-1}+y^{2-1}\right)$is divisible by $(x+y)$', i.e., '$(x+y)$ is divisible by $(x+y)$', which is true.

Therefore, $P(1)$ is true.

Let us now assume that $P(k)$ is true for some natural number k, i.e., $(x^{2k-1}+y^{2k-1})$ is divisible by $(x+y)$.

This means that for some natural number t, $x^{2k-1}+y^{2k-1}=(x+y)t$

Then, $x^{2k-1}=(x+y)t-y^{2k-1}$

We wish to prove that P $(k+1)$ is true, i.e., '$[x^{2(k+1)-1}+y^{2(k+1)-1}]$is divisible by $(x+y)$' is true.

Now,

$$x^{2(k+1)-1}+y^{2(k+1)-1}=x^{2k+1}+y^{2k+1}$$
$$=x^{2k-1+2}+y^{2k+1}$$
$$=x^{2}\cdot x^{2k-1}+y^{2k+1}$$
$$=x^{2}\cdot[(x+y)t-y^{2k-1}]+y$$
$$=x^{2}(x+y)t-x^{2}y^{2k-1}+y^{2k+1}$$
$$=x^{2}(x+y)t-x^{2}y^{2k-1}+y^{2}y^{2k-1}$$
$$=x^{2}(x+y)t-y^{2k-1}(x^{2}-y^{2})$$
$$=(x+y)[x^{2}t-(x-y)\,y^{2k-1}]$$

which is divisible by $(x+y)$.
Thus, $P\,(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Example 10.6 Prove that $2^{n}>n$ for every natural number n.

Solution: We have $P(n)$: $2^{n}>n$.

Therefore, $P(1)$: $2^{1}>1$, i.e., $2>1$, which is true.

We assume $P(k)$ to be true, that is,

$2^{k}>k$ ... (i)

We wish to prove that $P(k+1)$ is true, i.e. $2^{k+1}>k+1$.

Now, multiplying both sides of (i) by 2, we get, $2^{k+1}>2k$

$\Rightarrow 2^{k+1}>k+1$, since $k>1$. Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number n.

Sometimes, we need to prove a statement for all natural numbers greater than a particular natural number, say a (as in Example 10.7 below). In such a situation, we replace $P(1)$ by $P(a+1)$ in the statement of the principle.

Example 10.7 Prove that

$n^2 > 2(n+1)$ for all $n \geq 3$, where n is a natural number.

Solution: For $n \geq 3$, let us call the given statement, $P(n) : n^2 > 2\,(n+1)$

Since we have to prove the given statement for $n \geq 3$, the first relevant statement is $P(3)$. We, therefore, see whether $P(3)$ is true.

$P(3) : 3^2 > 2 \times 4$, i.e. $9 > 8$. So, $P(3)$ is true.

Let us assume that $P(k)$ is true, where $k \geq 3$, that is , $k^2 > 2(k+1)$ .... (i)

We wish to prove that $P(k+1)$ is true.

$P(k+1) : (k+1)^2 > 2(k+2)$

LHS of $P(k+1) = (k+1)^2 = k^2 + 2k + 1$

$> 2\,(k+1) + 2k + 1$ ... [By (i)]

$> 3 + 2k + 1$, since $2\,(k+1) > 3$. $= 2\,(k+2)$,

Thus, $(k+1)^2 > 2(k+2)$. Therefore, $P(k+1)$ is true.

Hence, by the principle of mathematical induction, the given statement is true for every natural number $n \geq 3$.

Example 10.8 Using principle of mathematical induction, prove that

$\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers n.

Solution : Let $P(n) : \left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ be a natural number.

$\therefore P(1) : \left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$ is a natural number.

or, $\frac{1}{5}+\frac{1}{3}+\frac{7}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1$, which is a natural number $\therefore$ $P(1)$ is true.

Let $P(k): \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)$ is a natural number be true ... (i)

Now $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$

$$=\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\left(\frac{7}{15}k+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$=\left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+\left(k^4+2k^3+3k^2+2k\right)+1 \quad \text{...(ii)}$$

By (i), $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

also $k^4+2k^3+3k^2+2k$ is a natural number and 1 is also a natural number.

$\therefore$ (ii) being sum of natural numbers is a natural number.

$\therefore$ $P(k+1)$ is true, whenever $P(k)$ is true.

$\therefore$ $P(n)$ is true for all natural numbers n.

Hence, $\left(\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\right)$ is a natural number for all natural numbers n.

LET US SUM UP

- Sentences which are either true or false are called statement or propasitions.
- The word induction means, formulating a general principle (or rule) based on several particular instances.
- The statement of the principle of mathematical indction.

  P (n), a statement involving a natural number n, is true for all $n \geq 1$, where n is a fixed natural number, if

  (i) P (1) is true, and

  (ii) whenever P (k) is true, then P (K+1) is true for $k \in n$