CHAPTER 5 : COMPLEX NUMBERS

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We started our study of number systems with the set of natural numbers, then the number zero was included to form the system of whole numbers; negative of natural numbers were defined. Thus, we extended our number system to whole numbers and integers.

To solve the problems of the type $p \div q$ we included rational numbers in the system of integers. The system of rational numbers has been extended further to irrational numbers as all lengths cannot be measured in terms of lengths expressed in rational numbers. Rational and irrational numbers taken together are termed as real numbers. But the system of real numbers is not sufficient to solve all algebraic equations. There are no real numbers which satisfy the equation $x^2+1 = 0$ or $x^2 = -1$. In order to solve such equations, i.e., to find square roots of negative numbers, we extend the system of real numbers to a new system of numbers known as complex numbers. In this lesson the learner will be acquinted with complex numbers, its representation and algebraic operations on complex numbers.

OBJECTIVES

After studying this lesson, you will be able to:

- describe the need for extending the set of real numbers to the set of complex numbers;
- define a complex number and cite examples;
- identify the real and imaginary parts of a complex number;
- state the condition for equality of two complex numbers;
- recognise that there is a unique complex number $x + iy$ associated with the point $P(x, y)$ in the Argand Plane and vice-versa;
- define and find the conjugate of a complex number;
- define and find the modulus and argument of a complex number;
- represent a complex number in the polar form;
- perform algebraic operations (addition, subtraction, multiplication and division) on complex numbers;
- state and use the properties of algebraic operations (closure, commutativity, associativity, identity, inverse and distributivity) of complex numbers; and

- state and use the following properties of complex numbers in solving problems:

  (i) $|z| = 0 \Leftrightarrow z = 0$ and $z_1 = z_2 \Rightarrow |z_1| = |z_2|$

  (ii) $|z| = |-z| = |\bar{z}|$ (iii) $|z_1 + z_2| \le |z_1| + |z_2|$

  (iv) $|z_1 . z_2| = |z_1| . |z_2|$ (v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

- to find the square root of a complex number.

EXPECTED BACKGROUND KNOWLEDGE

- Properties of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of point in a plane.

8.1 COMPLEX NUMBERS

Consider the equation $x^2 + 1 = 0$. ...(A)

This can be written as $x^2 = -1$ or $x = \pm\sqrt{-1}$

But there is no real number which satisfy $x^2 = -1$.In other words, we can say that there is no real number whose square is -1.In order to solve such equations, let us imagine that there exist a number *'i'* which equal to $\sqrt{-1}$.

In 1748, a great mathematician, L. Euler named a number 'i' as *Iota* whose square is -1. This *Iota or 'i'* is defined as imaginary unit. With the introduction of the new symbol *'i'*, we can interpret the square root of a negative number as a product of a real number with i.

Therefore, we can denote the solution of (A) as $x = \pm i$

Thus, $-4 = 4(-1)$

$$\therefore \quad \sqrt{-4} = \sqrt{(-1)(4)} = \sqrt{i^2 . 2^2} = i2$$

Conventionally written as 2i.

So, we have $\sqrt{-4} = 2i$, $\sqrt{-7} = \sqrt{7}i$

$\sqrt{-4}$, $\sqrt{-7}$ are all examples of complex numbers.

Consider another quadratic equation: $x^2 - 6x + 13 = 0$

This can be solved as under:

$$(x-3)^2 + 4 = 0 \text{ or, } (x-3)^2 = -4$$

or, $x - 3 = \pm 2i$ or, $x = 3 \pm 2i$

We get numbers of the form x + yi where x and y are real numbers and $i = \sqrt{-1}$.

Any number which can be expressed in the form a + bi where a,b are real numbers and $i = \sqrt{-1}$, is called a complex number.

A complex number is, generally, denoted by the letter z.

i.e. z = a + bi, 'a' is called the real part of z and is written as Re (a+bi) and 'b' is called the imaginary part of z and is written as Imag (a + bi).

If $a = 0$ and $b \neq 0$, then the complex number becomes bi which is a purely imaginary complex number.

$-7i$, $\frac{1}{2}i$, $\sqrt{3}i$ and πi are all examples of purely imaginary numbers.

If $a \neq 0$ and $b = 0$ then the complex number becomes 'a' which is a real number.

5, 2.5 and $\sqrt{7}$ are all examples of real numbers.

If a = 0 and b = 0, then the complex number becomes 0 (zero). Hence the real numbers are particular cases of complex numbers.

Example 8.1 Simplify each of the following using 'i'.

(i) $\sqrt{-36}$ (ii) $\sqrt{25}.\sqrt{-4}$

Solution: (i) $\sqrt{-36} = \sqrt{36(-1)} = 6i$

(ii) $\sqrt{25}.\sqrt{-4} = 5 \times 2i = 10i$

8.2 POSITIVE INTEGRAL POWERS OF i

We know that

$i^2 = -1,\ i^3 = i^2.i = -1.i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1,\ i^5 = (i^2)^2.i = 1.i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1,\ i^7 = (i^2)^3(i) = -i,\ i^8 = (i^2)^4 = 1$

Thus, we find that any higher powers of 'i' can be expressed in terms of one of four values i, –1, –i, 1

If n is a positive integer such that n>4, then to find i^n, we first divide n by 4.

Let m be the quotient and r be the remainder.

Then n = 4m + r. where $0 \leq r < 4$.

Thus, $i^n = i^{(4m+r)} = i^{4m}.i^r = (i^4)^m.i^r = i^r$ ($\because i^4=1$)

***Note :** For any two real numbers a and b, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ is true only when atleast one of a and b is either 0 or positive.*

If fact $\sqrt{-a}\times\sqrt{-b}$

$= i\sqrt{a}\times i\sqrt{b} = i^2\sqrt{ab}$

$= -\sqrt{ab}$ where a and b are positive real numbers.

Example 8.2 Find the value of $1 + i^{10} + i^{20} + i^{30}$

Solution: $1 + i^{10} + i^{20} + i^{30}$

$= 1 + (i^2)^5 + (i^2)^{10} + (i^2)^{15} = 1 + (-1)^5 + (-1)^{10} + (-1)^{15}$

$= 1 + (-1) + 1 + (-1) = 1 - 1 + 1 - 1 = 0$

Thus, $1 + i^{10} + i^{20} + i^{30} = 0$.

Example 8.3 Express $8i^3 + 6i^{16} - 12i^{11}$ in the form of a + bi

Solution: $8i^3 + 6i^{16} - 12i^{11}$ can be written as $8(i^2).i + 6(i^2)^8 - 12(i^2)^5.i$

$= 8(-1).i + 6(-1)^8 - 12(-1)^5.i = -8i + 6 - 12(-1).i$

$= -8i + 6 + 12i = 6 + 4i$

which is of the form a + bi where 'a' is 6 and 'b' is 4.

8.3 CONJUGATE OF A COMPLEX NUMBER

The complex conjugate (or simply conjugate) of a complex number z = a + bi is defined as the complex number a – bi and is denoted by $\bar{\ }$.

Thus, if z = a + bi then $\bar{z} = a - bi$.

Note : The conjugate of a complex number is obtained by changing the sing of the imaginary part.

Following are some examples of complex conjugates:

(i) If z = 2 + 3i, then $\bar{z} = 2 - 3i$

(ii) If z = 1–i, then $\bar{z} = 1 + i$

(iii) If z = –2 + 10i, then $\bar{z} = -2-10i$

8.3.1 PROPERTIES OF COMPLEX CONJUGATES

(i) If z is a real number then $z = \bar{z}$ i.e., the conjugate of a real number is the number itself.

For example, let $z = 5$

This can be written as $z = 5 + 0i$

$\therefore \quad \bar{z} = 5 - 0i = 5, \quad \therefore \quad z = 5 = \bar{z}$.

(ii) If z is a purely imaginary number then $\bar{z} = -z$

For example, if $z = 3i$. This can be written as $z = 0 + 3i$

$\therefore \quad \bar{z} = 0 - 3i = -3i = -z$

$\therefore \quad \bar{z} = -z$.

(iii) Conjugate of the conjugate of a complex number is the number itself.

i.e., $\overline{(\bar{z})} = z$

For example, if $z = a + bi$ then $\bar{z} = a - bi$

Again $\overline{(\bar{z})} = \overline{(a - bi)} = a + bi = z$

$\therefore \quad \overline{(\bar{z})} = z$

Example 8.4 Find the conjugate of each of the following complex numbers:

(i) $3 - 4i$ (ii) $(2 + i)^2$

Solution : (i) Let $z = 3 - 4i$ then $\bar{z} = \left(\overline{3 - 4i}\right) = 3 + 4i$

Hence, $3 + 4i$ is the conjugate of $3 - 4i$.

(iii) Let $z = (2 + i)^2$

i.e. $z = (2)^2 + (i)^2 + 2(2)(i) = 4 - 1 + 4i = 3 + 4i$

Then $\bar{z} = \left(\overline{3 + 4i}\right) = 3 - 4i$

Hence, $3 - 4i$ is the conjugate of $(2 + i)^2$

8.4 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

Let $z = a + bi$ be a complex number. Let two mutually perpendicular lines xox' and yoy' be taken as x-axis and y-axis respectively, O being the origin.

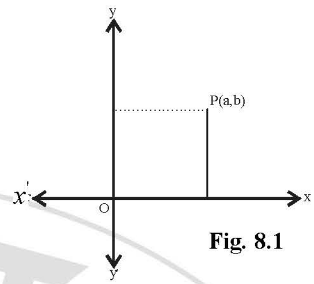

Fig. 8.1

Let P be any point whose coordinates are (a,b). We say that the complex number $z = a + bi$ is represented by the point P (a , b) as shown in Fig. 8.1

If $b = 0$, then z is real and the point representing complex number $z = a + 0i$ is denoted by (a, 0). This point (a, 0) lies on the x-axis.

So, xox' is called the real axis. In the Fig. 8.2 the point Q (a, 0) represent the complex number $z = a + 0i$.

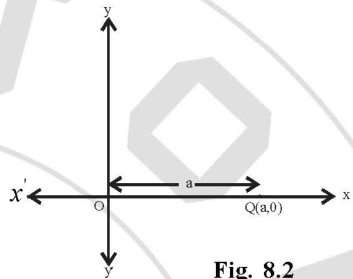

Fig. 8.2

If $a = 0$, then z is purely imaginary and the point representing complex number $z = 0 + bi$ is denoted by (0, b). The point (0, b) lies on the y-axis.

So, yoy' is called the imaginary axis. In Fig. 8.3, the point R (0, b) represents the complex number $z = 0 + bi$.

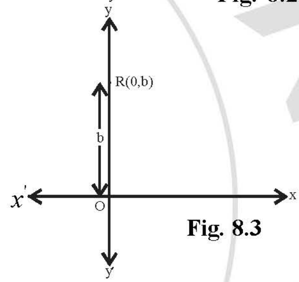

Fig. 8.3

The plane of two axes representing complex numbers as points is called the complex plane or Argand Plane.

The diagram which represents complex number in the Argand Plane is called Argand Diagram.

Example 8.5

Represent complex numbers $2 + 3i, -2-3i, 2-3i$ in the same Argand Plane

Solution: (a) $2+3i$ is represented by the point P (2, 3)

(b) $-2-3i$ is represented by the point Q $(-2,-3)$

(c) $2-3i$ is represented by the point R $(2, -3)$

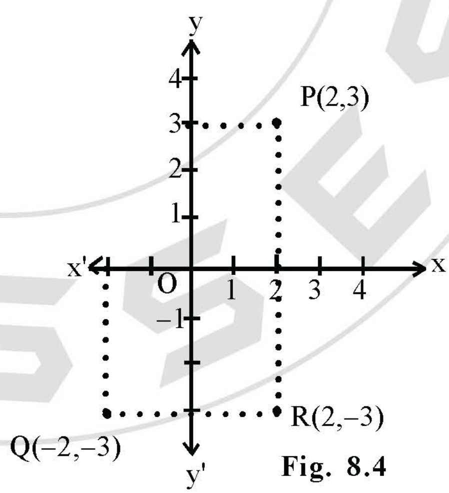

Fig. 8.4

8.5 MODULUS OF A COMPLEX NUMBER

We have learnt that any complex number $z = a + bi$ can be represented by a point in the Argand Plane. How can we find the distance of the point from the origin? Let P(a, b) be a point in the plane representing $a + bi$. Draw perpendiculars PM and PL on x-axis and y-axis respectively. Let OM = a and MP = b. We have to find the distance of P from the origin.

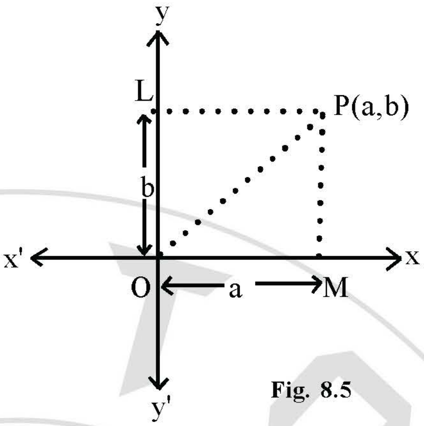

Fig. 8.5

$$\therefore \quad OP = \sqrt{OM^2 + MP^2}$$

$$= \sqrt{a^2 + b^2}$$

OP is called the modulus or absolute value of the complex number $a + bi$.

$\therefore$ Modulus of any complex number z such that $z = a + bi$, $a \in R$, $b \in R$ is denoted by $|z|$ and is given by $\sqrt{a^2 + b^2}$

$\therefore \quad |z| = |a + ib| = \sqrt{a^2 + b^2}$

8.5.1 Properties of Modulus

(a) $|z| = 0 \Leftrightarrow z = 0.$

Proof : Let $z = a + bi$, $a \in R$, $b \in R$

then $|z| = \sqrt{a^2 + b^2}$, $|z| = 0 \Leftrightarrow a^2 + b^2 = 0$

$\Leftrightarrow$ $a = 0$ and $b = 0$ (since a^2 and b^2 both are positive), $\Leftrightarrow z = 0$

(b) $|z| = |\overline{z}|$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

Now, $\overline{z} = a - bi \quad \therefore \quad |\overline{z}| = \sqrt{a^2 + (-b^2)} = \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |\overline{z}|$ ...(i)

(c) $|z| = |-z|$

Proof : Let $z = a + bi$ then $|z| = \sqrt{a^2 + b^2}$

$-z = -a - bi$ then $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Thus, $|z| = \sqrt{a^2 + b^2} = |-z|$ ...(ii)

By (i) and (ii) it can be proved that $|z| = |-z| = |\overline{z}|$ ...(iii)

Now, we consider the following examples:

Example 8.6 Find the modulus of z, –z and $\overline{z}$ where $z = 1 + 2i$

Solution : $z = 1 + 2i$ then $-z = -1 - 2i$ and $\overline{z} = 1 - 2i$

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5}, \quad |-z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

and $$|\overline{z}| = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

Thus, $$|z| = |-z| = \sqrt{5} = |\overline{z}|$$

Example 8.7 Find the modulus of the complex numbers shown in an Argand Plane (Fig. 8.6)

Solution: (i) P(4, 3) represents the complex number $z = 4 + 3i$

$\therefore \quad |z| = \sqrt{4^2 + 3^2} = \sqrt{25}$

or $\quad |z| = 5$

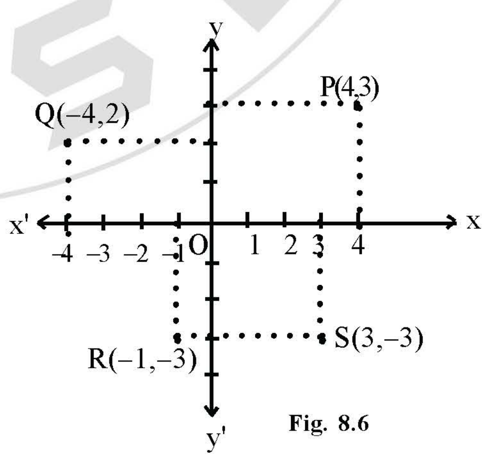

Fig. 8.6

(ii) Q(–4, 2) represents the complex number $z = -4 + 2i$

$\therefore \quad |z| = \sqrt{(-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20}$

or $\quad |z| = 2\sqrt{5}$

$\therefore \quad |z| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{1+9}$

or $\quad |z| = \sqrt{10}$

(iv) S(3, –3) represents the complex number z = 3 –3i

$\therefore \quad |z| = \sqrt{(3)^2 + (-3)^2} = \sqrt{9+9}$

or $\quad |z| = \sqrt{18} = 3\sqrt{2}$

.8.6 EQUALITY OF TWO COMPLEX NUMBERS

Two complex numbers are equal if and only if their real parts and imaginary parts are respectively equal.

In general a + bi = c + di if and only if a = c and b = d.

Example 8.8 For what value of x and y, 5x + 6yi and 10 + 18i are equal?

Solution : It is given that 5 x + 6y i = 10 + 18i

Comparing real and imaginary parts, we have

$5x = 10 \qquad$ or $x = 2$

and $\quad 6y = 18 \qquad$ or $y = 3$

For x = 2 and y = 3, the given complex numbers are equal.

.8.7 ADDITION OF COMPLEX NUMBERS

If $z_1 = a + bi$ and $z_2 = c + di$ are two complex numbers then their sum $z_1 + z_2$ is defined by

$$z_1 + z_2 = (a + c) + (b + d)i$$

For example, if $z_1 = 2 + 3i$ and $z_2 = -4 + 5i$,

then $\quad z_1 + z_2 = [2 + (-4)] + [3 + 5]i = -2 + 8i.$

Example 8.9 Simplify

(i) $(3 + 2i) + (4 - 3i)$ $\qquad$ (ii) $(2 + 5i) + (-3 - 7i) + (1 - i)$

Solution : (i) $(3 + 2i) + (4 - 3i) = (3 + 4) + (2 - 3)i = 7 - i$

(ii) $\quad (2 + 5i) + (-3 - 7i) + (1 - i) = (2 - 3 + 1) + (5 - 7 - 1)\, i = 0 - 3i$

or $\quad (2 + 5i) + (-3 - 7i) + (1 - i) = -3i$

8.7.1 Geometrical Represention of Addition of Two Complex Numbers

Let two complex numbers z_1 and z_2 be represented by the points P(a, b) and Q(c, d).

Their sum, $z_1 + z_2$ is represented by the point R (a + c, b + d) in the same Argand Plane.

Join OP, OQ, OR, PR and QR.

Draw perpendiculars PM, QN, RL from P, Q, R respectvely on X-axis.

Draw perpendicular PK to RL

In ΔQON

$ON = c$

and $QN = d$.

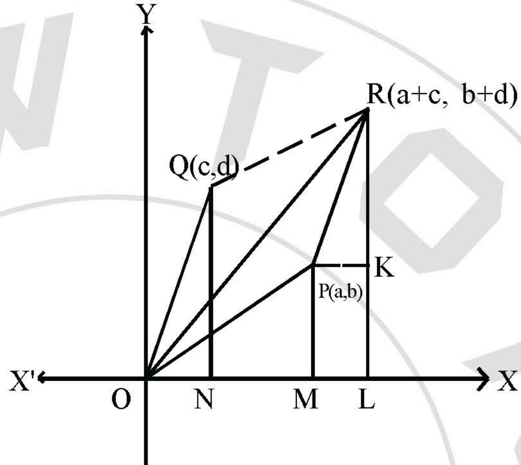

Fig. 8.7

In ΔROL	In ΔPOM
$RL = b + d$	$PM = b$
and $OL = a + c$	$OM = a$

Also $PK = ML = OL - OM$

$= a + c - a = c = ON$

$RK = RL - KL = RL - PM$

$= b + d - b = d = QN$.

In ΔQON and ΔRPK,

$ON = PK$, $QN = RK$ and $\angle QNO = \angle RKP = 90^\circ$

$\therefore$ $\Delta QON \cong \Delta RPK$

$\therefore$ $OQ = PR$ and $OQ \parallel PR$

$\Rightarrow$ OPRQ is a parallelogram and OR its diagonal.

Therefore, we can say that the sum of two complex numbers is represented by the diagonal of a parallelogram.

Example 8.10 Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$

Solution: We have proved that the sum of two complex numbers z_1 and z_2 represented by the diagonal of a parallelogram OPRQ (see fig. 8.8).

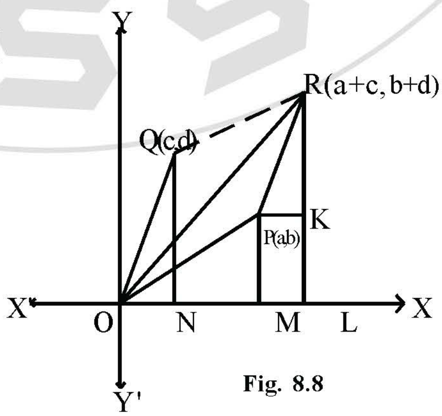

Fig. 8.8

In ΔOPR, $OR \leq OP + PR$

or $OR \leq OP + OQ$ (since $OQ = PR$)

or $|z_1 + z_2| \leq |z_1| + |z_2|$

Example 8.11 If $z_1 = 2 + 3i$ and $z_2 = 1 + i$,

verify that $| z_1 + z_2 | \leq | z_1 | + | z_1 |$

Solution: $z_1 = 2 + 3i$ and $z_2 = 1 + i$ represented by the points (2, 3) and (1, 1) respectively. Their sum $(z_1 + z_2)$ will be represented by the point (2+1, 3+1) i.e. (3, 4)

Verification

$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} = 3.6 \text{ approx.}$$

$$|z_2| = \sqrt{1^2 + 1^2} = \sqrt{2} = 1.41 \text{ approx.}$$

$$|z_1 + z_2| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$| z_1 | + | z_2 | = 3.6 + 1.41 = 5.01$$

$\therefore \quad | z_1 + z_2 | \leq | z_1 | + | z_2 |$

8.7.2 Subtraction of Complex Numbers

Let two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ be represented by the points (a, b) and (c, d) respectively.

$\therefore \quad (z_1) - (z_2) = (a + bi) - (c + di) = (a - c) + (b - d)i$

which represents a point $(a - c, b - d)$

$\therefore$ The difference i.e. $z_1 - z_2$ is represented by the point $(a - c, b - d)$.

Thus, to subtract a complex number from another, we subtract corresponding real and imaginary parts separately.

Example 8.12 Find $z_1 - z_2$ if:

$z_1 = 3 - 4i, \qquad z_2 = -3 + 7i$

Solution: $z_1 - z_2 = (3 - 4i) - (-3 + 7i) = (3 - 4i) + (3 - 7i)$

$= (3 + 3) + (-4 - 7)i = 6 + (-11i) = 6 - 11i$

Examle 8.13 What should be added to i to obtain 5 + 4i?

Solution: Let $z = a + bi$ be added to i to obtain $5 + 4i$

$\therefore \quad i + (a + bi) = 5 + 4i$

or, $\quad a + (b + 1)i = 5 + 4i$

Equating real and imaginary parts, we have

$a = 5$ and $b + 1 = 4$ or $b = 3$, $\therefore$ $z = 5 + 3i$ is to be added to i to obtain $5 + 4i$

8.8 PROPERTIES: WITH RESPECT TO ADDITION OF COMPLEX NUMBERS.

1. Closure : The sum of two complex numbers will always be a complex number.

Let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$, $a_1, b_1, a_2, b_2 \in R$.

Now, $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ which is again a complex nu mber.

This proves the closure property of complex numbers.

2. Commutative : If z_1 and z_2 are two complex numbers then

$z_1 + z_2 = z_2 + z_1$

Let $z_1 = a_1 + b_1i$ and $z_2 = a_2 + b_2i$

Now $z_1 + z_2 = (a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$

$= (a_2 + a_1) + (b_2 + b_1)i$ [commutative property of real numbers]

$= (a_2 + b_2i) + (a_2 + b_1i) = z_2 + z_1$

i.e. $z_1 + z_2 = z_2 + z_2$ Hence, addition of complex numbers is commutative.

3. Associative

If $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i$ and $z_3 = a_3 + b_3i$ are three complex numbers, then

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

Now $z_1 + (z_2 + z_3) = (a_1 + b_1i) + \{(a_2 + b_2i) + (a_3 + b_3i)\}$

$= (a_1 + b_1i) + \{(a_2 + a_3) + (b_2 + b_3)i\} = \{a_1 + (a_2 + a_3)\} + \{b_1 + (b_2 + b_3)\}i$

$= \{(a_1 + a_2) + (b_1 + b_2)i\} + (a_3 + b_3i) = \{(a_1 + b_1i) + (a_2 + b_2i)\} + (a_3 + b_3i)$

$= (z_1 + z_2) + z_3$

Hence, the associativity property holds good in the case of addition of complex numbers.

4. Existence of Additive Identitiy

if $z = a + bi$ is any complex number, then $(a + bi) + (0 + 0i) = a + bi$

i.e. $(0 + 0i)$ is called the additive identity for $a + ib$.

5. Existence of Additive Inverse

For every complex number $a + bi$ there exists a unique complex number $-a - bi$ such that $(a + bi) + (-a - bi) = 0 + 0i$. $-a - ib$ is called the additive inverse of $a + ib$.

In general, additive inverse of a complex number is obtained by changing the signs of real and imaginary parts.

8.9 ARGUMENT OF A COMPLEX NUMBER

Let P(a, b) represent the complex number $z = a + bi$, $a \in R$, $b \in R$, and OP makes an angle θ with the positive direction of x-axis. Draw $PM \perp OX$, Let $OP = r$

In right Δ OMP, OM = a, MP = b

$\therefore \quad r\cos\theta = a,\ r\sin\theta = b$

Then $z = a + bi$ can be written as $z = r(\cos\theta + i\sin\theta)$ ...(i)

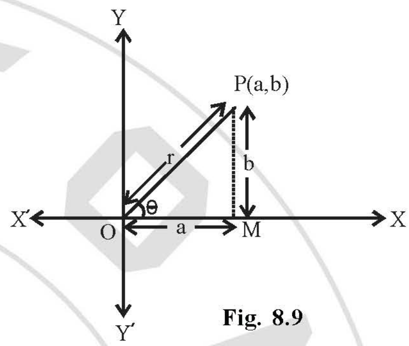

Fig. 8.9

where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$ or $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

(i) is known as the polar form of the complex number z, and r and θ are respectively called the modulus and argument of the complex number.

8.10 MULTIPLICATION OF TWO COMPLEX NUMBERS

Two complex numbers can be multiplied by the usual laws of addition and multiplication as is done in the case of numbers.

Let $\quad z_1 = (a + bi)$ and $z_2 = (c + di)$ then, $z_1.z_2 = (a + bi).(c + di)$

$= a(c + di) + bi(c + di)$

or $\quad = ac + adi + bci + bdi^2$

or $\quad = (ac - bd) + (ad + bc)i. \qquad$ [since $i^2 = -1$]

If $(a + bi)$ and $(c + di)$ are two complex numbers, their product is defined as the complex number $(ac - bd) + (ad + bc)i$

Example 8.14 Evaluate: $(1 + 2i)(1 - 3i)$

Solution:

$$(1 + 2i)(1 - 3i) = \{1 - (-6)\} + (-3 + 2)i = 7 - i$$

8.10.1 Prove that

$$|z_1.z_2| = |z_1|.|z_2|$$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\therefore \quad |z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$.

Now, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)i]$

$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

[Since $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_1 - \sin\theta_1\sin\theta_2$ and $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$]

$$|z_1.z_2| = r_1r_2\sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)} = r_1r_2$$

$$\therefore \quad |z_1.z_2| = r_1r_2 = |z_1|.|z_2|$$

and argument of $z_1z_2 = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

Example 8.15 Find the modulus of the complex number $(1 + i)(4 - 3i)$

Solution: Let $z = (1 + i)(4 - 3i)$

then $|z| = |(1 + i)(4 - 3i)|$

$= |(1 + i)|.|(4 - 3i)|$ (since $|z_1z_2| = |z_1|.|z_2|$)

But $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $|4 - 3i| = \sqrt{4^2 + (-3)^2} = 5$

$$\therefore \quad |z| = \sqrt{2}.5 = 5\sqrt{2}$$

8.11 DIVISION OF TWO COMPLEX NUMBERS

Division of complex numbers involves multiplying both numerator and denominator with the conjugate of the denominator. We will explain it through an example.

Let $z_1 = a + bi$ **and** $z_2 = c + di$**, then.**

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di}\left(c+di \neq 0\right)$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)\,(c-di)}{(c+di)(c-di)}$$

(multiplying numerator and denominator with the conjugate of the denominator)

$$= \frac{\left(ac+bd\right)+\left(bc-ad\right)i}{c^2+d^2}$$

Thus, $$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 8.16 Divide 3+i by 4–2i

Solution: $$\frac{3+i}{4-2i} = \frac{\left(3+i\right)\left(4+2i\right)}{\left(4-2i\right)\left(4+2i\right)}$$

Multiplying numerator and denominator by the conjugate of $(4-2i)$ we get

$$= \frac{10+10i}{20} = \frac{1}{2} + \frac{1}{2}i$$

Thus, $$\frac{3+i}{4-2i} = \frac{1}{2} + \frac{1}{2}i$$

8.11.1 Prove that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof: $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$|z_1| = r_1\sqrt{\cos^2\theta_1 + \sin^2\theta_1} = r_1$$

Similarly, $|z_2| = r_2$

and $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

Then, $$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$=\frac{r_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{r_2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$$=\frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2-i\cos\theta_1\sin\theta_2+i\sin\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)}{(\cos^2\theta_2+\sin^2\theta_2)}$$

$$=\frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)\right]$$

$$=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

Thus, $=\left|\frac{z_1}{z_2}\right|=\frac{r_1}{r_2}\sqrt{\cos^2(\theta_1-\theta_2)+\sin^2(\theta_1-\theta_2)}=\frac{r_1}{r_2}=\frac{|z_1|}{|z_2|}$

$\therefore$ Argument of $\left|\frac{z_1}{z_2}\right|=\theta_1-\theta_2=arg(z_1)-arg(z_2)$

Example 8.17 Find the modulus of the complex number $\frac{2+i}{3-i}$

Solution : Let $z=\frac{2+i}{3-i}$

$\therefore$ $|z|=\left|\frac{2+i}{3-i}\right|=\frac{|2+i|}{|3-i|}\left(\text{since}\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\right)$

$$=\frac{\sqrt{2^2+1^2}}{\sqrt{3^2+(-1)^2}}=\frac{\sqrt{5}}{\sqrt{10}}=\frac{1}{\sqrt{2}}\quad\therefore\quad |z|=\frac{1}{\sqrt{2}}$$

8.12 PROPERTIES OF MULTIPLICATION OF TWO COMPLEX NUMBERS

1. Closure If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then their product z_1z_2 is also a complex number.

2. Cummutative If $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers then $z_1z_2 = z_2z_1$.

3. Associativity If $z_1 = (a + bi)$, $z_2 = c + di$ and $z_3 = (e + fi)$ then

$$z_1(z_2.z_3) = (z_1.z_3).z_3$$

4. Existence of Multiplicative Identity: For every non-zero complex number $z_1 = a + bi$ there exists a unique complex number $(1 + 0i)$ such that

$$(a + bi).(1 + 0i) = (1 + 0i)(a + bi) = a + bi$$

Let $z_1 = x + yi$ be the multipicative identity of $z = a + bi$ Then $z.z_1 = z$.

i.e. $(a + bi)(x + yi) = a + bi$

or $(ax - by) + (ay + bx)i = a + bi$

or $ax - by = a$ and $ay + bx = b$

pr $x = 1$ and $y = 0$, i.e. $z_1 = x + yi = 1 + 0i$ is the multiplicative identity.

The complex number $1 + 0i$ is the identity for multiplication.

5. Existence of Multiplicative inverse: Multiplicative inverse is a complex number that when multiplied to a given non-zero complex munber yields one. In other words, for every non-zero complex number $z = a + bi$, there exists a unique complex number $(x + yi)$ such that their product is $(1 + 0i)$. i.e. $(a + bi)(x + yi) = 1 + 0i$ or $(ax - by) + (bx + ay)i = 1 + 0i$

Equating real and imaging parts, we have

$$ax - by = 1 \text{ and } bx + ay = 0$$

By cross multiplication

$$\frac{x}{a} = \frac{y}{-b} = \frac{1}{a^2 + b^2} \Rightarrow \quad x = \frac{a}{a^2 + b^2} = \frac{\operatorname{Re}(z)}{|z|^2} \text{ and } y = \frac{-b}{a^2 + b^2} = -\frac{\operatorname{Im}(z)}{|z|^2}$$

Thus, the multiplicative inverse of a non-zero compelx number $z = (a + bi)$ is

$$x + yi = \left(\frac{\operatorname{Re}(z)}{|z|^2} - \frac{\operatorname{Im}(z)}{|z|^2} i\right) = \frac{\bar{z}}{|z|^2}$$

Example 8. 18 Find the multiplication inverse of $2-4i$.

Solution: Let $z = 2 - 4i$ **We have,** $\bar{z} = 2 + 4i$ and $|z|^2 = |2^2 + (-4)^2| = 20$

$\therefore$ Required multiplicative inverse is $\dfrac{\bar{z}}{|z|^2} = \dfrac{2+4i}{20} = \dfrac{1}{10} + \dfrac{1}{5}i$

6. Distributive Property of Multiplication over Addition

Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$

Then $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

8.13 SQUARE ROOT OF A COMPLEX NUMBER

Let $a + ib$ be a complex number and $x + iy$ be its square root

i.e., $\sqrt{a+ib} = x + iy$

$\Rightarrow$ $a + ib = x^2 - y^2 + 2ixy$

Equating real and imaginary parts we have

$x^2 - y^2 = a$ ....(1)

and $2xy = b$ ...(2)

Using the algebraic identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$ we get

$(x^2 + y^2)^2 = a^2 + b^2 \Rightarrow x^2 + y^2 = \sqrt{a^2+b^2}$ ...(3)

From equations (1) and (3), we get

$$\left.\begin{aligned} 2x^2 &= \sqrt{a^2+b^2} + a \Rightarrow x = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)} \\ \text{and } 2y^2 &= \sqrt{a^2+b^2} - a \Rightarrow y = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)} \end{aligned}\right\} \quad \text{...(4)}$$

Out of these four pairs of values of x and y (given by equation (4) we have to choose the values which satisfy (1) and (2) both.

From (2) if b is + ve then both x and y should be of same sign and in that case

$$\sqrt{a+ib}=\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)}+i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$$

and $$-\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)}-i\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}-a)}$$

and if b is –ve then x and y should be of opposite sign. Therefore in that case

$$\sqrt{a+ib}=-\sqrt{\frac{1}{2}(\sqrt{a^2+b^2}+a)}+i\sqrt{\frac{1}{2}(\sqrt{b^2+b^2}-a)}$$

and $$\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}+a\right)}-i\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)}$$

Hence $a + ib$ has two square roots in each case and the two square roots just differ in sign.

Example 8.19 Find the square root of $7 + 24i$

Solution : Let $\sqrt{7+24i}=a+ib$ ...(1)

Squaring both sides, we get $7 + 24i = a^2 - b^2 + 2\,iab$

Comparing real and imaginary parts, we have $a^2 - b^2 = 7$ ...(2)

and $2\,ab = 24 \Rightarrow ab = 12$ ....(3)

Now $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$

$\Rightarrow \quad (a^2+b^2)^2=49+4\times 144$

$\Rightarrow \quad (a^2+b^2)^2=625$

$\Rightarrow \quad a^2 + b^2 = 25$ ...(4)

Solving (2) and (4), we get $2a^2 = 32 \Rightarrow a^2 = 16 \Rightarrow a = \pm 4$

and $2b^2 = 18 \Rightarrow b^2 = 9 \Rightarrow b = \pm 3$

From (3), $ab = 12$ which is +ve $\Rightarrow$ a and b should be of same sign

$\therefore$ Either $a = 4,\ b = 3$ or $= -4,\ b = -3$

Hence, the two square roots of $7 + 24i$ are $4 + 3i$ and $-4 - 3i$

Example 8.20 Find the square root of $-i$

Solution : Let $\sqrt{-i} = a + ib$

$\Rightarrow \quad -i = a^2 - b^2 + 2\,iab$ ...(1)

Equating real and imaginary parts of (1), we get $a^2 - b^2 = 0$ ...(2)

and $2\,ab = -1 \Rightarrow ab = -\dfrac{1}{2}$ ...(3)

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 = 0 + 4\left(\dfrac{1}{4}\right) = 1$

$\Rightarrow \quad a^2 + b^2 = 1$ ...(4)

From (2) and (4), $2b^2 = 1 \Rightarrow b^2 = \dfrac{1}{2} \Rightarrow b = \pm\dfrac{1}{\sqrt{2}}$

and $2a^2 = 1 \Rightarrow a^2 = \dfrac{1}{2} \Rightarrow a = \pm\dfrac{1}{\sqrt{2}}$.

Equation (3) suggests; that a and b should be of opposite sign therefore two square roots of $-i$ are $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$ and $-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

LET US SUM UP

- $z = a + bi$ is a complex number in the standard form where $a, b \in R$ and $i = \sqrt{-1}$.
- Any higher powers of 'i' can be expressed in terms of one of the four values i, –1, –i, 1.
- Conjugate of a complex number $z = a + bi$ is $a - bi$ and is denoted by $\bar{z}$.
- Modulus of a complex number $z = a + bi$ is $\sqrt{a^2 + b^2}$ i.e. $|z| = |a + bi| = \sqrt{a^2 + b^2}$

  (a) $|z| = 0 \Leftrightarrow z = 0$ (b) $|z| = |\bar{z}|$ (c) $|z_1 + z_2| \le |z_1| + |z_2|$
- $z = r(\cos\theta + i\sin\theta)$ represents the polar form of a complex number $z = a + bi$ where

  $r = \sqrt{a^2 + b^2}$ is modulus and $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$ is its argument.
- Multiplicative inverse of a complex number $z = a + bi$ is $\dfrac{\bar{z}}{|z|^2}$
- Square root of a complex number is also a complex number.
- Two square roots of a complex number only differ in sign.