CHAPTER 7 : PERMUTATIONS AND COMBINATIONS

One day, I wanted to travel from Bangalore to Allahabad by train. There is no direct train from Bangalore to Allahabad, but there are trains from Bangalore to Itarsi and from Itarsi to Allahabad. From the railway timetable I found that there are two trains from Bangalore to Itarsi and three trains from Itarsi to Allahabad. Now, in how many ways can I travel from Bangalore to Allahabad?

There are **counting problems** which come under the branch of Mathematics called **combinatorics.**

Suppose you have five jars of spices that you want to arrange on a shelf in your kitchen. You would like to arrange the jars, say three of them, that you will be using often in a more accessible position and the remaining two jars in a less accessible position. In this situation the order of jars is important. In how many ways can you do it?

In another situation suppose you are painting your house. If a particular shade or colour is not available, you may be able to create it by mixing different colours and shades. While creating new colours this way, the order of mixing is not important. It is the combination or choice of colours that determine the new colours; but not the order of mixing.

To give another similar example, when you go for a journey, you may not take all your dresses with you. You may have 4 sets of shirts and trousers, but you may take only 2 sets. In such a case you are choosing 2 out of 4 sets and the order of choosing the sets doesn't matter. In these examples, we need to find out the number of choices in which it can be done.

In this lesson we shall consider simple counting methods and use them in solving such simple counting problems.

OBJECTIVES

After studying this lesson, you will be able to :

- find out the number of ways in which a given number of objects can be arranged;
- state the Fundamental Principle of Counting;
- define n! and evaluate it for defferent values of n;
- state that permutation is an arrangement and write the meaning of ${}^nP_r$;
- state that ${}^nP_r = \dfrac{n!}{(n-r)!}$ and apply this to solve problems;

- show that $(i)\ (n+1)\,{}^nP_n = {}^{n+1}P_n$ $\quad (ii)\ {}^nP_{r+1} = (n-r)\,{}^nP_r$;
- state that a combination is a selection and write the meaning of ${}^nC_r$;
- distinguish between permutations and combinations;
- derive ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ and apply the result to solve problems;
- derive the relation ${}^nP_r = r!\,{}^nC_r$;
- verify that ${}^nC_r = {}^nC_{n-r}$ and give its interpretation; and
- derive ${}^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$ and apply the result to solve problems.

EXPECTED BACKGROUND KNOWLEDGE

- Number Systems
- Four Fundamental Operations

11.1 FUNDAMENTAL PRINCIPLE OF COUNTING

Let us now solve the problem mentioned in the introduction. We will write t_1, t_2 to denote trains from Bangalore to Itarsi and T_1, T_2, T_3, for the trains from Itarsi to Allahabad. Suppose I take t_1 to travel from Bangalore to Itarsi. Then from Itarsi I can take T_1 or T_2 or T_3. So the possibilities are t_1T_1, t_1T_2 and t_1T_3 where t_1T_1 denotes travel from Bangalore to Itarsi by t_1 and travel from Itarsi to Allahabad by T_1. Similarly, if I take t_2 to travel from Bangalore to Itarsi, then the possibilities are t_2T_1, t_2T_2 and t_2T_3. Thus, in all there are $6\,(2\times 3)$ possible ways of travelling from Bangalore to Allahabad.

Here we had a small number of trains and thus could list all possibilities. Had there been 10 trains from Bangalore to Itarsi and 15 trains from Itarsi to Allahabad, the task would have been very tedious. Here the **Fundamental Principle of Counting** or simply the **Counting Principle** comes in use :

If any event can occur in *m* ways and after it happens in any one of these ways, a second event can occur in *n* ways, then both the events together can occur in $m\times n$ ways.

Example 11.1 How many multiples of 5 are there from 10 to 95 ?

Solution : As you know, multiples of 5 are integers having 0 or 5 in the digit to the extreme right (i.e. the unit's place).

The first digit from the right can be chosen in 2 ways.

The second digit can be any one of 1,2,3,4,5,6,7,8,9.

i.e. There are 9 choices for the second digit.

Thus, there are $2\times 9 = 18$ multiples of 5 from 10 to 95.

Example 11.2 In a city, the bus route numbers consist of a natural number less than 100, followed by one of the letters A, B, C, D, E and F. How many different bus routes are possible?

Solution : The number can be any one of the natural numbers from 1 to 99.
There are 99 choices for the number.

The letter can be chosen in 6 ways.

$\therefore$ Number of possible bus routes are $99 \times 6 = 594$.

Example 11.3 There are 3 questions in a question paper. If the questions have 4,3 and 2 solutions respectively, find the total number of solutions.

Solution : Here question 1 has 4 solutions, question 2 has 3 solutions and question 3 has 2 solutions.

$\therefore$ By the multiplication (counting) rule,

total number of solutions = $4 \times 3 \times 2 = 24$

Example 11.4 Consider the word ROTOR. Whichever way you read it, from left to right or from right to left, you get the same word. Such a word is known as *palindrome*. Find the maximum possible number of 5-letter palindromes.

Solution : The first letter from the right can be chosen in 26 ways because there are 26 alphabets. Having chosen this, the second letter can be chosen in 26 ways

$\therefore$ The first two letters can be chosen in $26 \times 26 = 676$ ways

Having chosen the first two letters, the third letter can be chosen in 26 ways.
$\therefore$ All the three letters can be chosen in $676 \times 26 = 17576$ ways.

It implies that the maximum possible number of five letter palindromes is 17576 because the fourth letter is the same as the second letter and the fifth letter is the same as the first letter.

***Note :** In Example 11.4 we found the maximum possible number of five letter palindromes. There cannot be more than 17576. But this does not mean that there are 17576 palindromes. Because some of the choices like CCCCC may not be meaningful words in the English language.*

Example 11.5 How many 3-digit numbers can be formed with the digits 1,4,7,8 and 9 if the digits are not repeated.

Solution : Three digit number will have unit's, ten's and hundred's place.

Out of 5 given digits any one can take the unit's place.

This can be done in 5 ways. ... (*i*)

After filli ng the unit's place, any of the four remaining digits can take the ten's place.

This can be done in 4 ways. ... (*ii*)

After filling in ten's place, hundred's place can be filled from any of the three remaining digits.

This can be done in 3 ways. ... (*iii*)

$\therefore$ By counting principle, the number of 3 digit numbers = $5 \times 4 \times 3 = 60$

Let us now state the **General Counting Principle**

If there are *n* events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occured, the third event can occur in m_3 ways after the second event has ocurred, and so on, then all the *n* events can occur in $m_1 \times m_2 \times \ldots \times m_{n-1} \times m_n$ ways.

Example 11.6 Suppose you can travel from a place *A* to a place *B* by 3 buses, from place *B* to place *C* by 4 buses, from place *C* to place *D* by 2 buses and from place *D* to place *E* by 3 buses. In how many ways can you travel from *A* to *E*?

Solution : The bus from *A* to *B* can be selected in 3 ways.

The bus from *B* to *C* can be selected in 4 ways.

The bus from *C* to *D* can be selected in 2 ways.

The bus from *D* to *E* can be selected in 3 ways.

So, by the General Counting Principle, one can travel from *A* to *E* in $3 \times 4 \times 2 \times 3$ ways = 72 ways.

11.2 PERMUTATIONS

Suppose you want to arrange your books on a shelf. If you have only one book, there is only one way of arranging it. Suppose you have two books, one of History and one of Geography.

You can arrange the Geography and History books in two ways. Geography book first and the History book next, *GH* or History book first and Geography book next; *HG*. In other words, there are two arrangements of the two books.

Now, suppose you want to add a Mathematics book also to the shelf. After arranging History and Geography books in one of the two ways, say *GH*, you can put Mathematics book in one of the following ways: *MGH*, *GMH* or *GHM*. Similarly, corresponding to *HG*, you have three other ways of arranging the books. So, by the Counting Principle, you can arrange Mathematics, Geography and History books in 3×2 ways = 6 ways.

By permutation we mean an arrangement of objects in a particular order. In the above example, we were discussing the number of permutations of one book or two books.

In general, if you want to find the number of permutations of n objects $n \geq 1$, how can you do it? Let us see if we can find an answer to this.

Similar to what we saw in the case of books, there is one permutation of 1 object, 2×1 permutations of two objects and $3 \times 2 \times 1$ permutations of 3 objects. It may be that, there are $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ permutations of n objects. In fact, it is so, as you will see when we prove the following result.

Theorem 11.1 The total number of permutations of n objects is $n\,(n-1)\,....2.1$.

Proof : We have to find the number of possible arrangements of n different objects.

The first place in an arrangement can be filled in n different ways. Once it has been done, the second place can be filled by any of the remaining $(n-1)$ objects and so this can be done in $(n-1)$ ways. Similarly, once the first two places have been filled, the third can be filled in $(n-2)$ ways and so on. The last place in the arrangement can be filled only in one way, because in this case we are left with only one object.

Using the counting principle, the total number of arrangements of n different objects is $n(n-1)(n-2)\,........\,2.1.$ **.....(11.1)**

The product $n\,(n-1)\,...\,2.1$ occurs so often in Mathematics that it deserves a name and notation. It is usually denoted by $\boldsymbol{n!}$ (or by $\underline{|n}$ read as ***n* factorial**).

$$\boldsymbol{n! = n\,(n-1)\,...\,3.2.1}$$

Here is an example to help you familiarise yourself with this notation.

Example 11.7 Evaluate (a) 3! (b) 2! + 4! (c) $2! \times 3!$

Solution : (a) $3! = 3 \times 2 \times 1 = 6$

(b) $2! = 2 \times 1 = 2$, $4! = 4 \times 3 \times 2 \times 1 = 24$

Therefore, $2! + 4! = 2 + 24 = 26$

(c) $2! \times 3! = 2 \times 6 = 12$

Notice that $n!$ satisfies the relation, $n! = n \times (n-1)\,!$ **... (11.2)**

This is because, $n\,(n-1)! = n\,[(n-1)\,.\,(n-2)...\,2.1]$

$$= n\,.\,(n-1)\,.\,(n-2)\,...2.1 = n!$$

Of course, the above relation is valid only for $n \geq 2$ because 0! has not been defined so far. Let us see if we can define 0! to be consistent with the relation. In fact, if we define

$$0! = 1$$

then the relation 11.2 holds for $n = 1$ also.

Example 11.8 Suppose you want to arrange your English, Hindi, Mathematics, History, Geography and Science books on a shelf. In how many ways can you do it?

Solution : We have to arrange 6 books.

The number of permutations of n objects is $n! = n.\,(n-1)\,.\,(n-2)\,...\,2.1$

Here $n = 6$ and therefore, number of permutations is $6.5.4.3.2.1 = 720$

11.3 PERMUTATION OF *r* OBJECTS OUT OF *n* OBJECTS

Suppose you have five story books and you want to distribute one each to Asha, Akhtar and Jasvinder. In how many ways can you do it? You can give any one of the five books to Asha and after that you can give any one of the remaining four books to Akhtar. After that, you can give one of the remaining three books to Jasvinder. So, by the Counting Principle, you can distribute the books in $5 \times 4 \times 3$ *ie.*60 ways.

More generally, suppose you have to arrange r objects out of n objects. In how many ways can you do it? Let us view this in the following way. Suppose you have n objects and you have to arrange r of these in r boxes, one object in each box.

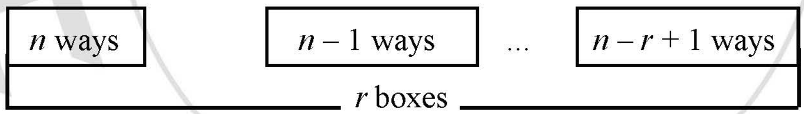

Fig. 11.1

Suppose there is one box. $r = 1$. You can put any of the n objects in it and this can be done in n ways. Suppose there are two boxes. $r = 2$. You can put any of the objects in the first box and after that the second box can be filled with any of the remaining $n - 1$ objects. So, by the counting principle, the two boxes can be filled in $n\,(n-1)$ ways. Similarly, 3 boxes can be filled in $n\,(n-1)\,(n-2)$ ways.

In general, we have the following theorem.

Theorem 11.2 The number of permutations of r objects out of n objects is

$$n\,(n-1)\cdots(n-r+1).$$

The number of permutations of r objects out of n objects is usually denoted by ${}^nP_r$.

Thus, $${}^nP_r = n(n-1)(n-2)...(n-r+1) \qquad \textbf{... (11.3)}$$

Proof : Suppose we have to arrange r objects out of n different objects. In fact it is equivalent to filling r places, each with one of the objects out of the given n objects.

The first place can be filled in n different ways. Once this has been done, the second place can be filled by any one of the remaining $(n-1)$ objects, in $(n-1)$ ways. Similarly, the third place can be filled in $(n-2)$ ways and so on. The last place, the rth place can be filled in $[n-(r-1)]$ i.e. $(n-r+1)$ different ways. You may easily see, as to why this is so.

Using the Counting Principle, we get the required number of arrangements of r out of n objects is $n(n-1)(n-2)..........(n-r+1)$

Example 11.9 Evaluate : (a) $^4P_2$ (b) $^6P_3$ (c) $\frac{^4P_3}{^3P_2}$ (d) $^6P_3 \times {^5P_2}$

Solution : (a) $^4P_2 = 4(4-1) = 4\times 3 = 12.$

(b) $^6P_3 = 6(6-1)\ (6-2) = 6\times 5\times 4 = 120.$

(c) $\dfrac{^4P_3}{^3P_2} = \dfrac{4(4-1)(4-2)}{3(3-1)} = \dfrac{4\times 3\times 2}{3\times 2} = 4$

(d) $^6P_3 \times {^5P_2} = 6\ (6-1)\ (6-2)\times 5\ (5-1),\ = 6\times 5\times 4\times 5\times 4\ = \ 2400$

Example 11.10 If you have 6 New Year greeting cards and you want to send them to 4 of your friends, in how many ways can this be done?

Solution : We have to find number of permutations of 4 objects out of 6 objects.

This number is $^6P_4 = 6(6-1)(6-2)(6-3) = 6.5.4.3 = 360$

Therefore, cards can be sent in 360 ways.

Consider the formula for ^nP_r, namely, $^nP_r = n\ (n-1)\ ...\ (n-r+1)$. This can be obtained by removing the terms $n-r, n-r-1,...,2, 1$ from the product for n!. The product of these terms is $(n-r)\ (n-r-1)\ ...2.1$, i.e., $(n-r)!$.

$$\text{Now, } \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)...(n-r+1)(n-r)...2.1}{(n-r)\ (n-r-1)...2.1}$$

$$= n(n-1)(n-2)...(n-r+1) = {^nP_r}$$

So, using the factorial notation, this formula can be written as follows : $^nP_r = \dfrac{n!}{(n-r)!}$ .....**(11.4)**

Example 11.11 Find the value of $^nP_0$.

Solution : Here $r = 0$. Using relation 11.4, we get ${}^nP_0 = \frac{n!}{n!} = 1$

Example 11.12 Show that $(n+1)\,{}^nP_r = {}^{n+1}P_{r+1}$

Solution : $(n+1)\,{}^nP_r = (n+1)\frac{n!}{(n-r)!} = \frac{(n+1)n!}{(n-r)!}$

$$= \frac{(n+1)!}{[(n+1)-(r+1)]!} \text{ [writing } n-r \text{ as } [(n+1)-(r+1)]$$

$$= {}^{n+1}P_{r+1} \qquad \text{(By definition)}$$

11.4 PERMUTATIONS UNDER SOME CONDITIONS

We will now see examples involving permutations with some extra conditions.

Example 11.13 Suppose 7 students are staying in a hall in a hostel and they are allotted 7 beds. Among them, Parvin does not want a bed next to Anju because Anju snores. Then, in how many ways can you allot the beds?

Solution : Let the beds be numbered 1 to 7.

Case 1 : Suppose Anju is allotted bed number 1.

Then, Parvin cannot be allotted bed number 2.

So Parvin can be allotted a bed in 5 ways.

After alloting a bed to Parvin, the remaining 5 students can be allotted beds in 5! ways.

So, in this case the beds can be allotted in $5 \times 5! ways = 600$ ways.

Case 2 : Anju is allotted bed number 7.

Then, Parvin cannot be allotted bed number 6

As in Case 1, the beds can be allotted in 600 ways.

Case 3 : Anju is allotted one of the beds numbered 2,3,4,5 or 6.

Parvin cannot be allotted the beds on the right hand side and left hand side of Anju's bed. For example, if Anju is allotted bed number 2, beds numbered 1 or 3 cannot be allotted to Parvin.

Therefore, Parvin can be allotted a bed in 4 ways in all these cases.

After allotting a bed to Parvin, the other 5 can be allotted a bed in 5! ways.

Therefore, in each of these cases, the beds can be allotted in $4 \times 5! = 480$ ways.

$\therefore$ The beds can be allotted in

$(2\times 600+5\times 480)\ ways = (1200+2400)\ ways = 3600$ ways.

Example 11.14 In how many ways can an animal trainer arrange 5 lions and 4 tigers in a row so that no two lions are together?

Solution : They have to be arranged in the following way :

L	T	L	T	L	T	L	T	L

The 5 lions should be arranged in the 5 places marked 'L'. This can be done in 5! ways.

The 4 tigers should be in the 4 places marked 'T'. This can be done in 4! ways.

Therefore, the lions and the tigers can be arranged in $5!\times 4!\ ways = 2880$ ways.

Example 11.15 There are 4 books on fairy tales, 5 novels and 3 plays. In how many ways can you arrange these so that books on fairy tales are together, novels are together and plays are together and in the order, books on fairytales, novels and plays.

Solution : There are 4 books on fairy tales and they have to be put together.

They can be arranged in 4! ways. Similarly, there are 5 novels.

They can be arranged in 5! ways. And there are 3 plays.

They can be arranged in 3! ways.

So, by the counting principle all of them together can be arranged in $4!\times 5!\times 3!$ ways = 17280 ways.

Example 11.16 Suppose there are 4 books on fairy tales, 5 novels and 3 plays as in Example 11.15. They have to be arranged so that the books on fairy tales are together, novels are together and plays are together, but we no longer require that they should be in a specific order. In how many ways can this be done?

Solution : First, we consider the books on fairy tales, novels and plays as single objects.

These three objects can be arranged in 3!ways = 6 ways.

Let us fix one of these 6 arrangements.

This may give us a specific order, say, novels $\rightarrow$ fairy tales $\rightarrow$ plays.

Given this order, the books on the same subject can be arranged as follows.

The 4 books on fairy tales can be arranged among themselves in 4! = 24 ways.

The 5 novels can be arranged in 5! = 120 ways.

The 3 plays can be arranged in 3! = 6 ways.

For a given order, the books can be arranged in $24 \times 120 \times 6 = 17280$ ways.

Therefore, for all the 6 possible orders the books can be arranged in $6 \times 17280 = 103680$ ways.

Example 11.17 In how many ways can 4 girls and 5 boys be arranged in a row so that all the four girls are together?

Solution : Let 4 girls be one unit and now there are 6 units in all.

They can be arranged in 6! ways.

In each of these arrangements 4 girls can be arranged in 4! ways.

$\therefore$ Total number of arrangements in which girls are always together

$= 6! \times 4! = 720 \times 24 = 17280$

Example 11.18 How many arrangements of the letters of the word 'BENGALI' can be made if the vowels are always together.

Solution : There are 7 letters in the word 'Bengali; of these 3 are vowels and 4 consonants. Considering vowels *a*, *e*, *i* as one letter, we can arrange 4+1 letters in 5! ways in each of which vowels are together. These 3 vowels can be arranged among themselves in 3! ways.

$\therefore$ Total number of words $= 5! \times 3! = 120 \times 6 = 720$

undergone a knee surgery and needs a lower berth while Ms. Gupta wants to rest during the journey and needs an upper berth. In how many ways can the berths be shared by the family?

2. Consider the word UNBIASED. How many words can be formed with the letters of the word in which no two vowels are together?

3. There are 4 books on Mathematics, 5 books on English and 6 books on Science. In how many ways can you arrange them so that books on the same subject are together and they are arranged in the order Mathematics $\rightarrow$ English $\rightarrow$ Science.

4. There are 3 Physics books, 4 Chemistry books, 5 Botany books and 3 Zoology books. In how many ways can you arrange them so that the books on the same subject are together?

5. 4 boys and 3 girls are to be seated in 7 chairs such that no two boys are together. In how many ways can this be done?

6. Find the number of permutations of the letters of the word 'TENDULKAR', in each of the following cases :

(i) beginning with T and ending with R. (ii) vowels are always together.

(iii) vowels are never together.

11.5 COMBINATIONS

Let us consider the example of shirts and trousers as stated in the introduction. There you have 4 sets of shirts and trousers and you want to take 2 sets with you while going on a trip. In how many ways can you do it?

Let us denote the sets by S_1, S_2, S_3, S_4. Then you can choose two pairs in the following ways :

1. $\{S_1, S_2\}$	2. $\{S_1, S_3\}$	3. $\{S_1, S_4\}$
4. $\{S_2, S_3\}$	5. $\{S_2, S_4\}$	6. $\{S_3, S_4\}$

[Observe that $\{S_1, S_2\}$ is the same as $\{S_2, S_1\}$]. So, there are 6 ways of choosing the two sets that you want to take with you. Of course, if you had 10 pairs and you wanted to take 7 pairs, it will be much more difficult to work out the number of pairs in this way.

Now as you may want to know the number of ways of wearing 2 out of 4 sets for two days, say Monday and Tuesday, and the order of wearing is also important to you. We know from section 11.3, that it can be done in ${}^4P_2 = 12$ ways. But note that each choice of 2 sets gives us two ways of wearing 2 sets out of 4 sets as shown below :

1. $\{S_1, S_2\} \rightarrow S_1$ on Monday and S_2 on Tuesday or S_2 on Monday and S_1 on Tuesday
2. $\{S_1, S_3\} \rightarrow S_1$ on Monday and S_3 on Tuesday or S_3 on Monday and S_1 on Tuesday
3. $\{S_1, S_4\} \rightarrow S_1$ on Monday and S_4 on Tuesday or S_4 on Monday and S_1 on Tuesday
4. $\{S_2, S_3\} \rightarrow S_2$ on Monday and S_3 on Tuesday or S_3 on Monday and S_2 on Tuesday
5. $\{S_2, S_4\} \rightarrow S_2$ on Monday and S_4 on Tuesday or S_4 on Monday and S_2 on Tuesday
6. $\{S_3, S_4\} \rightarrow S_3$ on Monday and S_4 on Tuesday or S_4 on Monday and S_3 on Tuesday

Thus, there are 12 ways of wearing 2 out of 4 pairs.

This argument holds good in general as we can see from the following theorem.

Theorem 11.3 Let $n \geq 1$ be an integer and $r \leq n$. Let us denote the number of ways of choosing r objects out of n objects by ${}^nC_r$. Then

$$ {}^nC_r = \frac{{}^nP_r}{r!} \qquad \text{... (11.5)} $$

Proof : We can choose r objects out of n objects in ${}^nC_r$ ways. Each of the r objects chosen can be arranged in $r!$ ways. The number of ways of arranging r objects is $r!$. Thus, by the counting principle, the number of ways of choosing r objects and arranging the r objects chosen can be done in ${}^nC_r r!$ ways. But, this is precisely ${}^nP_r$. In other words, we have

$${}^nP_r = r!.{}^nC_r \qquad \textbf{... (11.6)}$$

Dividing both sides by $r!$, we get the result in the theorem.

Here is an example to help you to familiarise yourself with ${}^nC_r$.

Example 11.19 Evaluate each of the following :

(a) ${}^5C_2$ (b) ${}^5C_3$ (c) ${}^4C_3 + {}^4C_2$ (d) $\dfrac{{}^6C_3}{{}^4C_2}$

Solution : (a) ${}^5C_2 = \dfrac{{}^5P_2}{2!} = \dfrac{5.4}{1.2} = 10$. (b) ${}^5C_3 = \dfrac{{}^5P_3}{3!} = \dfrac{5.4.3}{1.2.3} = 10$.

(c) ${}^4C_3 + {}^4C_2 = \dfrac{{}^4P_3}{3!} + \dfrac{{}^4P_2}{2!} = \dfrac{4.3.2}{1.2.3} + \dfrac{4.3}{1.2} = 4 + 6 = 10$

(d) ${}^6C_3 = \dfrac{{}^6P_3}{3!} = \dfrac{6.5.4}{1.2.3} = 20$ and ${}^4C_2 = \dfrac{4.3}{1.2} = 6$

$\therefore \dfrac{{}^6C_3}{{}^4C_2} = \dfrac{20}{6} = \dfrac{10}{3}$.

Example 11.20 Find the number of subsets of the set $\{1,2,3,4,5,6,7,8,9,10,11\}$ having 4 elements.

Solution : Here the order of choosing the elements doesn't matter and this is a problem in combinations.

We have to find the number of ways of choosing 4 elements of this set which has 11 elements.

By relation (11.5), this can be done in ${}^{11}C_4 = \dfrac{11.10.9.8}{1.2.3.4} = 330$ ways.

Example 11.21 12 points lie on a circle. How many cyclic quadrilaterals can be drawn by using these points?

Solution : For any set of 4 points we get a cyclic quadrilateral. Number of ways of choosing 4 points out of 12 points is ${}^{12}C_4 = 495$. Therefore, we can draw 495 quadrilaterals.

Example 11.22 In a box, there are 5 black pens, 3 white pens and 4 red pens. In how many ways can 2 black pens, 2 white pens and 2 red pens can be chosen?

Solution : Number of ways of choosing 2 black pens from 5 black pens

$$= {}^5C_2 = \frac{{}^5P_2}{2!} = \frac{5.4}{1.2} = 10.$$

Number of ways of choosing 2 white pens from 3 white pens , $= {}^3C_2 = \frac{{}^3P_2}{2!} = \frac{3.2}{1.2} = 3$.

Number of ways of choosing 2 red pens from 4 red pens , $= {}^4C_2 = \frac{{}^4P_2}{2!} = \frac{4.3}{1.2} = 6$.

$\therefore$ By the Counting Principle, 2 black pens, 2 white pens, and 2 red pens can be chosen in $10 \times 3 \times 6 = 180$ ways.

Example 11.23 A question paper consists of 10 questions divided into two parts A and B. Each part contains five questions. A candidate is required to attempt six questions in all of which at least 2 should be from part A and at least 2 from part B. In how many ways can the candidate select the questions if he can answer all questions equally well?

Solution : The candidate has to select six questions in all of which at least two should be from Part A and two should be from Part B. He can select questions in any of the following ways :

	Part A	**Part B**
(i)	2	4
(ii)	3	3
(iii)	4	2

If the candidate follows choice (i), the number of ways in which he can do so is ${}^5C_2 \times {}^5C_4 = 10 \times 5 = 50$

If the candidate follows choice (ii), the number of ways in which he can do so is ${}^5C_3 \times {}^5C_3 = 10 \times 10 = 100$.

Similarly, if the candidate follows choice (iii), then the number of ways in which he can do so is ${}^5C_4 \times {}^5C_2 = 50$.

Therefore, the candidate can select the question in $50 + 100 + 50 = 200$ ways.

Example 11.24 A committee of 5 persons is to be formed from 6 men and 4 women. In how many ways can this be done when

(i) at least 2 women are included?

(ii) atmost 2 women are included?

Solution : (i) When at least 2 women are included.

The committee may consist of

2 women, 3 men : It can be done in ${}^4C_2 \times {}^6C_3$ ways.

or, 3 women, 2 men : It can be done in ${}^4C_3 \times {}^6C_2$ ways.

or, 4 women, 1 man : It can be done in ${}^4C_4 \times {}^6C_1$ ways.

$\therefore$ Total number of ways of forming the committee

$= {}^4C_2.{}^6C_3 + {}^4C_3.{}^6C_2 + {}^4C_4.{}^6C_1 = 6\times 20 + 4\times 15 + 1\times 6 = 120 + 60 + 6 = 186$

(ii) When atmost 2 women are included

The committee may consist of

2 women, 3 men : It can be done in ${}^4C_2.{}^6C_3$ ways

or, 1 woman, 4 men : It can be done in ${}^4C_1.{}^6C_4$ ways

or, 5 men : It can be done in ${}^6C_5$ ways

$\therefore$ Total number of ways of forming the committee

$= {}^4C_2.{}^6C_3 + {}^4C_1.{}^6C_4 + {}^6C_5 = 6\times 20 + 4\times 15 + 6 = 120 + 60 + 6 = 186$

Example 9.25 The Indian Cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and atleast 4 bowlers?

Solution : We are to choose 11 players including 1 wicket keeper and 4 bowlers

or, 1 wicket keeper and 5 bowlers.

Number of ways of selecting 1 wicket keeper, 4 bowlers and 6 other players

$= {}^2C_1.{}^5C_4.{}^9C_6$

$$= 2\times\frac{5\times 4\times 3\times 2.1}{4.3.2.1.}\times\frac{9\times 8\times 7\times 6\times 5\times 4}{6\times 5\times 4\times 3\times 2\times 1} = 2\times 5\times\frac{9\times 8\times 7}{3\times 2\times 1} = 840$$

Number of ways of selecting 1 wicket keeper, 5 bowlers and 5 other players

$$= {}^2C_1.{}^5C_5.{}^9C_5 = 2\times 1\times\frac{9\times 8\times 7\times 6\times 5}{5\times 4\times 3\times 2\times 1} = 2\times 1\times\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1} = 252$$

$\therefore$ Total number of ways of selecting the team $= 840 + 252 = 1092$

11.6 SOME SIMPLE PROPERTIES OF ${}^nC_r$

In this section we will prove some simple properties of ${}^nC_r$ which will make the computations of these coefficients simpler. Let us go back again to Theorem 11.3. Using relation 11.6 we can rewrite the formula for ${}^nC_r$ as : ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ ....**(11.7)**

Example 11.26 Find the value of ${}^nC_0$.

Solution : Here $r = 0$. Therefore, ${}^nC_0 = \dfrac{n!}{0!n!} = \dfrac{1}{0!} = 1$,

since we have defined $0! = 1$.

The formula given in Theorem 11.3 was used in the previous section. As we will see shortly, the formula given in Equation 11.7 will be useful for proving certain properties of ${}^nC_r$.

$${}^nC_r = {}^nC_{n-r} \qquad \textbf{...(11.8)}$$

This means just that the number of ways of choosing r objects out of n objects is the same as the number of ways of not choosing $(n-r)$ objects out of n objects. In the example described in the introduction, it just means that the number of ways of selecting 2 sets of dresses is the same as the number of ways of rejecting $4-2=2$ dresses. In Example 11.20, this means that the number of ways of choosing subsets with 4 elements is the same as the number of ways of rejecting 8 elements since choosing a particular subset of 4 elements is equivalent to rejecting its complement, which has 8 elements.

Let us now prove this relation using Equation 11.7. The denominator of the right hand side of this equation is $r!\,(n-r)!$. This does not change when we replace r by $n-r$.

$$(n-r)!.[n-(n-r)]! = (n-r)!.r!$$

The numerator is independent of r. Therefore, replacing r by $n-r$ in Equation 11.7 we get result.

How is the relation 11.8 useful? Using this formula, we get, for example, ${}^{100}C_{98}$ is the same as ${}^{100}C_2$. The second value is much more easier to calculate than the first one.

Example 11.27 Evaluate :

(a) ${}^7C_5$ (b) ${}^{11}C_9$ (c) ${}^{10}C_9$ (d) ${}^{12}C_9$

Solution : (a) From relation 11.8, we have

$${}^7C_5 = {}^7C_{7-5} = {}^7C_2 = \frac{7.6}{1.2} = 21$$

(b) Similarly $\quad {}^{10}C_9 = {}^{10}C_{10-9} = {}^{10}C_1 = 10$

(c) $^{11}C_9 = {}^{11}C_{11-9} = {}^{11}C_2 = \dfrac{11.10}{1.2} = 55$

(d) $^{12}C_{10} = {}^{12}C_{12-10} = {}^{12}C_2 = \dfrac{12.11}{1.2} = 66$

There is another relation satisfied by ^nC_r which is also useful. We have the following relation:

$$^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r \qquad \text{....(11.9)}$$

$$^{n-1}C_{r-1} + {}^{n-1}C_r = \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-r-1)!r!}$$

$$= \frac{(n-1)!}{(n-r)(n-r-1)!(r-1)!} + \frac{(n-1)!}{r(n-r-1)!(r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!(r-1)!}\left[\frac{n}{(n-r)r}\right]$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!r(r-1)!}$$

$$= \frac{n!}{(n-r)!r!} = {}^nC_r$$

Example 11.28 Evaluate :

(a) $^6C_2 + {}^6C_1$ (b) $^8C_2 + {}^8C_1$ (c) $^5C_3 + {}^5C_2$ (d) $^{10}C_2 + {}^{10}C_3$

Solution : (a) Let us use relation (11.9) with $n = 7, r = 2$. So, $^6C_2 + {}^6C_1 = {}^7C_2 = 21$

(b) Here $n = 9$, $r = 2$. Therefore, $^8C_2 + {}^8C_1 = {}^9C_2 = 36$

(c) Here $n = 6$, $r = 3$. Therefore, $^5C_3 + {}^5C_2 = {}^6C_3 = 20$

(d) Here $n = 11$, $r = 3$. Therefore, $^{10}C_2 + {}^{10}C_3 = {}^{11}C_3 = 165$

Example 11.29 If $^nC_{10} = {}^nC_{12}$ find n,

Solution : Using $^nC_r = {}^nC_{n-r}$ we get $n - 10 = 12$ or, $n = 12 + 10 = 22$

11.7 PROBLEMS INVOLVING BOTH PERMUTATIONS AND COMBINATIONS

So far, we have studied problems that involve either permutation alone or combination alone. In this section, we will consider some examples that need both of these concepts.

Example 11.30 There are 5 novels and 4 biographies. In how many ways can 4 novels and 2 biographies can be arranged on a shelf ?

Soluton : 4 novels can be selected out of 5 in ${}^5C_4$ ways. 2 biographies can be selected out of 4 in ${}^4C_2$ ways.

Number of ways of arranging novels and biographies = ${}^5C_4 \times {}^4C_2 = 5 \times 6 = 30$

After selecting any 6 books (4 novels and 2 biographies) in one of the 30 ways, they can be arranged on the shelf in 6! = 720 ways.

By the Counting Principle, the total number of arrangements = $30 \times 720 = 21600$

Example 11.31 From 5 consonants and 4 vowels, how many words can be formed using 3 consonants and 2 vowels ?

Solution : From 5 consonants, 3 consonants can be selected in ${}^5C_3$ ways.

From 4 vowels, 2 vowels can be selected in ${}^4C_2$ ways.

Now with every selection, number of ways of arranging 5 letters is ${}^5P_5$

$\therefore$ Total number of words $= {}^5C_3 \times {}^4C_2 \times {}^5P_5 = \dfrac{5\times 4}{2\times 1} \times \dfrac{4\times 3}{2\times 1} \times 5!$

$= 10\times 6\times 5\times 4\times 3\times 2\times 1 = 7200$

LET US SUM UP

- Fundamental principle of counting states.

  If there are n events and if the first event can occur in m_1 ways, the second event can occur in m_2 ways after the first event has occurred, the third event can occur in m_3 ways after the second event has occurred and so on, then all the n events can occur in

  $m_1 \times m_2 \times m_3 \times \ldots\ldots \times m_{n-1} \times m_n$ ways.
- The number of permutations of n objects taken all at a time is $n!$
- ${}^nP_r = \dfrac{n!}{(n-r)!}$
- ${}^nP_n = n!$
- The number of ways of selecting r objects out of n objects is ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
- ${}^nC_r = {}^nC_{n-r}$
- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$