Lecture 7 : Derivative AS a Function

In the previous section we defined the derivative of a function f at a number a (when the function f is defined in an open interval containing a) to be

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

when this limit exists. This gives the **slope of the tangent** to the curve $y = f(x)$ when $x = a$

Example Last day we saw that if $f(x) = x^2 + 5x$, then $f'(a) = 2a + 5$ for any value of a. Therefore $f'(1) = 7, f'(2) = 9, f'(2.5) = 10$ etc....

The value of $f'(a)$ varies as the number a varies, hence f' is a function of a. We can change the variable from a to x to get a new function, called **The derivative of** f

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

($f'(x)$ is defined when f is defined in an open interval containing x and the above limit exists). Note that when calculating this limit for a particular value of x, $h \to 0$ and the value of x remains constant.

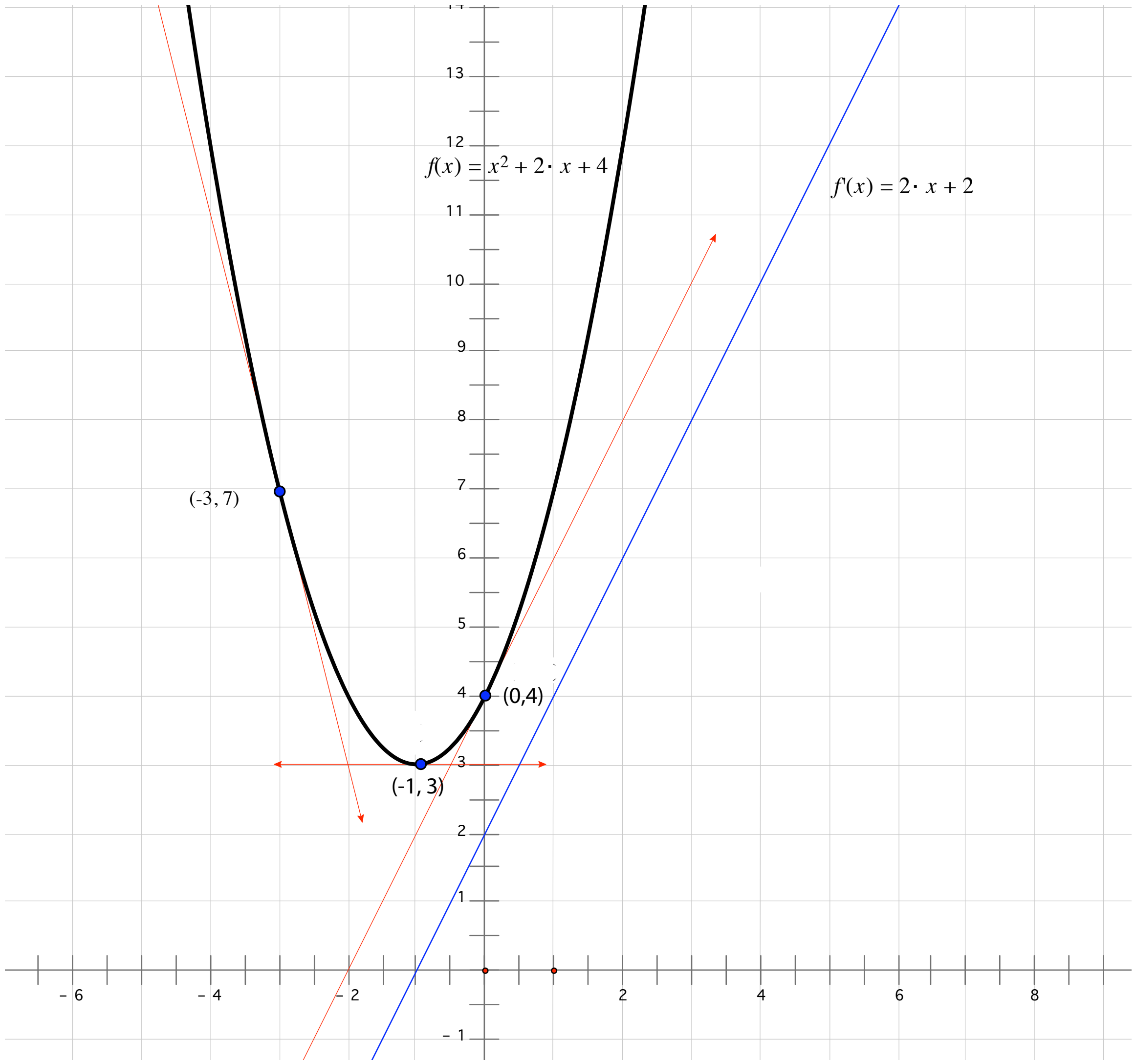
Note also that **if x is in the domain of f'**, it must satisfy the following 3 conditions:

1. x must be in the domain of f.
2. $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ must exist at x.
3. f must be defined in an open interval containing x.

The domain of the function f' may be smaller than the domain of the function f since 2 or 3 may fail for some values of x in the domain of f.

Example What is $f'(x)$ when $f(x) = x^2 + 2x + 4$?. What is the domain of $f'(x)$?

Example Consider the function in the example above $f(x) = x^2 + 2x + 4$. The graph, $y = f(x)$ is shown below along with the graph of the new function $f'(x) = 2x + 2$. We can see how the graph of $f'(x)$ is related to the slope of the tangents to the graph of f.



Fill in $<, >$ *or* $=$ as appropriate:

When $f(x)$ is decreasing the function $f'(x)$___0

When $f(x)$ is increasing the function $f'(x)$___0

At the turning point $x = -1$, $f'(x)$___0

Example Consider the function $f(x) = |x|$.

Does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x > 0$?

Does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x < 0$?

Does $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist when $x = 0$?

What is the domain of $f'(x)$?

Alternative Notation

Using $y = f(x)$, to denote that the independent variable is y, there are a number of notations used to denote the derivative of $f(x)$:

$$f'(x) \ = \ y' \ = \ \frac{dy}{dx} \ = \ \frac{df}{dx} \ = \ \frac{d}{dx}f(x) \ = \ Df(x) \ = \ D_x f(x).$$

The symbols D and $\frac{d}{dx}$ are called differential operators, because when they are applied to a function, they transform the function to its derivative. The symbol $\frac{dy}{dx}$ should not be interpreted as a quotient rather it is a limit originating from the notation

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

When we evaluate the derivative at a number a, we use the following notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}.$$

Differentiability

Definition When a function f is defined in an open interval containing a, we say a function f is **differentiable** at a if $f'(a)$ exists. [That is, conditions 1, 2 and 3 from page 1 must be satisfied for $x = a$.] It is **differentiable on an open interval** , (a, b) (or (a,∞) or $(-\infty, a)$) if it is differentiable at every number in the interval.

Example Let $f(x) = |x|$. Is $f(x)$ differentiable at 0?

If $f(x)$ differentiable on the intervals $(-\infty, 0)$ and $(0,\infty)$. .

Is $f(x)$ continuous at 0?

The following theorem shows that if a function has a discontinuity at a point a, then it cannot be differentiable at a. (Note by the previous example, the converse is not true; a function can be continuous at a, but not differentiable at a).

Theorem	If f is differentiable at a, then f is continuous at a.

Proof Lets assume that f is a function which is differentiable at a. Then we know that

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)$$

exists. To show that f is continuous at a, we must show that

$$\lim_{x\to a} f(x)=f(a)$$

or equivalently that $\lim_{x\to a}(f(x)-f(a))=0$.

We have $\lim_{x\to a}(x-a)=0$. So by our rules of limits we have

$$\lim_{x\to a}(f(x)-f(a))=\lim_{x\to a}\left(\frac{f(x)-f(a))}{x-a}\cdot(x-a)\right)=\lim_{x\to a}\left(\frac{f(x)-f(a))}{x-a}\right)\cdot\lim_{x\to a}(x-a)$$

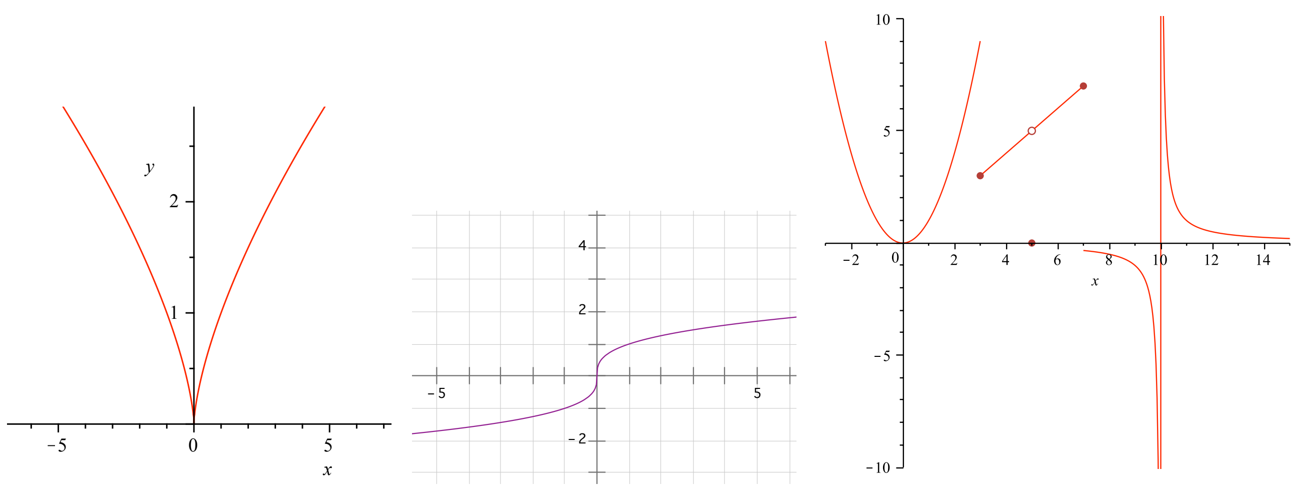
$$=f'(a)\cdot 0=0$$

Points where functions are Not Differentiable

A function f can fail to be differentiable at a point a in a number of ways.

- The function might be continuous at a, but have a sharp point or kink in the graph, like the graph of $f(x)=|x|$ at 0.
- The function might not be continuous or might be undefined at a.
- The function might be continuous but the tangent line may be vertical, i.e. $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\pm\infty$.

Example Identify the points in the graphs below where the functions are not differentiable.



Higher Derivatives

We have seen that given a function $f(x)$, we can define a new function $f'(x)$. We can continue this process by defining a new function,

$$f''(x) = \frac{d}{dx} f'(x).$$

This is the second derivative of the function $f(x)$. This function gives the slope of the tangent to the curve $y = f'(x)$ at each value of x.

We can then define the third derivative of $f(x)$ as the derivative of the second derivative, etc...

Example Let $f(x) = x^2 + 2x + 4$. We saw above that the derivative of $f(x)$ is $f'(x) = 2x + 2$. Find and interpret the second derivative of $f(x)$;

$f''(x) =$

The second derivative gives us the rate of change of the rate of change. In the case of a position function $s = s(t)$ of an object moving in a straight line, the derivative $v(t) = s'(t)$ gives us the velocity of the moving object at time t and the second derivative $a(t) = v'(t) = s''(t)$ gives us the **acceleration** of the moving object at time t. This is the rate of change of the velocity at time t.

Example The position of an object moving in a straight line at time t is given by $s(t) = t^2 + 2t + 4$. What is the velocity and acceleration of the object after $t = 5$ seconds?

Notation

The second derivative is also denoted by

$$f''(x) \;=\; \frac{d}{dx}\left(\frac{dy}{dx}\right) \;=\; \frac{d^2y}{dx^2} \;=\; y''.$$

The third derivative of f is the derivative of the second derivative, denoted

$$\frac{d}{dx} f''(x) \;=\; f'''(x) \;=\; y''' \;= y^{(3)} \;=\; \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \;=\; \frac{d^3y}{dx^3}$$

Higher derivative are denoted

$$f^{(4)}(x) \;=\; y^{(4)} \;=\; \frac{d^4y}{dx^4}, \qquad f^{(5)}(x) \;=\; y^{(5)} \;=\; \frac{d^5y}{dx^5}, \quad etc\ldots$$

Example If $f(x) = x^2 + 2x + 4$, find $f^{(4)}(x)$ and $f^{(5)}(x)$.

Old Exam Questions

1. Find the derivative of the function

$$f(x) = \frac{x}{x-5}$$

using the **limit** definition of the derivative.

2. Which of the statements given below is false?

(a) If f is differentiable at $x = a$, then $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ must equal $f(a)$.

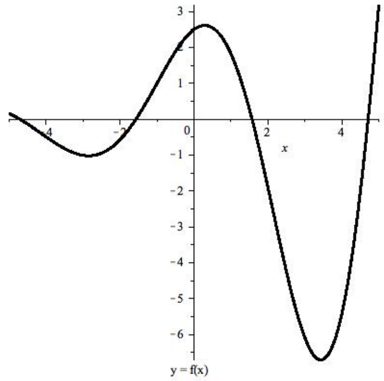
(b) If f is differentiable at $x = a$, then a must be in the domain of f.

(c) If f is differentiable at $x = a$, then $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ must exist.

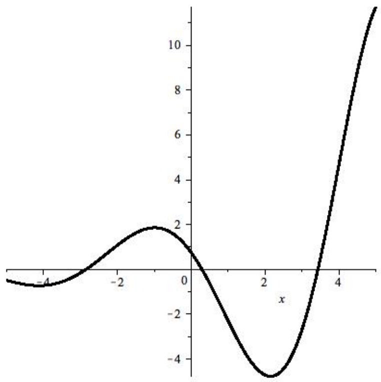
(d) If f is differentiable at $x = a$, then f must be continuous at $x = a$.

(e) If f is differentiable at $x = a$, then $\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}$

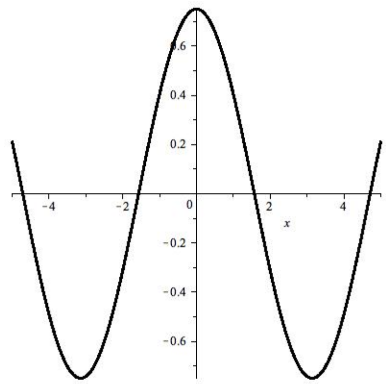
1. The graph of the function $f(x)$ is shown below:



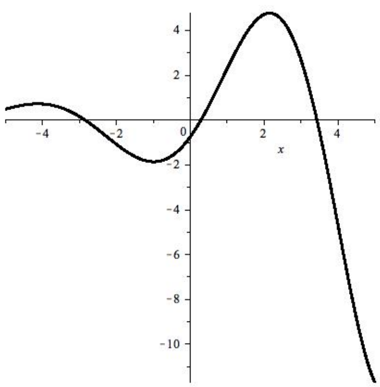
Which of the following gives the graph of $f'(x)$?



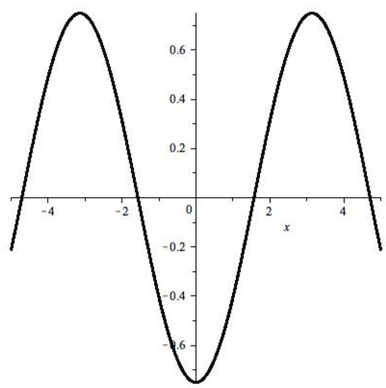
(a)



(b)



(c)



(d)

(e) None of the above

Old Exam Question , Sample Solution

1. Find the derivative of the function
$$f(x) = \frac{x}{x-5}$$
using the **limit** definition of the derivative.

Note the format of the solution below. It is important to carry the limits and show all calculations in order to recieve full credit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+h}{x+h-5} - \frac{x}{x-5}}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)(x-5) - x(x+h-5)}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + hx - 5x - 5h - x^2 - xh + 5x}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + \cancel{hx} - \cancel{5x} - 5h - \cancel{x^2} - \cancel{xh} + \cancel{5x}}{(x+h-5)(x-5)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-5\cancel{h}}{(x+h-5)(x-5)} \cdot \frac{1}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-5}{(x+h-5)(x-5)}$$

$$= \frac{-5}{(x-5)(x-5)}$$

$$= \frac{-5}{(x-5)^2}$$

2. Which of the statements given below is false?

If f is differentiable at a,

1. a must be in the domain of f.
2. $\lim_{h\to 0} \frac{f(a+h)-f(x)}{h}$ must exist at a.
3. f must be defined in an open interval containing a.

(a) If f is differentiable at $x = a$, then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ must equal $f(a)$. false, it is not required that this limit is $f(a)$. For example consider $f(x) = x^2 + 2x + 4$ from the notes. $f'(x) = 2x + 2$. $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 4 \neq f(1) = 7$.

(b) If f is differentiable at $x = a$, then a must be in the domain of f. True see 1 above.

(c) If f is differentiable at $x = a$, then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ must exist. True see 2 above.

(d) If f is differentiable at $x = a$, then f must be continuous at $x = a$. True by the theorem given in notes.

(e) If f is differentiable at $x = a$, then $\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ True since the limit exists only if the laft and right hand limits exist and are equal.

3. The derivative must be positive when $f(x)$ is increasing and negative when it is decreasing. In particular $f'(x) > 0$ for all values of x bigger than 4 in this instance. Therefore the answer is (a).