CHAPTER 8 : BINOMIAL THEOREM

Suppose you need to calculate the amount of interest you will get after 5 years on a sum of money that you have invested at the rate of 15% compound interest per year. Or suppose we need to find the size of the population of a country after 10 years if we know the annual growth rate. A result that will help in finding these quantities is the **binomial theorem.** This theorem, as you will see, helps us to calculate positive integral powers of any real binomial expression, that is, any expression involving two terms.

The binomial theorem, was known to Indian and Greek mathematicians in the 3rd century B.C. for some cases. The credit for the result for natural exponents goes to the Arab poet and mathematician Omar Khayyam (A.D. 1048-1122). Further generalisation to rational exponents was done by the British mathematician Newton (A.D. 1642-1727).

There was a reason for looking for further generalisation, apart from mathematical interest. The reason was its many applications. Apart from the ones we mentioned at the beginning, the binomial theorem has several applications in probability theory, calculus, and in approximating numbers like $(1.02)^7$, etc. We shall discuss them in this lesson.

OBJECTIVES

After studying this lesson, you will be able to:

- state the binomial theorem for a positive integral index and prove it using the principle of mathematical induction;
- write the binomial expansion for expressions like $(x+y)^n$ for different values of x and y using binomial theorem;
- write the general term and middle term (s) of a binomial expansion;

EXPECTED BACKGROUND KNOWLEDGE

- Number System
- Four fundamental operations on numbers and expressions.
- Algebraic expressions and their simplifications.
- Indices and exponents.

12.1 THE BINOMIAL THEOREM FOR A NATURAL EXPONENT

You must have multiplied a binomial by itself, or by another binomial. Let us use this knowledge to do some expansions. Consider the binomial $(x + y)$. Now,

$(x + y)^1 = x + y$

$(x + y)^2 = (x + y)(x + y) = x^2 + 2xy + y^2$

$(x + y)^3 = (x + y)(x + y)^2 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = (x + y)(x + y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = (x + y)(x + y)^4 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ and so on.

In each of the equations above, the right hand side is called the binomial expansion of the left hand side.

Note that in each of the above expansions, we have written the power of a binomial in the expanded form in such a way that the terms are in descending powers of the first term of the binomial (which is x in the above examples). If you look closely at these expansions, you would also observe the following:

1. The number of terms in the expansion is one more than the exponent of the binomial. For example, in the expansion of $(x + y)^4$, the number of terms is 5.
2. The exponent of x in the first term is the same as the exponent of the binomial, and the exponent decreases by 1 in each successive term of the expansion.
3. The exponent of y in the first term is zero (as $y^0 = 1$). The exponent of y in the second term is 1, and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.
4. The sum of the exponents of x and y in each term is equal to the exponent of the binomial. For example, in the expansion of $(x + y)^5$, the sum of the exponents of x and y in each term is 5.

If we use the combinatorial co-efficients, we can write the expansion as

$(x + y)^3 = {}^3C_0\, x^3 + {}^3C_1 x^2y + {}^3C_2 x\, y^2 + {}^3C_3 y^3$

$(x + y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3y + {}^4C_2 x^2y^2 + {}^4C_3 xy^3 + {}^4C_4 y^4$

$(x + y)^5 = {}^5C_0 x^5 + {}^5C_1 x^4y + {}^5C_2 x^3y^2 + {}^5C_3 x^2y^3 + {}^5C_4 xy^4 + {}^5C_5 y^5$, and so on.

More generally, we can write the binomial expansion of $(x + y)^n$, where n is a positive integer, as given in the following theorem. This statement is called the **binomial theorem for a natural** (or **positive integral**) **exponent**.

Theorem

$(x + y)^n = {}^nC_0\, x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_{n-1} xy^{n-1} + {}^nC_n y^n \ldots$(A)

where $n \in N$ and $x, y \in R$.

Proof : Let us try to prove this theorem, using the principle of mathematical induction.

Let statement (A) be denoted by $P(n)$, i.e.,

$P(n)$: $(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2} y^2 + {}^nC_3 x^{n-3} y^3 + \ldots$

$$+\,^{n}C_{n-1}\,xy^{\,n-1} + {}^{n}C_{n}y^{\,n} \qquad ...(i)$$

Let us examine whether $P(1)$ is true or not.

From (i), we have $P(1)$: $(x + y)^1 = {}^1C_0x + {}^1C_1\,y = 1\times x + 1\times y$

i.e., $(x + y)^1 = x + y$ Thus, $P(1)$ holds.

Now, let us assume that $P(k)$ is true, i.e.,

$P(k)$: $(x + y)^k = {}^kC_0x^{\,k} + {}^kC_1x^{\,k-1}y + {}^kC_2x^{\,k-2}y^{\,2} + {}^kC_3x^{\,k-3}y^{\,3} + \ldots + {}^kC_{k-1}xy^{\,k-1} + {}^kC_k\,y^{\,k}$ ...(ii)

Assuming that $P(k)$ is true, if we prove that $P(k + 1)$ is true, then $P(n)$ holds, for all n. Now,

$$(x+y)^{k+1} = (x+y)(x+y)^k = (x+y)({}^kC_0x^k + {}^kC_1x^{k-1}y + {}^kC_2x^{k-2}y^2 +$$

$$\ldots + {}^kC_{k-1}xy^{k-1} + {}^kC_ky^k)$$

$$= {}^kC_0x^{k+1} + {}^kC_0x^ky + {}^kC_1x^ky + {}^kC_1x^{k-1}y^2 + {}^kC_2x^{k-1}y^2 + {}^kC_2x^{k-2}y^3 +$$

$$\ldots + {}^kC_{k-1}x^2y^{k-1} + {}^kC_{k-1}xy^k + {}^kC_kxy^k + {}^kC_ky^{k+1}$$

i.e. $(x+y)^{k+1} = {}^kC_0x^{k+1} + ({}^kC_0 + {}^kC_1)x^ky + ({}^kC_1 + {}^kC_2)x^{k-1}y^2 +$

$$\ldots + ({}^kC_{k-1} + {}^kC_k)xy^k + {}^kC_ky^{k+1} \qquad ...(iii)$$

From Lesson 11, you know that ${}^kC_0 = 1 = {}^{k+1}C_0$, and ${}^kC_k = 1 = {}^{k+1}C_{k+1}$ ....(iv)

Also, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Therefore, ${}^kC_0 + {}^kC_1 = {}^{k+1}C_1$, ${}^kC_1 + {}^kC_2 = {}^{k+1}C_2$, ${}^kC_2 + {}^kC_3 = {}^{k+1}C_3$ .....(v)

....

.... and so on

Using (iv) and (v), we can write (iii) as

$$(x+y)^{k+1} = {}^{k+1}C_0x^{k+1} + {}^{k+1}C_1x^ky + {}^{k+1}C_2x^{k-1}y^2 + \,,\, \ldots + {}^{k+1}C_kxy^k + {}^{k+1}C_{k+1}y^{k+1}$$

which shows that P$(k+1)$ is true.

Thus, we have shown that (a) $P(1)$ is true, and (b) if $P(k)$ is true, then $P(k+1)$ is also true.

Therefore, by the principle of mathematical induction, $P(n)$ holds for any value of n. So, we have proved the binomial theorem for any natural exponent.

This result is supported to have been proved first by the famous Arab poet Omar Khayyam, though no one has been able to trace his proof so far.

We will now take some examples to illustrate the theorem.

Example 12.1 Write the binomial expansion of $(x+3y)^5$.

Solution : Here the first term in the binomial is x and the second term is $3y$. Using the binomial theorem, we have

$$(x+3y)^5 = {}^5C_0x^5 + {}^5C_1x^4(3y)^1 + {}^5C_2x^3(3y)^2 + {}^5C_3x^2(3y)^3 + {}^5C_4x(3y)^4 + {}^5C_5(3y)^5$$

$$= 1\times x^5 + 5x^4\times 3y + 10x^3\times(9y^2) + 10x^2\times(27y^3) + 5x\times(81y^4) + 1\times 243y^5$$

$$= x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$$

Thus, $(x+3y)^5 = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$

Example 12.2 Expand $(1+a)^n$ in terms of powers of a, where a is a real number.

Solution : Taking $x = 1$ and $y = a$ in the statement of the binomial theorem, we have

$$(1+a)^n = {}^nC_0(1)^n + {}^nC_1(1)^{n-1}a + {}^nC_2(1)^{n-2}a^2 + \ldots + {}^nC_{n-1}(1)a^{n-1} + {}^nC_na^n$$

i.e., $$(1+a)^n = 1 + {}^nC_1a + {}^nC_2a^2 + \ldots + {}^nC_{n-1}a^{n-1} + {}^nC_na^n \qquad \ldots \text{(B)}$$

(B) is another form of the statement of the binomial theorem.

The theorem can also be used in obtaining the expansions of expressions of the type

$$\left(x+\frac{1}{x}\right)^5, \left(\frac{y}{x}+\frac{1}{y}\right)^5, \left(\frac{a}{4}+\frac{2}{a}\right)^5, \left(\frac{2t}{3}-\frac{3}{2t}\right)^6, \text{ etc.}$$

Let us illustrate it through an example.

Example 12.3 Write the expansion of $\left(\frac{y}{x}+\frac{1}{y}\right)^4$, where $x, y \neq 0$.

Solution : We have :

$$\left(\frac{y}{x}+\frac{1}{y}\right)^4 = {}^4C_0\left(\frac{y}{x}\right)^4 + {}^4C_1\left(\frac{y}{x}\right)^3\left(\frac{1}{y}\right) + {}^4C_2\left(\frac{y}{x}\right)^2\left(\frac{1}{y}\right)^2 + {}^4C_3\left(\frac{y}{x}\right)\left(\frac{1}{y}\right)^3 + {}^4C_4\left(\frac{1}{y}\right)^4$$

$$= 1\times\frac{y^4}{x^4} + 4\times\frac{y^3}{x^3}\times\frac{1}{y} + 6\times\frac{y^2}{x^2}\times\frac{1}{y^2} + 4\times\left(\frac{y}{x}\right)\times\frac{1}{y^3} + 1\times\frac{1}{y^4}$$

$$= \frac{y^4}{x^4} + 4\frac{y^2}{x^3} + \frac{6}{x^2} + \frac{4}{xy^2} + \frac{1}{y^4}$$

Example 12.4 The population of a city grows at the annual rate of 3%. What percentage increase is expected in 5 years ? Give the answer up to 2 decimal places.

Solution : Suppose the population is a at present. After 1 year it will be

$$a+\frac{3}{100}a=a\left(1+\frac{3}{100}\right)$$

After 2 years, it will be $a\left(1+\frac{3}{100}\right)+\frac{3}{100}\left[a\left(1+\frac{3}{100}\right)\right]$

$$=a\left(1+\frac{3}{100}\right)\left(1+\frac{3}{100}\right)=a\left(1+\frac{3}{100}\right)^2$$

Similarly, after 5 years, it will be $a\left(1+\frac{3}{100}\right)^5$

Using the binomial theorem, and ignoring terms involving more than 3 decimal places,

we get , $a\left(1+\frac{3}{100}\right)^5\approx a\left[1+5(0.03)+10(0.03)^2\right]=a\times 1.159$

So, the increase is $0.159\times 100\%=\frac{159}{1000}\times 100\times\frac{1}{100}=15.9\%$ in 5 years.

Example 12.5 Using binomial theorem, evaluate , (*i*) 102^4 (*ii*) 97^3

Solution : (*i*) $102^4 = (100+2)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1(100)^3.2 + {}^4C_2(100)^2.2^2 + {}^4C_3\,(100).2^3 + {}^4C_4\,.2^4$

$= 100000000 + 8000000 + 240000 + 3200 + 16 = 108243216$

(*ii*) $(97)^3 = (100-3)^3 = {}^3C_0(100)^3 - {}^3C_1(100)^2.3 + {}^3C_2(100).3^2 - {}^3C_3.3^3$

$= 1000000 - 90000 + 2700 - 27 = 1002700 - 90027 = 912673$

12.2 GENERAL TERM IN A BINOMIAL EXPANSION

Let us examine various terms in the expansion (A) of $(x+y)^n$, i.e., in

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n$$

We observe that , the first term is ${}^nC_0x^n$, i.e., ${}^nC_{1-1}x^ny^0$;

the second term is ${}^nC_1x^{n-1}y$, i.e., ${}^nC_{2-1}x^{n-1}y^1$;

the third term is ${}^nC_2x^{n-2}y^2$, i.e., ${}^nC_{3-1}x^{n-2}y^2$; and so on.

From the above, we can generalise that

the $(r+1)^{\text{th}}$ term is ${}^nC_{(r+1)-1}x^{n-r}y^r$, i.e., ${}^nC_r x^{n-r}y^r$.

If we denote this term by T_{r+1}, we have, $T_{r+1} = {}^nC_r x^{n-r}y^r$

T_{r+1} is generally referred to as the **general term** of the binomial expansion.

Let us now consider some examples and find the general terms of some expansions.

Example 12.6 Find the $(r+1)^{\text{th}}$ term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$, where n is a natural number. Verify your answer for the first term of the expansion.

Solution : The general term of the expansion is given by :

$$T_{r+1} = {}^nC_r (x^2)^{(n-r)}\left(\frac{1}{x}\right)^r$$

$$= {}^nC_r x^{2n-2r}\frac{1}{x^r} = {}^nC_r x^{2n-3r} \qquad ...(i)$$

Hence, the $(r+1)$th term in the expansion is ${}^nC_r x^{2n-3r}$.

On expanding $\left(x^2+\frac{1}{x}\right)^n$, we note that the first term is $(x^2)^n$ or x^{2n}.

Using (i), we find the first term by putting $r = 0$.

Since $T_1 = T_{0+1}$ $\therefore$ $T_1 = {}^nC_0 x^{2n-0} = x^{2n}$

This verifies that the expression for T_{r+1} is correct for $r+1 = 1$.

Example 12.7 Find the fifth term in the expansion of

$$\left(1-\frac{2}{3}x^3\right)^6$$

Solution : Using here $T_{r+1} = T_5$ which gives $r+1 = 5$, i.e., $r = 4$.

Also $n = 6$ and let $x = 1$ and $y = \frac{-2}{3}x^3$.

$$T_5 = {}^6C_4\left(-\frac{2}{3}x^3\right)^4 = {}^6C_2\left(\frac{16}{81}x^{12}\right) = \frac{6\times 5}{2}\times\frac{16}{81}\times x^{12} = \frac{80}{27}x^{12}$$

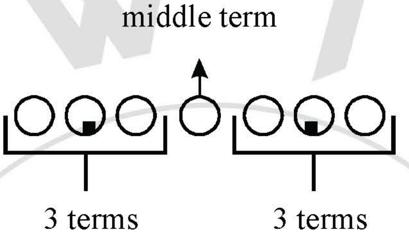
Thus, the fifth term in the expansion is $\frac{80}{27}x^{12}$.

12.3 MIDDLE TERMS IN A BINOMIAL EXPANSION

Now you are familiar with the general term of an expansion, let us see how we can obtain the **middle term** (or terms) of a binomial expansion. Recall that the number of terms in a binomial expansion is always one more than the exponent of the binomial. This implies that if the exponent is even, the number of terms is odd, and if the exponent is odd, the number of terms is even.

Thus, while finding the middle term in a binomial expansion, we come across two cases:

Case 1 : When n is even. To study such a situation, let us look at a particular value of n, say $n = 6$. Then the number of terms in the expansion will be 7. From Fig. 12.1, you can see that there are three terms on either side of the fourth term.



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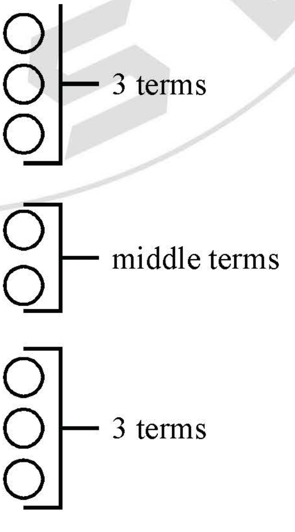
In general, when the exponent n of the binomial is even, there are $\frac{n}{2}$ terms on either side of the $\left(\frac{n}{2}+1\right)$th term. Therefore, the $\left(\frac{n}{2}+1\right)$th term is the middle term.

Case 2: When n is odd , Let us take $n = 7$ as an example to see what happens in this case. The number of terms in the expansion will be 8. Looking at Fig. 12.2, do you find any one middle term in it? There is not. But we can partition the terms into two equal parts by a line as shown in the figure. We call the terms on either side of the partitioning line taken together, the middle terms. This is because there are an equal number of terms on either side of the two, taken together.



Thus, in this case, there are two middle terms, namely, the fourth,

i.e., $\left(\frac{7+1}{2}\right)$ and the fifth, i.e., $\left(\frac{7+3}{2}\right)$ terms

Similarly, if $n = 13$, then the $\left(\frac{13+1}{2}\right)$th and the $\left(\frac{13+3}{2}\right)$th terms, i.e., the 7th and 8th terms are two middle terms, as is evident from Fig. 12.3.

From the above, we conclude that

Fig. 12.3

When the exponent n of a binomial is an odd natural number, then the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms are two middle terms in the corresponding binomial expansion.

Let us now consider some examples.

Example 12.8 Find the middle term in the expansion of $(x^2+y^2)^8$.

Solutuion : Here $n = 8$ (an even number).

Therefore, the $\left(\frac{8}{2}+1\right)$th, i.e., the 5th term is the middle term.

Putting $r = 4$ in the general term $T_{r+1} = {}^8C_r(x^2)^{8-r}y^r$, $T_5 = {}^8C_4(x^2)^{8-4}(y^2)^4 = 70x^8y^8$

Example 12.9 Find the middle term(s) in the expansion of $\left(2x^2+\frac{1}{x}\right)^9$.

Solution : Here $n = 9$ (an odd number). Therefore, the $\left(\frac{9+1}{2}\right)th$ and $\left(\frac{9+3}{2}\right)th$ are middle terms. i.e. T_5 and T_6 are middle terms.

For finding T_5 and T_6, putting $r = 4$ and $r = 5$ in the general term , $T_{r+1} = {}^9C_r(2x^2)^{9-r}\left(\frac{1}{x}\right)^6$,

$$T_5 = {}^9C_4(2x^2)^{9-4}\left(\frac{1}{x}\right)^4 \qquad = \frac{9\times8\times7\times6}{4\times3\times2}\times(32x^{10})\times\left(\frac{1}{x}\right)^4 = 4032\,x^6$$

and $T_6 = {}^9C_5\,(2x^2)^{9-5}\left(\dfrac{1}{x}\right)^5 = 2016\,x^3$

Thus, the two middle terms are $4032\,x^6$ and $2016\,x^3$.

LET US SUM UP

- For a natural number n,

  $$(x+y)^n = {}^nC_o\,x^n + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}\,y^2 + \ldots + {}^nC_{n-1}\,xy^{n-1} + {}^nC_n\,y^n$$

  This is called the **Binomial Theorem for a positive integral (or natural) exponent.**

- Another form of the Binomical Theorem for a positive integral exponent is $(1+a)^n = {}^nC_o + {}^nC_1\,a + {}^nC_2\,a^2 + \ldots. + {}^nC_{n-1}\,a^{n-1} + {}^nC_n\,a^n$

- The general term in the expansion of $(x+y)^n$ is ${}^nC_r\,x^{n-r}y^r$ and in the expansion of $(1+a)^n$ is ${}^nC_r a^r$, where n is a natural number and $0 \le r \le n$.

- If n is an even natural number, there is only one middle term in the expansion of $(x+y)^n$. If n is odd, there are two middle trems in the expansion.

- The formula for the general term can be used for finding the middle term(s) and some other specific terms in an expansion.