

## VOLUME AND SURFACE AREA

### Objectives:

After completing this section, you should be able to do the following:

- Calculate the volume of given geometric figures.
- Calculate the surface area of given geometric figures.
- Solve word problems involving volume and surface area.

### Vocabulary:

As you read, you should be looking for the following vocabulary words and their definitions:

- volume
- surface area
- sphere
- great circle of a sphere
- pyramid
- cone

### Formulas:

You should be looking for the following formulas as you read:

- volume of a rectangle solid
- surface area of a rectangular solid
- volume of a cylinder
- surface area of a cylinder
- volume of a solid with a matching base and top
- volume of a sphere
- surface area of a sphere
- volume of a pyramid
- volume of a cone

We will continue our study of geometry by studying three-dimensional figures. We will look at the two-dimensional aspect of the outside covering of the figure and also look at the three-dimensional space that the figure encompasses.

The *surface area* of a figure is defined as the sum of the areas of the exposed sides of an object. A good way to think about this would be as the

surface area

# Geometry Notes

area of the paper that it would take to cover the outside of an object without any overlap. In most of our examples, the exposed sides of our objects will be polygons whose areas we learned how to find in the previous section. When we talk about the surface area of a sphere, we will need a completely new formula.

The *volume* of an object is the amount of three-dimensional space an object takes up. It can be thought of as the number of cubes that are one unit by one unit by one unit that it takes to fill up an object. Hopefully this idea of cubes will help you remember that the units for volume are cubic units.

volume

## Surface Area of a Rectangular Solid (Box)

$$SA = 2(lw + lh + wh)$$

$l$  = length of the base of the solid  
 $w$  = width of the base of the solid  
 $h$  = height of the solid

## Volume of a Solid with a Matching Base and Top

$$V = Ah$$

$A$  = area of the base of the solid  
 $h$  = height of the solid

## Volume of a Rectangular Solid (specific type of solid with matching base and top)

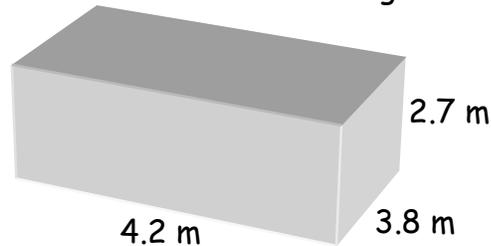
$$V = lwh$$

$l$  = length of the base of the solid  
 $w$  = width of the base of the solid  
 $h$  = height of the solid

## Geometry Notes

### Example 1:

Find the volume and the surface area of the figure below



Solution:

This figure is a box (officially called a rectangular prism). We are given the lengths of each of the length, width, and height of the box, thus we only need to plug into the formula. Based on the way our box is sitting, we can say that the length of the base is 4.2 m; the width of the base is 3.8 m; and the height of the solid is 2.7 m. Thus we can quickly find the volume of the box to be

$$V = lwh = (4.2)(3.8)(2.7) = 43.092 \text{ cubic meters.}$$

Although there is a formula that we can use to find the surface area of this box, you should notice that each of the six faces (outside surfaces) of the box is a rectangle. Thus, the surface area is the sum of the areas of each of these surfaces, and each of these areas is fairly straight-forward to calculate. We will use the formula in the problem. It will give us

$$SA = 2(lw + lh + wh) = 2(4.2 * 3.8 + 4.2 * 2.7 + 3.8 * 2.7) = 75.12$$

square meters.

A *cylinder* is an object with straight sides and circular ends of the same size. The volume of a cylinder can be found in the same way you find the volume of a solid with a matching base and top. The surface area of a cylinder can be easily found when you realize that you have to find the area of the circular base and top and add that to the area of the sides. If you slice the side of the cylinder in a straight line from top to bottom and open it up, you will see that it makes a rectangle. The base of the rectangle is the circumference of the circular base, and the height of the rectangle is the height of the cylinder.

cylinder

# Geometry Notes

## Volume of a Cylinder

$$V = Ah$$

$A$  = the area of the base of the cylinder

$h$  = the height of the cylinder

## Surface Area of a Cylinder

$$SA = 2(\pi r^2) + 2\pi rh$$

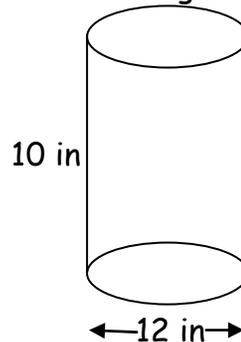
$r$  = the radius of the circular base of the cylinder

$h$  = the height of the cylinder

$\pi$  = the number that is approximated by 3.141593

### Example 2:

Find the volume and surface area of the figure below



### Solution:

This figure is a cylinder. The diameter of its circular base is 12 inches. This means that the radius of the circular base is

$r = \frac{1}{2}d = \frac{1}{2}(12) = 6$  inches. The height of the cylinder is 10 inches.

To calculate the volume and surface area, we simply need to plug into the formulas.

Surface Area:

$SA = 2(\pi r^2) + 2\pi rh = 2(\pi \cdot 6^2) + 2\pi(6)(10) = 72\pi + 120\pi = 192\pi$  square units. This is an exact answer. An approximate answer is 603.18579 square units.

# Geometry Notes

## Area of a Circle

$$A = \pi r^2$$

$r$  = radius of the circle  
 $\pi$  = the number that is approximated by 3.141593

Volume:

In order to plug into the formula, we need to recall how to find the area of a circle (the base of the cylinder is a circle). We will then replace  $A$  in the formula with the formula for the area of a circle.  
 $V = Ah = \pi r^2 h = \pi(6^2)(10) = 360\pi$  cubic inches. An approximation of this exact answer would be 1130.97336 cubic inches.

Our next set of formulas is going to be for spheres. A *sphere* is most easily thought of as a ball. The official definition of a sphere is a three-dimensional surface, all points of which are equidistant from a fixed point called the center of the sphere. A circle that runs along the surface of a sphere so that it cuts the sphere into two equal halves is called a *great circle of that sphere*. A great circle of a sphere would have a diameter that is equal to the diameter of the sphere.

sphere

great circle of a sphere

## Surface Area of a Sphere

$$SA = 4\pi r^2$$

$r$  = the radius of the sphere  
 $\pi$  = the number that is approximated by 3.141593

## Volume of a Sphere

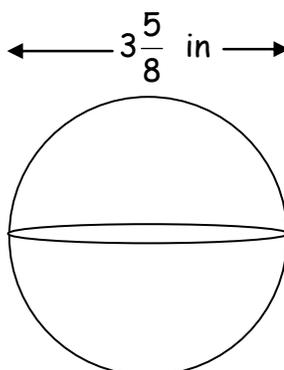
$$V = \frac{4}{3}\pi r^3$$

$r$  = the radius of the sphere  
 $\pi$  = the number that is approximated by 3.141593

## Geometry Notes

### Example 3:

Find the volume and surface area of the figure below



Solution:

This is a sphere. We are given that the diameter of the sphere is  $3\frac{5}{8}$  inches. We need to calculate the radius of the sphere to calculate the volume and surface area. The radius of a sphere is half of its diameter. This means that the radius is

$$r = \frac{1}{2}d = \frac{1}{2}\left(3\frac{5}{8}\right) = \frac{1}{2}\left(\frac{29}{8}\right) = \frac{29}{16} = 1.8125 \text{ inches.}$$

We can now just plug this number in to the formulas to calculate the volume and surface area.

$$\text{Volume: } V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1.8125^3) \approx 24.941505 \text{ cubic inches.}$$

$$\text{Surface Area: } SA = 4\pi r^2 = 4\pi(1.8125^2) \approx 41.28219 \text{ square inches.}$$

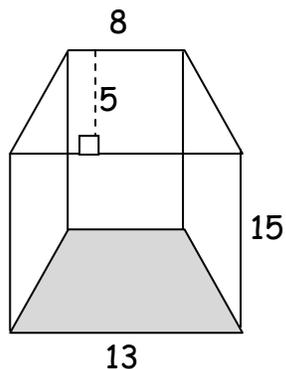
### Example 4:

Find the volume of the figure

Volume of a  
Solid with a  
Matching Base  
and Top

$$V = Ah$$

$A$  = area of the  
base of the solid  
 $h$  = height of the  
solid



Solution:

**Area of a Trapezoid**

$$A = \frac{1}{2}(b_1 + b_2)h$$

$b_1$  = the one base of the trapezoid

$b_2$  = the other base of the trapezoid

$h$  = the height of the trapezoid

This figure is a solid with the same shape base and top. The shape of the base and top is a trapezoid. Thus we will need to remember the formula for the area of a trapezoid. For this trapezoid, the lengths of the bases are 13 and 8 units. It does not matter which of these we say is  $b_1$  and which is  $b_2$ . The height of the trapezoid is 5 units. The height of the solid is 15 units. We will start by plugging the information about the trapezoidal base into the formula for the area of a trapezoid. Once we have this area, we will plug that and the height of the solid into the volume formula.

Area of the trapezoidal base:

$$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(13 + 8)5 = 52.5 \text{ square units.}$$

Volume of trapezoidal solid:

$$V = Ah = (52.5)(15) = 787.5 \text{ cubic units.}$$

Our next formulas will be for finding the volume of a *cone* or a *pyramid*. These two formulas are grouped together since they are very similar. Each is basically 1/3 times the area of the base of the solid times the height of the solid. In the case of the cone, the base is a circle. In the case of the pyramid, we will have a base that is a rectangle. The height in both cases is the perpendicular distance from the apex to the plane which contains the base.

A pyramid is a solid figure with a polygonal base (in our case a rectangle) and triangular faces that meet at a common point (the apex). A cone is the surface of a conic solid whose base is a circle. This is more easily thought of as a pointed ice-cream cone whose top is circular and level.

pyramid  
cone

**Volume of a Rectangular Pyramid**

$$V = \frac{1}{3}lwh$$

$l$  = the length of the base of the pyramid

$w$  = the width of the base of the pyramid

$h$  = the perpendicular height of the pyramid

## Volume of a Cone

$$V = \frac{1}{3}\pi r^2 h$$

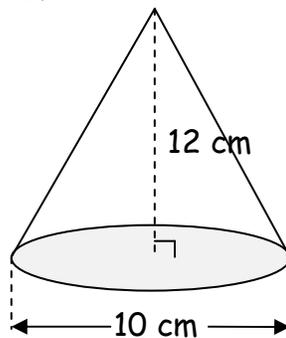
$r$  = the radius of the circular base of the cone

$\pi$  = the number that is approximated by 3.141593

$h$  = the perpendicular height of the cone

### Example 5:

Find the volume of the figure.



Solution:

Since the figure has a circular base and looks like an ice cream cone, this must be a cone. In order to find the volume of a cone, we need the radius of the circular base and the height (perpendicular height) of the cone. The height is given as 12 centimeters. The other measurement of 10 centimeters is the diameter of the circular base. We thus must calculate the radius

to get  $r = \frac{1}{2}d = \frac{1}{2}(10) = 5$  centimeters. We are now ready to plug

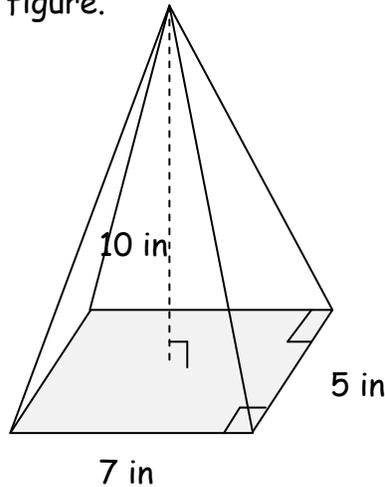
into the volume of a cone formula.  $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(5^2)(12) = 100\pi$

cubic centimeters is the exact volume. An approximation of this volume would be 314.159266 cubic centimeters.

# Geometry Notes

## Example 6:

Find the volume of the figure.



Solution:

The base of this figure is a rectangle and the sides of the figure are triangles, thus this figure is a rectangular pyramid. The height (perpendicular height) is 10 inches. The length of the base is 7 inches, and the width of the base is 5 inches. Since we have all of the parts for the volume formula, we can just plug into the volume of a rectangular pyramid formula to get

$V = \frac{1}{3}lwh = \frac{1}{3}(7)(5)(10) = \frac{350}{2}$  cubic inches. An approximation of this volume would be 116.66667 cubic inches.

Just like with areas, we can add and subtract volumes of different solids to find the volume of a solid that is a combination of more than one solid or that have one solid removed from another.

## Example 7:

Find the volume of the figure.



Solution:

## Geometry Notes

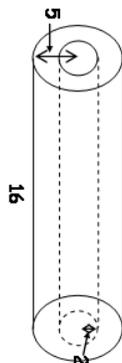
### Volume of a Cylinder

$$V = Ah$$

$A$  = the area of the base of the cylinder

$h$  = the height of the cylinder

The first thing that we need to do is figure out what type of figure this is. If we rotate the solid 90 degrees to the right, we get a figure that looks like this



This looks like a cylinder with the middle missing. A good way to think about this figure is a roll of paper towels. We are trying to find out the volume of the paper towels. The best way to do this is to figure out what the volume of the larger cylinder is without the missing part. We can then find the volume of the smaller cylinder or missing part. Finally, we will subtract the volume of the smaller cylinder from the volume of the larger cylinder to get the volume of our current solid.

We will start with the larger cylinder. The radius of the circular base is given as 5 units. The height of the larger cylinder is 16 units. We can then calculate the volume of the larger cylinder to be  $V = Ah = \pi r^2 h = \pi(5^2)(16) = 400\pi$  cubic units.

Next we will calculate the volume of the smaller cylinder. The radius of the circular base of the smaller cylinder is 2 units. The height of the smaller cylinder is 16 units. We can calculate the volume of the smaller cylinder would be

$$V = Ah = \pi r^2 h = \pi(2^2)(16) = 64\pi \text{ cubic units.}$$

We now subtract the volume of the smaller cylinder from the volume of the larger cylinder to get the volume of our solid.

Volume of larger cylinder - volume of smaller cylinder =  
 $400\pi - 64\pi = 336\pi$  cubic units. This is approximately 1055.57513 cubic units.

## Geometry Notes

### Example 8:

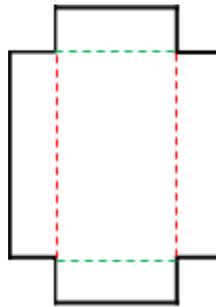
From an 8.5-inch by 11-inch piece of cardboard, 2-inch square corners are cut out and the resulting flaps are folded up to form an open box. Find the volume and surface area of the box.

#### Solution:

For this problem, it will be really helpful to make the box described above. You start with a standard piece of paper. You cut out the dashed square indicated below from each corner. You make sure each side of the square is 2 inches in length.



What you are left with is the shape below.



You now fold up along the dashed lines to create a box. The box that we have created is a rectangular solid. This box has no top. Not having a top will not affect the volume of the box.

We only need to determine the length of the base of the box, the width of the base of the box, and the height of the box. The red dashed lines represents the length of the base of the box. The original length of the paper was 11 inches. We removed 2 inches from the top of the page and we also removed 2 inches from the

## Geometry Notes

bottom of the page. Thus the red dashed line is  $11 - 2 - 2 = 7$  inches.

The green dashed line represents the width of the base of the box. The original width of the paper was 8.5 inches. We removed 2 inches from the left side of the page and also removed 2 inches from the right side of the page. Thus the green dashed line is  $8.5 - 2 - 2 = 4.5$  inches.

We now need to think about the height of the box. Since we have folded up the sides for form the height of the box, we just need to determine how tall those sides are. Since they were made by cutting out 2-inch square from each corner, these sides must be 2 inches high.

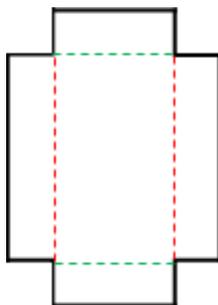
Now we are ready to calculate the volumes

$$V = lwh = (7)(4.5)(2) = 63 \text{ cubic inches.}$$

Now we need to calculate the surface area of our box. Since there is no top to this box, we can start formula for the surface area of a box. We will then need to subtract off the area of the top of the box. This will give us

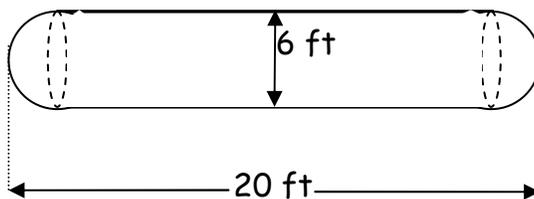
$SA = 2(lw + wh + lh) = 2(7 \cdot 4.5 + 4.5 \cdot 2 + 7 \cdot 2) = 109$  square inches for the box with the top included. The top would have the same area as the base of the box. This would be  $A = lw = (7)(4.5) = 31.5$  square inches. Thus the surface area of our figure is total surface area - area of the top =  $109 - 31.5 = 77.5$  square inches.

There is another way to calculate the surface area of this box. The surface area is the amount of paper it would take to cover the box without overlap. You should notice that this is the same as the amount of paper we used to make the box. Thus, it is enough to calculate the area of the paper as shown here.



## Example 9:

A propane gas tank consists of a cylinder with a hemisphere at each end. Find the volume of the tank if the overall length is 20 feet and the diameter of the cylinder is 6 feet.



**Solution:**

We are told this tank consists of a cylinder (one its side) with a hemisphere at each end. A hemisphere is half of a sphere. To find the volume, we need to find the volume of the cylinder and the volumes of each hemisphere and then adding them together.

Let's start with the hemisphere sections. The diameter of the circular base of the cylinder is indicated to be 6 feet. This would also be the diameter of the hemispheres at each end of the cylinder. We need the radius of the sphere to find its volume. Once we calculate the volume of the whole sphere, we multiply it by  $\frac{1}{2}$  to find the volume of the hemisphere (half of a sphere). We calculate the radius to be  $r = \frac{1}{2}d = \frac{1}{2}(6) = 3$  feet. Now we can calculate the volume of the two hemispheres at the ends of the tank.

Left Hemisphere:

The volume of a whole sphere is  $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3^3) = 36\pi$  cubic feet. We now multiply this volume by  $\frac{1}{2}$  to find the volume of the hemisphere to get  $\frac{1}{2}V = \frac{1}{2}(36\pi) = 18\pi$  cubic feet.

### Volume of a Sphere

$$V = \frac{4}{3}\pi r^3$$

$r$  = the radius of the sphere  
 $\pi$  = the number that is approximated by 3.141593

## Geometry Notes

Right Hemisphere:

The volume of a whole sphere is  $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3^3) = 36\pi$  cubic feet. We now multiply this volume by  $\frac{1}{2}$  to find the volume of the hemisphere to get  $\frac{1}{2}V = \frac{1}{2}(36\pi) = 18\pi$  cubic feet.

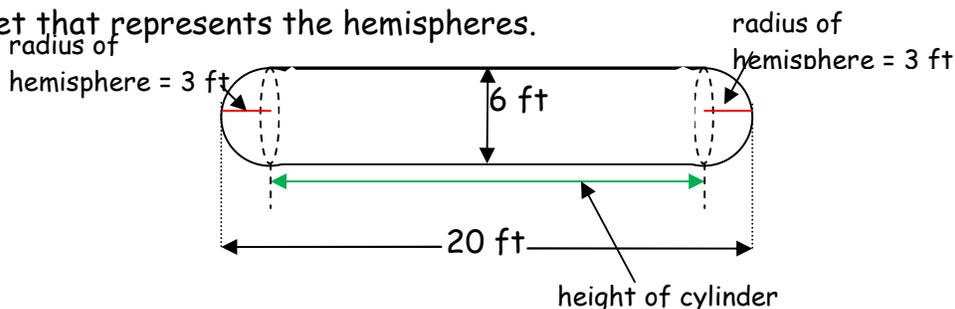
You might notice that we went through exactly the same process with exactly the same number for each hemisphere. We could have shortened this process by realizing that putting together the two hemispheres on each end, which were of the same diameter, would create a whole sphere. We could just have calculated the volume of this whole sphere.

### Volume of a Cylinder

$$V = Ah$$

$A$  = the area of the base of the cylinder  
 $h$  = the height of the cylinder

Now we need to calculate the volume of the cylinder. We need the radius of the cylinder and the height of the cylinder to find its volume. The radius of the cylinder is the same as the radius of the hemispheres at each end. Thus the radius of the cylinder is 3 feet. It may look like the height of the cylinder is 20 feet. It turns out that this is not the case. The 20 feet includes the hemispheres at each end. We need to subtract the part of the 20 feet that represents the hemispheres.



Looking at the figure above, we see that the distance from the end of the left hemisphere to the left end of the cylinder is the radius of the hemisphere. The radius of the hemisphere is 3 feet. Similarly, the distance from the right end of the cylinder to the end of the right hemisphere is also the radius of the hemisphere. The radius of the hemisphere is 3 feet. If we now subtract these two distances from the overall length of the tanks, we will have the

## Geometry Notes

height of the cylinder to be  $h = 20 - 3 - 3 = 14$  feet. We can now calculate the volume of the cylinder.

Cylinder

$$V = Ah = \pi r^2 h = \pi(3^2)(14) = 126\pi \text{ cubic feet.}$$

We can now find the volume of the tank by adding together the volumes of the cylinder, the right hemisphere, and the left hemisphere. We get the volume to be  $V = 126\pi + 18\pi + 18\pi = 162\pi$  cubic feet. This is approximated by 508.938 cubic feet.

### Example 10:

A regulation baseball (hardball) has a great circle circumference of 9 inches; a regulation softball has a great circle circumference of 12 inches.

- Find the volumes of the two types of balls.
- Find the surface areas of the two types of balls.

Solution:

Part a:

In order to find the volume of a sphere, we need the radius of the sphere. In this problem, we are not given the radius. Instead we are given the circumference of a great circle of the sphere. From this information, we can calculate the radius of the great circle, which is the radius of the sphere.

Radius of the baseball:

We calculate the radius of the baseball by plugging in the circumference of the great circle of the baseball into the formula for the circumference of the circle and solve for  $r$  (the radius).

$$C = 2\pi r$$

$$9 = 2\pi r$$

$$\frac{9}{2\pi} = r$$

Circumference  
(Perimeter) of a  
Circle

$$C = 2\pi r$$

$r$  = radius of the  
circle

$\pi$  = the number

## Geometry Notes

### Volume of a Sphere

$$V = \frac{4}{3} \pi r^3$$

$r$  = the radius of the sphere  
 $\pi$  = the number that is approximated by 3.141593

Now that we have the radius of the baseball, we can calculate the volume, by plugging the radius into the formula for the volume of a sphere.

$$V = \frac{4}{3} \pi r^3$$

$$V = \frac{4}{3} \pi \left( \frac{9}{2\pi} \right)^3$$

$$V = \frac{4 \cdot 9^3 \pi}{3 \cdot 2^3 \pi^3}$$

$$V = \frac{2916\pi}{24\pi^3}$$

$$V = \frac{243}{2\pi^2} \text{ cubic inches}$$

This is approximately 12.3105 cubic inches.

Radius of the softball:

We will go through the same process with the softball to calculate the radius of the softball.

$$C = 2\pi r$$

$$12 = 2\pi r$$

$$\frac{12}{2\pi} = r$$

$$\frac{6}{\pi} = r$$

Now that we have the radius of the softball, we can calculate the volume, by plugging the radius into the formula for the volume of a sphere.

$$V = \frac{4}{3} \pi r^3$$

$$V = \frac{4}{3} \pi \left( \frac{6}{\pi} \right)^3$$

$$V = \frac{4 \cdot 6^3 \pi}{3 \cdot \pi^3}$$

$$V = \frac{864\pi}{3\pi^3}$$

$$V = \frac{288}{\pi^2} \text{ cubic inches}$$

This is approximately 29.1805 cubic inches.

## Geometry Notes

### Surface Area of a Sphere

$$SA = 4\pi r^2$$

$r$  = the radius of  
the sphere  
 $\pi$  = the number  
that is  
approximated by  
3.141593

Part b:

In order to find the surface area of a sphere, we need the radius of the sphere. We calculated the radius for each type of ball in part a. We only need to plug this information in to the formula for surface area of a sphere.

Surface area of a baseball:

$$SA = 4\pi \left(\frac{9}{2\pi}\right)^2$$

$$SA = \frac{4\pi(9)^2}{2^2 \pi^2}$$

$$SA = \frac{324\pi}{4\pi^2}$$

$$SA = \frac{81}{\pi} \text{ square inches}$$

This is approximately 25.7831 square inches.

Surface area of a softball:

$$SA = 4\pi \left(\frac{6}{\pi}\right)^2$$

$$SA = \frac{4\pi(6)^2}{\pi^2}$$

$$SA = \frac{144\pi}{\pi^2}$$

$$SA = \frac{144}{\pi} \text{ square inches}$$

This is approximately 45.83662 square inches.

Our final example is an application problem. We will need to be able to use dimensional analysis, volumes, and common sense in order to be able to answer the question.

### Example 11:

Mike Jones bought an older house and wants to put in a new concrete driveway. The driveway will be 30 feet long, 10 feet wide, and 9 inches thick. Concrete (a mixture of sand, gravel, and cement) is measured by the cubic yard. One sack of dry cement mix costs \$7.30, and it takes

## Geometry Notes

four sacks to mix up 1 cubic yard of concrete. How much will it cost Mike to buy the cement?

Solution:

The driveway that is being poured will be a rectangular solid or box. Thus in order to answer this question, we will first need to find the volume of this box (the amount of cubic units it will take to fill this box). The problem tells us that concrete is measured by the cubic yard. This lets us know that those are the units we will want to calculate with. None of the dimensions of the driveway are given in yards. We will need to use our dimensional analysis (unit conversion) to convert all of the measurements to yards. This could be done at a later point, but this is the easiest place to take care of the conversion.

Convert length:

$$\frac{30 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = 10 \text{ yards}$$

Convert width:

$$\frac{10 \text{ feet}}{1} \cdot \frac{1 \text{ yard}}{3 \text{ feet}} = \frac{10}{3} \text{ yards}$$
 We are not going to estimate this

value since the approximation with introduce error. As in the finance section, we don't want to round until the end of the problem or in a place where it is absolutely necessary.

Convert height:

$$\frac{9 \text{ inches}}{1} \cdot \frac{1 \text{ yard}}{36 \text{ inches}} = \frac{1}{4} \text{ yards} = .25 \text{ yards}$$
 Since the decimal

representation of this number is a terminating decimal, we can use this representation in our calculations.

Volume of driveway:

We are now ready to calculate the volume of the driveway.

$$V = lwh = (10) \left( \frac{10}{3} \right) (.25) = \frac{25}{3} \text{ cubic yards.}$$

## Geometry Notes

Now, we are told that it takes 4 bags of cement to make one cubic yard of concrete. So we will now calculate how many bags of cement to buy. Mike's driveway is  $\frac{25}{3}$  cubic yards and each of those cubic yards requires 4 bags of cement. Thus Mike will need  $\frac{25}{3} \cdot 4 = \frac{100}{3}$  bags or approximately 33.333 bags. Here is where common sense needs to come in. There is no store that will sell .333 bags of cement mix. Stores only sell whole bags of cement mix. Thus we will need to round up to the next whole bag (we can't round down or we will not have enough cement mix to complete the driveway). This means that Mike will need to buy 34 bags of cement mix. Each of these bags will cost \$7.30. This means that Mike will pay  $7.30 \cdot 34 = \$248.20$  for the cement mix for this driveway.