FACTORISATION AND QUADRATIC EQUATIONS

If at any stage you want to return to where you came from use the Back Button

1 . Removing Brackets

Scientists and Engineers use algebra as a concise way of describing relationships between physical quantities . It follows that the mathematical analysis of Engineering/Science problems frequently requires skill in manipulating algebraic expressions. In these notes we describe various methods for changing the appearance of algebraic expressions. Our starting point is **the distributive laws** of arithmetic .

$$a(b + c) = ab + ac$$

$$\text{and } (a + b)(c + d) = ac + ad + bc + bd$$

where a, b, c and d represent real numbers.

In addition we shall be using the laws of signs which we covered in the notes on **Numbers and the Laws of Arithmetic.**

Examples

1. $4(x + y) = 4x + 4y$ 2. $4(x - y) = 4x - 4y$

3. $5(x + 7) = 5x + 35$ 4. $7(-x + 2) = -7x + 14$

5. $-3(x + y) = -3x - 3y$ 6. $-5(x - y) = -5x + 5y$

6. $(x + 3)(x + 4) = x^2 + 3x + 4x + 12 = x^2 + 7x + 12$
      Notice how the terms $3x + 4x$ are combined .

7. $(x - 6)(x + 4) = x^2 + 4x - 6x - 24 = x^2 - 2x - 24$

8. $(x - 5)(x - 2) = x^2 - 7x + 10$
       Here we have combined the like terms in one step.

9.   We can extend the application to brackets containing three or more terms:-
      $$(2x + y - 2)(x - y) = 2x^2 - 2xy + yx - y^2 - 2x + 2y$$
      $$= 2x^2 - xy - y^2 - 2x + 2y$$
      To follow this think of it as $2x(x - y) + y(x - y) - 2(x - y)$

10.   To deal with expressions like $(x - 3)(x + 2)(2x + y)$ multiply two
      of the terms first .(It doesn't matter which two you choose)
      $$(x - 3)(x + 2)(2x + y) = (x^2 + 2x - 3x - 6)(2x + y) = (x^2 - x - 6)(2x + y)$$
      $$= 2x^3 + x^2 y - 2x^2 - xy - 12x - \quad y \qquad 6$$

      or
      $$(x - 3)(x + 2)(2x + y) = (x - 3)(2x^2 + xy + 4x + 2y)$$
      $$= 2x^3 + x^2 y + 4x^2 + 2xy - 6x^2 - 3xy - 12x - 6y$$
      $$= 2x^3 + x^2 y - 2x^2 - xy - 12x - \quad y \qquad 6$$

Tutorial 1

Multiply out and simplify your answers where possible .

(1) $3(x+2)$

(2) $3(x+2)+4(x+3)$

(3) $2(x+2)+(x-1)(x-2)$

(4) $4(x+1)-4(x-1)$

(5) $(x+1)^2$

(6) $(x+1)(x+2)$

(7) $(x+1)(x-1)$

(8) $(x^2+2x+3)-(x+1)^2$

(9) $(x+1)^2-(x^2-3x+2)$

(10) $(x+1)^2-(x-1)^2$

(11) $(x-1)^2-(x+1)^2$

(12) $(2x-1)(3x+1)$

(13) $(x-1)^3$ (This is another way of writing $(x-1)(x-1)(x-1)$)

(14) $(2x^2-x+2)(x^2+x-2)$

(15) $(x+y)(x+2y)$

(16) $(x-y)(x^2+xy-y^2)$

(17) $(x+2y-2)^2$

(18) $(x+2y)^3$

[Click here for the solutions to Tutorial 1](#). You can use the back button in the Explorer to return here.

2. Factorisation

What we are concerned with here is the reverse of the procedure followed in the previous section.
i.e. Given the answers to the questions how to work back to the original expression. This process is called factorisation .
For example we know from Example 10 above that

$$2x^3 + x^2y - 2x^2 - xy - 12x - 6y = (x-3)(x+2)(2x+y)$$

The terms $x-3, x+2,$ and $2x+y$ are called the factors of the expression on the left and in this case finding these factors would be a very difficult task. First something simpler.

2.1 Common Factors
We start with some examples in which we are looking for what are called **common factors**.

Examples

1. $4x + 4y = 4(x+y)$ Both terms on the left are divisible by 4 . We say that 4 is a **common factor** of both terms .

2. $3a + 12b = 3(a+4b)$ 3. $21a - 14b = 7(3a-2b)$ 4. $-9x - 6y = -3(3x+2y)$

5. $a^2b + a = a(ab+1)$ 6. $a^3b - a^2b^2 = a^2b(a-b)$ 7.
$-2x^2y - 14xy^2 = -2xy(x+7y)$

8. $(x+1)x + (x+1)^2 = (x+1)(x+(x+1)) = (x+1)(2x+1)$

9. $3a^2b + 9ab^2 - 6ab = 3ab(a+3b-3)$

10. $4a^3b^2c - 12a^2b^3c^2 + 2ab^2c^2 = 2ab^2c(2a^2 - 6abc + c)$

Tutorial 2.1

Find the common factors and hence factorise the following:

1. $3x^2 + 24xy$

2. $-5xy^2 + 10x^2y$

3. $4p^2q + 12pq^2 + 24pq$

4. $9a^2b^3c^3 + 15a^3b^2c^2 + 21abc^2$

5. $-8p^2q - 12pq^2 + 16p^2q^2$

6. $(2x-y)(x+y) - (2x-y)^2$

7. $a^2bx + a^2by + a^3b^2 + a^2b$

8. $(x^2 - 1) + (x+1)$ Hint: $x^2 - 1 = (x+1)(x-1)$

9. $-(a+2b)^2 + a(a+2b) + 2b(a+2b)$

10       $8ax^2 + 6bx^2 - 4a - 3b$ Hint: Factorise the first two terms .

[Click here for the solutions to Tutorial 2.1](#). You can use the back button in the Explorer to return here.

2.2 Factorising Quadratic Expressions

Any expression of the form $ax^2 + bx + c$ where a, b and c are real numbers is said to be a quadratic expression. The numbers a, b and c are called **the coefficients** of the expression . a is the coefficient of x^2 , b is the coefficient of x and c is called the constant term .

In all the examples we shall look at the coefficients will be integers.

Eg $2x^2 - 4x + 3$, $x^2 + 2x + 1$, $3x^2 + 4$ etc

In the second of these $a = 1$ but of course we do not write $1x^2$, and in the third $b = 0$.

In some cases the factors can be found by a process of inspection . To see how to do this we shall look at two different categories.

TYPE (A)   a =1

Study the following examples:-
    (i)   By multiplying out the left hand side we see that $(x+2)(x+3) = x^2 + 5x + 6$

On the right hand side $b = 5$ and $c = 6$. Notice that $b = 2 + 3$ and $c = 2 \times 3$

    (ii) By multiplying out the left hand side we see that $(x-2)(x-3) = x^2 - 5x + 6$

This time $b = -5$ and $c = 6$ and notice that $b = (-2) + (-3)$ and $c = (-2) \times (-3)$

    (iii) By multiplying out the left hand side we see that $(x-2)(x+3) = x^2 + x - 6$

This time $b = 1$ and $c = -6$ and notice that $b = (-2) + 3$ and $c = (-2) \times 3$

    (iv) By multiplying out the left hand side we see that $(x+2)(x-3) = x^2 - x - 6$

This time $b = -1$ and $c = -6$ and notice that $b = 2 + (-3)$ and $c = 2 \times (-3)$

In all these examples b is the sum of two numbers which when multiplied give c .These two numbers may be both positive , both negative or one positive and one negative .

In example (i) c and b are both positive and we have two positive factors of c .

In example (ii) c is positive and b is negative and we have two negative factors of c .

In examples (iii) and (iv) c is negative and therefore we must have one positive and one negative factor of c ,and b may be positive or negative .

Examples

1. Factorise $x^2 + 6x + 8$. In this case b and c are both positive .
   We look at all the possible factors of 8 and choose the pair that add to 6
   $8 = 1 \times 8 = 2 \times 4$ and $6 = 2 + 4$. Hence $x^2 + 6x + 8 = (x+2)(x+4)$

2. Factorise $x^2 - 6x + 8$. In this case b is negative and c is positive .
   $8 = (-1) \times (-8) = (-2) \times (-4)$ and $-6 = (-2) + (-4$ Hence
   $x^2 - 6x + 8 = (x-2)(x-4)$

3. Factorise $x^2 + 2x - 8$. In this case c is negative .
   $-8 = (-1) \times 8 = 1 \times (-8) = (-2) \times 4 = 2 \times (-4)$ and $2 = (-2) + 4$.
   Hence $x^2 + 2x - 8 = (x-2)(x+4)$

4. Clearly $x^2 - 2x - 8 = (x+2)(x-4)$

5. Factorise $x^2 + 5x - 24$.
   $-24 = (-1) \times 24 = 1 \times (-24) = (-2) \times 12 = 2 \times (-12)$
   $\qquad = (-3) \times 8 = 3 \times (-8) = (-4) \times 6 = 4 \times (-6)$
   Hence $x^2 + 5x - 24 = (x-3)(x+8)$

With practice it becomes unnecessary to write down all the possible factors of c and in the following examples we go straight to the answer .

$$x^2 + 8x + 12 = (x+6)(x+2)$$
$$x^2 - 8x + 12 = (x-6)(x-2)$$
$$x^2 + 4x - 12 = (x+6)(x-2)$$
$$x^2 - 4x - 12 = (x-6)(x+2)$$
$$x^2 + 7x + 12 = (x+4)(x+3)$$
$$x^2 - 7x + 12 = (x-4)(x-3)$$
$$x^2 + x - 12 = (x+4)(x-3)$$
$$x^2 - x - 12 = (x-4)(x+3)$$
$$x^2 + 13x + 12 = (x+12)(x+1)$$
$$x^2 + 13x + 12 = (x-12)(x-1)$$
$$x^2 + 11x - 12 = (x+12)(x-1)$$
$$x^2 - 11x - 12 = (x-12)(x+1)$$

Type (B) a≠1

To factorise $2x^2 + 9x - 18$ note first that $2x^2 = (2x) \times x$ so that the factors , if they exist ,will possibly be of the form: $(2x \quad)(x \quad)$

Also we have $-18 = (-1) \times 18 = 1 \times (-18) = (-2) \times 9 = 2 \times (-9) = (-3) \times 6 = 3 \times (-6)$.

After some trial and error we arrive at:

$$2x^2 + 9x - 18 = (2x - 3)(x + 6)$$

In such cases one writes down the various possibilities and proceeds by a process of ' intelligent ' trial and error.

Here are some more examples which you should verify by expanding the right hand sides

$$2x^2 - 15x + 18 = (2x - 3)(x - 6)$$
$$2x^2 - 9x - 18 = (2x + 3)(x - 6)$$
$$2x^2 + 9x - 18 = (2x - 3)(x + 6)$$
$$2x^2 + 13x + 15 = (2x + 3)(x + 5)$$
$$2x^2 - 13x + 15 = (2x - 3)(x - 5)$$
$$2x^2 + 7x - 15 = (2x + 3)(x - 5)$$
$$2x^2 - 7x - 15 = (2x - 3)(x + 5)$$
$$2x^2 + 11x + 15 = (x + 3)(2x + 5)$$
$$2x^2 - 11x + 15 = (x - 3)(2x - 5)$$
$$2x^2 + x - 15 = (x + 3)(2x - 5)$$
$$2x^2 - x - 15 = (x - 3)(2x + 5)$$
$$3x^2 + 5x - 2 = (3x - 1)(x + 2)$$

Finally in this section we need to point out that factorisation in this way is not always possible .

For example $x^2 + 12x + 8$. Integer factors of 8 which add up to 12 do not exist. However the expression does have real factors. In fact

$$x^2 + 12x + 8 = (x + 6 + 2\sqrt{7})(x + 6 - 2\sqrt{7})$$

as you can verify by carefully multiplying out the right hand side but it is impossible to find these factors by the inspection method described above. Problems like this are covered in Section 3 .

Tutorial 2.2

Factorise the following by inspection :-

(i) $x^2 + 3x + 2$

(ii) $x^2 - 3x + 2$

(iii) $x^2 + x - 2$

(iv) $\quad x^2 - x - 2$

(v) $\quad x^2 - 3x - 18$

(vi) $\quad x^2 + 3x - 4$

(vii) $\quad x^2 - 5x + 6$

(viii) $\quad x^2 - 2x - 8$

(ix) $\quad 2x^2 + 5x - 12$

(x) $\quad 2x^2 + x - 21$

(xi) $\quad 3x^2 + 2x - 8$

(xii) $\quad x^2 - 9x + 20$

(xiii) $\quad x^2 + 12x - 24$

(xiv) $\quad x^2 + 5x + 6$

(xv) $\quad x^2 + 5x - 6$

(xvi) $\quad x^2 - 8x + 15$

(xvii) $\quad x^2 + 11x + 18$

(xviii) $\quad x^2 - 11x + 18$

(xix) $\quad 3x^2 - 7x - 6$

(xx) $\quad 6 - x - x^2$

(xxi) $\quad 2x^2 - x - 15$

(xxii) $\quad 2x^2 - x - 3$

(xxiii) $\quad 6x^2 - 19x + 10$

(xxiv) $\quad 2x^2 + 11x + 15$

(xxv) $\quad 12 + x - 6x^2$

(xxvi) $\quad x^4 + 10x^2 + 24$

(xxvii) $\quad x^2 + 6x + 8$

(xxviii) $\quad x^2 - 7x - 18$

(xxix) $\quad 3 + x - 2x^2$

(xxx) $\quad 4x^2 + 12x + 9$

Click here for the solutions to Tutorial 2.2. You can use the back button in the Explorer to return here.

3 . Completing the Square

Completing the square is the name given to an algebraic process which shows whether or not a quadratic expression has real factors and finds the factors if they exist.

Before we introduce the process we need to recap three results which we have seen earlier.

(a) **The difference of two squares :-**
$$x^2 - k^2 = (x + k)(x - k)$$

Examples $\quad x^2 - 16 = (x + 4)(x - 4)$
$$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$$
$$(x + 2)^2 - 9 = (x + 2 + 3)(x + 2 - 3) = (x + 5)(x - 1)$$

(b)   **Perfect square**
$$x^2 + 2kx + k^2 = (x+k)^2 \ and \ x^2 - 2kx + k^2 = (x-k)^2$$

Examples $x^2 + 4x + 4 = (x+2)^2$
$$x^2 - 6x + 9 = (x-3)^2$$

(c)   **Sum of two squares**
$$x^2 + k^2 \quad \text{has no real factors.}$$

For example try to factorise $x^2 + 9$ using the method of section 2 above . You need to find two real numbers which give 9 when multiplied and zero when summed . Clearly impossible .

We now introduce the process known as **completing the square** by means of the following example .

Consider the quadratic expression $x^2 + 10x + 21$.

We know from (b) above that $x^2 + 10x + 25 = (x+5)^2$.

Therefore $x^2 + 10x + 21 = x^2 + 10x + 25 + 21 - 25 = (x+5)^2 + 21 - 25$

Thus $x^2 + 10x + 21 = (x+5)^2 - 4$

Changing a quadratic expression in this way is called **completing the square .**

Let us analyse the example :- The given quadratic expression is replaced by

$$\left(x + \frac{coefficient\ of\ x}{2} \right)^2 + corrected\ constant\ term$$

Examples

1. $x^2 + 2x + 4 = (x+1)^2 + 4 - 1$
$$= (x+1)^2 + 3$$

2. $x^2 + 2x - 4 = (x+1)^2 - 4 - 1$
$$= (x+1)^2 - 5$$

3. $x^2 - 6x + 2 = (x-3)^2 + 2 - 9$
$$= (x-3)^2 - 7$$

4.
$$x^2 + 5x + 2 = \left(x + \frac{5}{2}\right)^2 + 2 - \frac{25}{4}$$
$$= \left(x + \frac{5}{2}\right)^2 - \frac{17}{4}$$

5.
$$x^2 + 7x + 13 = \left(x + \frac{7}{2}\right)^2 + 13 - \frac{49}{4}$$
$$= \left(x + \frac{7}{2}\right)^2 + \frac{3}{4}$$

So much for the process of completing the square . We now look at some examples of its application in finding factors of quadratic expressions.

Examples

1. $x^2 + 4x - 12 = (x + 2)^2 - 12 - 4 = (x + 2)^2 - 16$

The right hand side is the difference of two squares , and as we saw at the beginning of Section 3 it can be factorised to give :

$$x^2 + 4x - 12 = (x + 2)^2 - 16 = (x + 2 + 4)(x + 2 - 4) = (x + 6)(x - 2)$$

In this particular example the factors could have easily been found by inspection as we did in Section 2. But as the following examples show , completing the square will find factors that could not possibly be found by inspection.

2. $x^2 + 12x + 8 = (x + 6)^2 + 8 - 36 = (x + 6)^2 - 28 = (x + 6)^2 - \left(\sqrt{28}\right)^2$

Therefore $x^2 + 12x + 8 = (x + 6 + \sqrt{28})(x + 6 - \sqrt{28})$

3. $x^2 - 7x + 8 = \left(x - \frac{7}{2}\right)^2 + 8 - \frac{49}{4} = \left(x - \frac{7}{2}\right)^2 - \frac{17}{4}$
$$= \left(x - \frac{7}{2} + \frac{\sqrt{17}}{2}\right)\left(x - \frac{7}{2} - \frac{\sqrt{17}}{2}\right)$$

4. $x^2 + 8x + 20 = (x + 4)^2 + 20 - 16 = (x + 4)^2 + 4$

In this case the right hand side is the sum of two squares and as we saw in result (c) at the start of this section such expressions have no real factors.
Expressions that have no real factors are called **irreducible .**

So far we have only considered quadratics in which the coefficient of x^2 has been unity .The following examples show how to deal with general quadratics .

5. $2x^2 + 7x + 3 = 2\left(x^2 - \frac{7}{2}x + \frac{3}{2}\right) = 2\left(\left(x - \frac{7}{4}\right)^2 + \frac{3}{2} - \frac{49}{16}\right) = 2\left(\left(x - \frac{7}{4}\right)^2 - \frac{25}{16}\right)$

$$= 2\left(x + \frac{7}{4} + \frac{5}{4}\right)\left(x + \frac{7}{4} - \frac{5}{4}\right) = 2\left(x + \frac{12}{4}\right)\left(x + \frac{2}{4}\right)$$

$$= 2(x+3)(x + \frac{1}{2}) = (x+3)(2x+1)$$

These factors could have been found by inspection of course so the question arises which method should be used ?

It is always worth spending a minute trying to factorise by inspection then if nothing seems to work proceed by completing the square .

6. $3x^2 - 15x + 12 = 3\left(x^2 - 5x + 4\right) = 3\left((x - \frac{5}{2})^2 + 4 - \frac{25}{4}\right) = 3\left((x - \frac{5}{2})^2 - \frac{9}{4}\right)$

$$= 3\left(x - \frac{5}{2} + \frac{3}{2}\right)\left(x - \frac{5}{2} - \frac{3}{2}\right) = 3(x-1)(x-4)$$

In this case the 3 would be left outside the brackets since multiplying it into one of the factors does not simplify the final answer .

By applying the process of completing the square to the general quadratic expression we can obtain a very useful result .

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a} + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right)$$

From this we can deduce that :-

If the numerical value of $b^2 - 4ac$ is positive then we have the difference of two squares and the quadratic expression factorises .

On the other hand if $b^2 - 4ac$ is negative we have the sum of two squares and the quadratic expression has no real factors .

For example :- $x^2 + 3x + 3$, $a = 1,\ b = 3, c = 3,\ \therefore\ b^2 - 4ac = -3$ and the given expression has no real factors .

$$x^2 + 3x - 3 \quad , \quad a = 1,\ b = 3, c = -3,\ \therefore\ b^2 - 4ac = 21 \quad \text{and factors}$$

exist
which we could find by completing the square .

It is always worthwhile using the above test . It only takes a second and can save a lot of unnecessary work .

Tutorial 3

Factorise the following expressions , where possible ,either by inspection or by completing the square

    (i) $x^2 + 2x - 3$ (ii) $4 - 3x - x^2$

(iii)	$2x^2 + 5x - 7$	(iv)	$20x^2 + 33x - 36$
(v)	$4x^2 - 9$	(vi)	$3x^2 - 5$
(vii)	$x^2 - 4x + 4$	(viii)	$x^2 - 2px + p^2$
(ix)	$x^2 + 4x - 5$	(x)	$x^2 - 2x - 8$
(xi)	$x^2 - 6x - 7$	(xii)	$2x^2 + 5x + 3$
(xiii)	$3x^2 + 14x - 5$	(xiv)	$6x^2 - x - 2$
(xv)	$x^2 + 2x - 1$	(xvi)	$x^2 - 4x + 6$
(xvii)	$x^2 + 8x - 5$	(xviii)	$x^2 + x - 1$
(xix)	$x^2 + 3x + 1$	(xx)	$2x^2 - 3x - 1$
(xxi)	$3x^2 + 7x + 1$	(xxii)	$5x^2 + 5x + 1$
(xxiii)	$x^2 - 2qx + q$	(xxiv)	$x^2 + px + p$

You can use the back button in the Explorer to return here.

4. Quadratic Equations

Consider the quadratic expression $x^2 - x - 6$.

If we put $x = 3$ the expression becomes $3^2 - 3 - 6$ which has the value zero and if we put $x = -2$ the expression becomes $(-2)^2 - (-2) - 6$ which also has the value zero . It is a fact as we shall see that no other values of x make the expression equal to zero .

To put this another way we say that $x = 3 \; and \; x = -2$ both satisfy the equation $x^2 - x - 6 = 0$.

These two values are called the roots of the equation $x^2 - x - 6 = 0$.

We find these roots as follows:-

Either by inspection or by completing the square we see that $x^2 - x - 6 = (x - 3)(x + 2)$

so that the equation $x^2 - x - 6 = 0$ is equivalent to $(x - 3)(x + 2) = 0$.

To satisfy this we must have either $x - 3 = 0 \; or \; x + 2 = 0$

The first of these is satisfied by putting $x = 3$ and the second by putting $x = -2$.

Generalising from this example :-

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation .

If the expression $ax^2 + bx + c$ factorises into two real factors the equation will have two real roots . If the expression is irreducible the equation has no real roots .

Examples

1.
$$x^2 - 7x + 12 = 0$$

By inspection $(x - 4)(x - 3) = 0$

The roots of the equation are $x = 4$ and $x = 3$.
We can verify this by substituting these values into the quadratic equation .
$$4^2 - 28 + 12 = 0 \quad and \quad 3^2 - 21 + 12 = 0$$

2. $x^2 + 10x + 4 = 0$

By completing the square $(x+5)^2 + 4 - 25 = 0$
$$(x+5)^2 - 21 = 0$$
$$(x+5+\sqrt{21})(x+5-\sqrt{21}) = 0$$

The roots of the equation are $x = -5 - \sqrt{21}$ and $x = -5 + \sqrt{21}$

We could verify these answers by substituting into the given equation but provided   you are careful it is not necessary to do this every time.

3. $x^2 - 4x - 4 = 0$

By inspection $(x-2)^2 = 0$

There is apparently only one value $x = 2$ which satifies this equation but since

$(x-2)^2 = (x-2)(x-2)$ we say that $x = 2$ is a **repeated root** of the equation .

By applying the process of completing the square to the general quadratic equation we can derive a formula which can be used to solve any quadratic equation .
You may ask the question , if there is a formula why do we need all that has gone before ? The answer is that the processes of factorising by inspection and of completing the square have important applications in many other areas of the Mathematics of Engineering and Science , not just in solving quadratic equations , and they are skills which you need to practice.

Consider the general quadratic equation $ax^2 + bx + c = 0$. By completing the square we obtain

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a} \right) = 0$$

$$a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \right) = 0$$

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} \right) = 0$$

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 \right) = 0$$

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 \right) = 0$$

The term inside the brackets is the difference of two squares so we obtain
$$a\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \right) = 0$$

Hence either $\quad x = -\dfrac{b}{2a} - \dfrac{\sqrt{b^2-4ac}}{2a} \quad or \quad x = -\dfrac{b}{2a} + \dfrac{\sqrt{b^2-4ac}}{2a} \quad$ which we write as

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

This is the formula for solving any quadratic equation . The algebra used in obtaining the formula is rather heavy but it is easy to use and you should certainly memorise the result .
In fact , from now on , if you have a quadratic equation to solve and the factors don't jump out at you quickly you should use the formula .

Examples

1. $\qquad\qquad x^2 + 5x + 6 = 0 \quad , \quad a = 1, b = 5, c = 6$

   In this case the factors are easily seen :- $x^2 + 5x + 6 = (x+3)(x+2)$
   Hence we obtain the two roots $x = -3 \quad and \quad x = -2$.

   However we will use the formula just for practice :-
   $$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 6}}{2 \times 1} = \frac{-5 \pm \sqrt{1}}{2} = \frac{-5 \pm 1}{2}$$
   i.e. $\quad x = \dfrac{-5+1}{2} = -2 \quad or \quad x = \dfrac{-5-1}{2} = -3$

2. $\qquad\qquad x^2 - 2x - 7 = 0 \quad , \quad a = 1, b = -2, c = -7$

   $$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times (-7)}}{2 \times 1} = \frac{2 \pm \sqrt{4 + 28}}{2} = \frac{2 \pm \sqrt{32}}{2} = \frac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$$

   i.e. $\quad x = 1 + 2\sqrt{2} \quad or \quad x = 1 - 2\sqrt{2}$

   Whether you leave the answers in this form or evaluate using your calculator will depend on the context .

3. $\qquad\qquad 0.25x^2 + 0.75x - 0.32 = 0 \quad , \quad a = 0.25, b = 0.75, c = -0.32$

   $$x = \frac{-0.75 \pm \sqrt{(0.75)^2 - 4 \times 0.25 \times (-0.32)}}{2 \times 0.25} = \frac{-0.75 \pm \sqrt{0.5625 + 0.32}}{0.5} = \frac{-0.75 \pm 0.9394}{0.5}$$

   i.e. $\quad x = 0.3788 \quad or \quad x = -3.3788$

4. $\qquad 3x^2 + 6x + 5 = 0 \qquad a = 3, b = 6, c = 5$

   $$x = \frac{-6 \pm \sqrt{36 - 4 \times 3 \times 5}}{2 \times 3} = \frac{-6 \pm \sqrt{36 - 60}}{6} = \frac{-6 \pm \sqrt{-24}}{6}$$

   A negative number does not have a real square root so the given equation has

no real roots.

5. $0.16x^2 - 0.04x + 0.25 = 0$ $a = 0.16, b = -0.04, c = 0.25$

$$x = \frac{-(-0.4) \pm \sqrt{(-0.4)^2 - 4 \times 0.16 \times 0.25}}{2 \times 0.16} = \frac{0.4 \pm \sqrt{0.16 - 0.16}}{0.32}$$

Therefore $x = \dfrac{0.4}{0.32} = 1.25$ is a repeated root of the equation .

Tutorial 4

Solve the equations below either by factorisation or by using the formula :-

(i) $x^2 - 2x - 2 = 0$ (ii) $x^2 + 5x + 5 = 0$

(iii) $x^2 - x - 3 = 0$ (iv) $5x^2 + 3x - 1 = 0$

(v) $4x^2 - 8x - 1 = 0$ (vi) $x^2 - 4x + 12 = 0$

(vii) $x^2 - 4x + 3 = 0$ (viii) $x^2 - 4x + 4 = 0$

(ix) $3x^2 + 5x = 0$ (x) $4x^2 - 9 = 0$

(xi) $x^3 + 4x^2 - 8x = 0$ Hint. x is a common factor .

(xii) $x^4 - 4x^2 + 1 = 0$ Hint.   Put $y = x^2$.

The equation in (xi) is said to be of third degree and is called a cubic equation . Notice that it has three roots .

The equation in (xii) is of fourth degree and is called a quartic equation . Notice that it has four roots .

. You can use the back button in the Explorer to return here.

There are more tutorial questions like these in the notes on **Equations and Transposition of Formulae .**