

LESSON 7

ALTERNATING CURRENT

Alternating current

As we have seen earlier a rotating coil in a magnetic field, induces an alternating emf and hence an alternating current. Since the emf induced in the coil varies in magnitude and direction periodically, it is called an alternating emf. The significance of an alternating emf is that it can be changed to lower or higher voltages conveniently and efficiently using a transformer. Also the frequency of the induced emf can be altered by changing the speed of the coil. This enables us to utilize the whole range of electromagnetic spectrum for one purpose or the other. For example domestic power in India is supplied at a frequency of 50 Hz. For transmission of audio and video signals, the required frequency range of radio waves is between 100 KHz and 100 MHz. Thus owing to its wide applicability most of the countries in the world use alternating current.

Measurement of AC

Since alternating current varies continuously with time, its average value over one complete cycle is zero. Hence its effect is measured by rms value of a.c.

RMS value of a.c.

The rms value of alternating current is defined as that value of the steady current, which when passed through a resistor for a given time, will generate the same amount of heat as generated by an alternating current when passed through the same resistor for the same time.

The rms value is also called effective value of an a.c. and is denoted by I_{rms} when an alternating current $i = i_0 \sin \omega t$ flows through a resistor of resistance R , the amount of heat produced in the resistor in a small time dt is $dH = i^2 R dt$. We know that alternating current is given $i = i_0 \sin \omega t$

The total amount of heat produced in the resistance in one complete cycle is

$$\begin{aligned}
 H &= \int_0^T i^2 R dt \\
 H &= \int_0^T i_0^2 \sin^2 \omega t R dt \\
 H &= i_0^2 R \int_0^T \left(\frac{1 - \cos 2\omega t}{2} \right) dt \\
 H &= \frac{i_0^2 R}{2} \left[\int_0^T dt - \int_0^T \cos 2\omega t dt \right]
 \end{aligned}$$

$$\text{As } \int_0^T \cos 2\omega t dt = 0$$

$$H = \frac{i_0^2 R}{2} T$$

But this heat is also equal to the heat produced by rms value of AC in the same resistor (R) and in the same time (T),

$$H = I_{rms}^2 RT$$

Thus

$$I_{rms}^2 RT = \frac{i_0^2 R}{2} T$$

$$\therefore I_{rms} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$$

We can calculate rms value as root mean square value :

The mean value or average value of ac over time T is given by

$$i_{rms}^2 = \frac{\int_0^T i^2 dt}{\int_0^T dt}$$

$$i_{rms}^2 = \frac{\int_0^T i_0^2 \sin^2(\omega t) dt}{\int_0^T dt}$$

$$i_{rms}^2 = \frac{i_0^2 \int_0^T [1 - \cos 2\omega t] dt}{2T}$$

$$\text{As } \int_0^T \cos 2\omega t dt = 0$$

$$i_{rms}^2 = \frac{i_0^2 T}{2T} = \frac{i_0^2}{2}$$

$$\therefore i_{rms} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$$

Similarly

$$E_{rms} = \frac{E_0}{\sqrt{2}}$$

Solved numerical

Q) If the voltage in ac circuit is represented by the equation

$$V = 220\sqrt{2}\sin(314t - \phi)$$

Calculate (a) peak and rms value of the voltage

(b) average voltage

(c) frequency of ac

Solution:

(a) For ac voltage

$$V = V_0 \sin(\omega t - \phi)$$

The peak value

$$V_0 = 220\sqrt{2} = 311V$$

The rms value of voltage

$$V_{rms} = \frac{V_0}{\sqrt{2}} = \frac{220\sqrt{2}}{\sqrt{2}} = 220V$$

(b) Average voltage in full cycle is zero, Average voltage in half cycle is

$$V_{ave} = \frac{2}{\pi} V_0 = \frac{2}{\pi} 311 = 198.71V$$

(c) As $\omega = 2\pi f$

$$2\pi f = 314$$

$$f = 314/2\pi = 50\text{Hz}$$

Q) Write the equation of a 25 cycle current sine wave having rms value of 30 A.

Solution:

Given: frequency $f = 25\text{ Hz}$ and $I_{rms} = 30\text{A}$ or $i_0 = 30\sqrt{2}$

$$I = i_0 \sin(2\pi f)t$$

$$I = 30\sqrt{2} \sin(2\pi \times 25)t$$

$$I = 30\sqrt{2} \sin(50\pi)t$$

Q) An electric current has both A.C. and D.C. components. The value of the D.C component is equal to 12A while the A.C. component is given as $I = 9\sin\omega t$ A. Determine the formula for the resultant current and also calculate the value of I_{rms}

Solution: Resultant current at any instant of time will be $I = 12 + 9\sin\omega t$

$$\text{Now } I_{rms} = \sqrt{\langle I^2 \rangle} = \sqrt{(12 + 9\sin\omega t)^2}$$

$$I_{rms} = \sqrt{\langle 144 + 216\sin\omega t + 81\sin^2\omega t \rangle}$$

Here, the average is taken over a time interval equal to the periodic time

$$\text{Now } \langle 144 \rangle = 144$$

$$216\langle \sin\omega t \rangle = 0$$

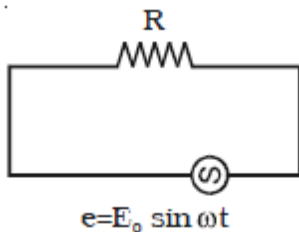
$$\text{And } 81\langle \sin^2\omega t \rangle = 81 \times (1/2) = 40.5$$

$$\therefore I_{rms} = \sqrt{144 + 40.5} = 13.58\text{ A}$$

Series AC Circuit

1) When only resistance is in an ac circuit

Consider a simple ac circuit consisting of resistor of resistance R and an ac generator, as shown in figure



According to Kirchhoff's loop law at any instant, the algebraic sum of the potential difference around a closed loop in a circuit must be zero

$$e - V_R = 0$$

$$e - I_R R = 0$$

$$E_0 \sin \omega t - I_R R = 0$$

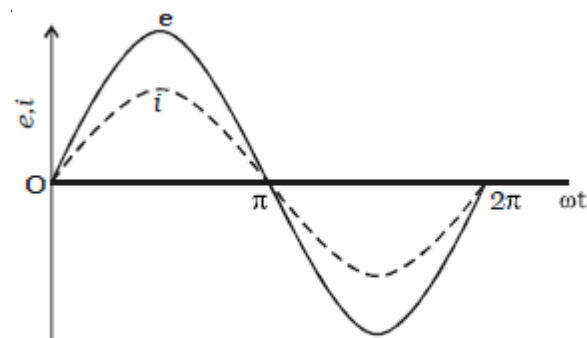
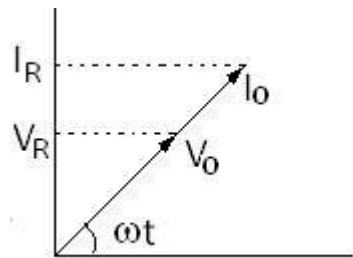
$$I_R = E_0 \sin \omega t / R = I_0 \sin \omega t \text{ ---(1)}$$

Where I_0 is the maximum current $I_0 = E_0 / R$

From above equations, we see that the instantaneous voltage drop across the resistor is

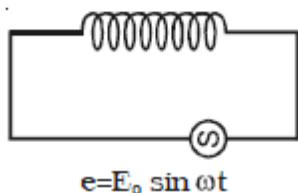
$$V_R = I_0 R \sin \omega t \text{ ---(2)}$$

We see in equation (1) and (2) I_R and V_R both vary as $\sin \omega t$ and reach their maximum values at the same time as shown in graph they are said to be in phase.



2) When only Inductor is in an ac circuit

Consider an ac circuit consisting only of an inductor of inductance L connected to the terminals of ac generator, as shown in figure



The induced emf across the inductor is given by $L(di/dt)$. On applying Kirchhoff's loop rule to the circuit

$$e - V_L = 0$$

$$e = L \frac{di}{dt}$$

$$E_0 \sin \omega t = L \frac{di}{dt}$$

Integrating above expression as a function of time

$$i_L = \frac{E_0}{L} \int \sin \omega t dt = -\frac{E_0}{\omega L} \cos \omega t + C$$

For average value of current over one time period to be zero, $C = 0$

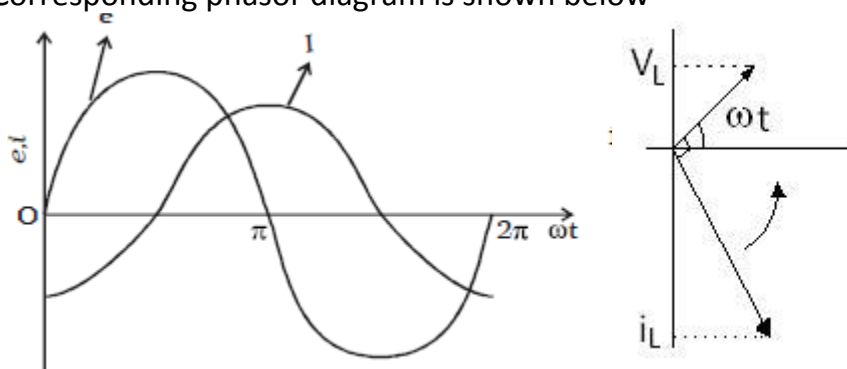
$$\therefore i_L = -\frac{E_0}{\omega L} \cos \omega t$$

When we use the trigonometric identity $\cos \omega t = -\sin \left(\omega t - \frac{\pi}{2} \right)$

We can express equation as

$$i_L = \frac{E_0}{\omega L} \sin \left(\omega t - \frac{\pi}{2} \right)$$

From above equation it is clear that current lags by $\pi/2$ to voltage. The voltage reaches maximum, one quarter of than oscillation period before current reaches maximum value. Corresponding phasor diagram is shown below



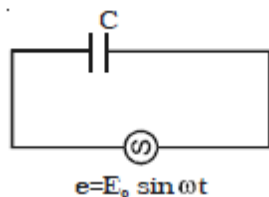
Secondly current is maximum when $\cos \omega t = 1$

$$i_0 = \frac{E_0}{\omega L}$$

ωL is known as inductive reactance denoted by X_L

3) When only capacitor is in an ac circuit

Figure shows an ac circuit consisting of a capacitor of capacitance C connected across the terminals of an ac generator.



On applying Kirchhoff's rule to this circuit, we get

$$e - V_C = 0$$

$$V_C = e$$

$$V_C = E_0 \sin \omega t$$

Where V_C is the instantaneous voltage drop across the capacitor. From the definition of capacitance $V_C = Q/C$, and this value of V_C substituted into equation gives

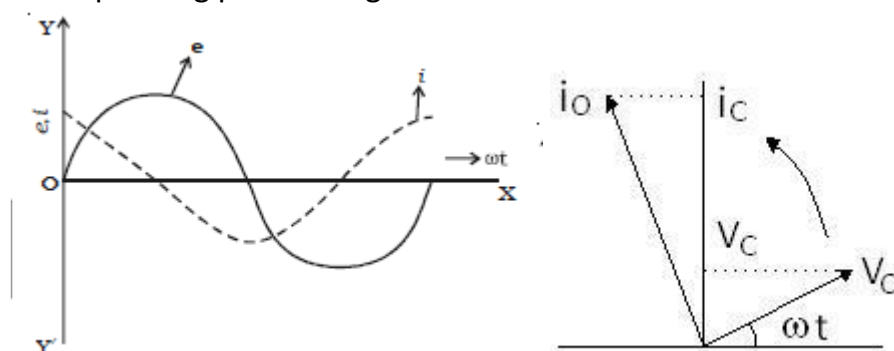
$$Q = C E_0 \sin \omega t$$

Since $i = dQ/dt$, on differentiating above equation gives the instantaneous current in the circuit

$$i_c = \frac{dQ}{dt} = C E_0 \omega \cos \omega t$$

From above equation it is clear that current leads the voltage by $\pi/2$

A plot of current and voltage versus times, shows that the current reaches its maximum value one quarter of a cycle sooner than the voltage reaches maximum value. The corresponding phasor diagram is shown



Secondly when $\cos \omega t = 1$, in equation $i_c = C E_0 \omega \cos \omega t$ the current in circuit is maximum

$$i_c = C E_0 \omega = \frac{E_0}{X_C}$$

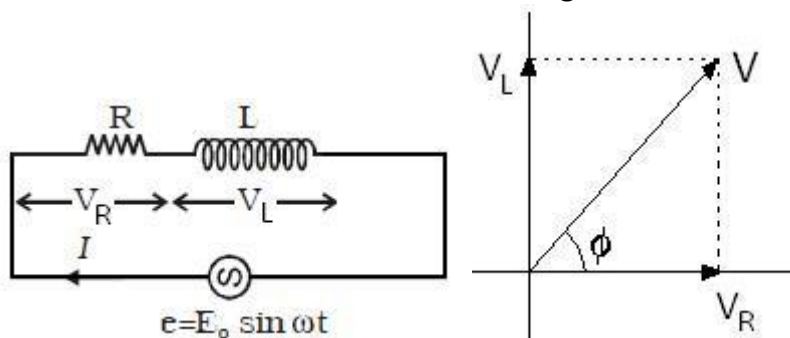
X_C is called the capacitive reactance

$$X_C = \frac{1}{\omega C}$$

For DC supply, $\omega = 0$ therefore X_C will be infinite and current will not flow through capacitor once it is fully charged.

SERIES L-R Circuit

Now consider an ac circuit consisting of a resistor or resistance R and an inductor of inductance L in series with an ac source generator



Suppose in phasor diagram, current is taken along positive direction. The V_R is also along positive x-direction as there is no phase difference between i_R and V_R . While V_L will be along y direction as we know that current lags behind the voltage by 90°

So we can write

$$V = V_R + jV_L$$

$$V = i_R R + j(i X_L)$$

$$V = iZ$$

Here $Z = R + jX_L = R + j(\omega L)$ is called as impedance of the circuit. Impedance plays the same role in ac circuit as the ohmic resistance does in DC circuit. The modulus of impedance is

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

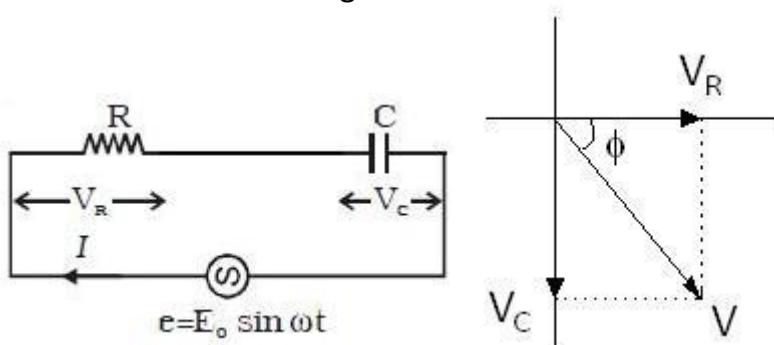
The potential difference leads the current by an angle

$$\phi = \tan^{-1} \left| \frac{V_L}{V_R} \right| = \tan^{-1} \left(\frac{X_L}{R} \right)$$

$$\phi = \tan^{-1} \left(\frac{\omega L}{R} \right)$$

SERIES R-C Circuit

Now consider an ac circuit consisting of resistance R and a capacitor of capacitance C in series with an ac source generator



Suppose in phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction but V_C is along negative y-direction as current leads the potential by phase 90° so we can write

$$V = V_R - jV_C$$

$$V = iR - j\left(\frac{i}{\omega C}\right) = iZ$$

Here impedance

$$Z = R - j\left(\frac{1}{\omega C}\right)$$

And the potential difference lags the current by an angle

$$\phi = \tan^{-1} \left| \frac{V_C}{V_R} \right| = \tan^{-1} \left(\frac{X_C}{R} \right)$$

$$\phi = \tan^{-1} \left(\frac{1/\omega C}{R} \right) = \tan^{-1} \left(\frac{1}{\omega RC} \right)$$

Solved Numerical

Q) An alternating current voltage of 220 V r.m.s. at frequency of 40 cycles/ second is supplied to a circuit containing a pure inductance of 0.01H and a pure resistance of 6 ohm in series. Calculate (i) the current (ii) potential difference across the resistance (iii) potential difference across the inductance (iv) the time lag

Solution

The impedance of L-R circuit is given by

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

$$|Z| = \sqrt{R^2 + (2\pi fL)^2}$$

$$|Z| = \sqrt{(6)^2 + (2 \times 3.14 \times 40 \times 0.01)^2}$$

$$Z = 6.504 \text{ ohms}$$

(i) r.m.s value of current

$$i_{rms} = \frac{E_{rms}}{Z} = \frac{220}{6.504} = 33.83 \text{ amp}$$

(ii) The potential difference across the resistance is given by

$$V_R = i_{rms} \times R = 33.83 \times 6 = 202.98 \text{ Volt}$$

(iii) Potential difference across inductance is given by

$$V_L = i_{rms} \times (\omega L) = 33.83 \times 6 = 202.98 \text{ volts}$$

(iv) Phase angle

$$\phi = \tan^{-1} \left(\frac{\omega L}{R} \right)$$

$$\phi = \tan^{-1} \left(\frac{2\pi fL}{R} \right) = \tan^{-1} \left(\frac{2 \times 3.14 \times 40 \times 0.01}{6} \right)$$

$$\phi = \tan^{-1}(0.4189) = 22^\circ 73'$$

Now time lag =

$$\frac{\varphi}{360} \times T = \frac{\varphi}{360} \times \frac{1}{f} = \frac{22^\circ 73'}{360 \times 40} = 0.001579 \text{ s}$$

Q) An ac source of an angular frequency ω is fed across a resistor R and a capacitor C in series. The current registered is i. If now the frequency of source is changed to $\omega/3$ (but maintaining the same voltage), the current in the circuit is found to be halved. Calculate the ratio of reactance to resistance at the original frequency

Solution:

At angular frequency ω , the current in R-C circuit is given by

$$i_{rms} = \frac{E_{rms}}{\sqrt{R^2 + \left(\frac{1}{\omega^2 C^2}\right)}} \quad \text{--- eq(1)}$$

When frequency changed to $\omega/3$, the current is halved. Thus

$$\frac{i_{rms}}{2} = \frac{E_{rms}}{\sqrt{\left\{R^2 + \left(\frac{1}{\frac{\omega^2 C^2}{3^2}}\right)\right\}}}$$

$$\frac{i_{rms}}{2} = \frac{E_{rms}}{\left\{R^2 + \frac{9}{\omega^2 C^2}\right\}} \quad \text{--- eq(2)}$$

From above equation (1) and (2) we have

$$\frac{E_{rms}}{\sqrt{R^2 + \left(\frac{1}{\omega^2 C^2}\right)}} = \frac{2E_{rms}}{\sqrt{\left\{R^2 + \left(\frac{9}{\omega^2 C^2}\right)\right\}}}$$

Solving the equation we get

$$3R^2 = \frac{5}{\omega^2 C^2}$$

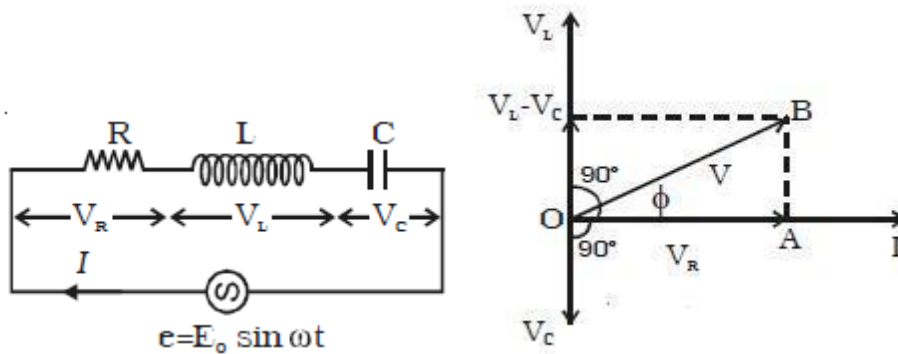
Hence ratio of reactance to resistance

$$\frac{1/\omega C}{R} = \sqrt{\frac{3}{5}}$$

SERIES L – C – R CIRCUIT

Consider an ac circuit consisting of resistance R, capacitor of capacitance C and an inductor of inductance L are in series with ac source generator

Suppose in a phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction, V_L along positive y-direction and V_C along negative y-direction, as potential difference across an inductor leads the current by 90° in phase while that across a capacitor, lags by 90°



$$V = \sqrt{V_R^2 + (V_L - V_C)^2}$$

We can write $V = V_R + jV_L - jV_C$

$$V = iR + j(iX_L) - j(iX_C)$$

$$V = iR + j[i(X_L - X_C)] = iZ$$

Here impedance is

$$Z = R + j(X_L - X_C)$$

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The potential difference leads the current by an angle

$$\phi = \tan^{-1} \left| \frac{V_L - V_C}{V_R} \right|$$

$$\phi = \tan^{-1} \left| \frac{X_L - X_C}{R} \right|$$

$$\phi = \tan^{-1} \left| \frac{\omega L - \frac{1}{\omega C}}{R} \right|$$

The steady current is given by

$$i = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi)$$

The peak current is

$$i_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

It depends on angular frequency ω of ac source and it will be maximum when

$$\omega L - \frac{1}{\omega C} = 0$$

$$\omega = \sqrt{\frac{1}{LC}}$$

And corresponding frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

This frequency is known as resonant frequency of the given circuit. At this frequency peak current will be $i_0 = \frac{V_0}{R}$

This resistance R in the LCR circuit is zero, the peak current at resonance is $i_0 = \frac{V_0}{R}$

It means, there can be a finite current in pure LC circuit even without any applied emf.

When a charged capacitor is connected to pure inductor

This current in the circuit is at frequency $f = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$

Solved Numerical

Q) A resistor of resistance R, an inductor of inductance L and a capacitor of capacitance C all are connected in series with an a.c. supply. The resistance of R is 16 ohm. And for a given frequency, the inductive reactance of L is 24 ohms and capacitive reactance of C is 12 ohms. If the current in circuit is 5amp, find

- The potential difference across R, L and C
- the impedance of the circuit
- the voltage of ac supply
- Phase angle

Solution:

(a) Potential difference across resistance $V_R = iR = 5 \times 16 = 80$ volt

Potential difference across inductance

$$V_L = i \times (\omega L) = 5 \times 24 = 120 \text{ volt}$$

Potential across condenser

$$V_C = i \times (1/\omega C) = 5 \times 12 = 60 \text{ volts}$$

(b) Impedance

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$|Z| = \sqrt{16^2 + (24 - 12)^2} = 20 \text{ ohm}$$

(c) The voltage of ac supply is given by

$$V = iZ = 5 \times 20 = 100 \text{ volt}$$

(c) Phase angle

$$\varphi = \tan^{-1} \left| \frac{\omega L - \frac{1}{\omega C}}{R} \right|$$

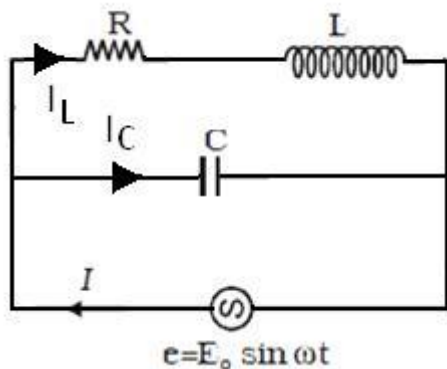
$$\varphi = \tan^{-1} \left| \frac{24 - 12}{16} \right|$$

$$\varphi = \tan^{-1}(0.75) = 36^{\circ} 87'$$

PARALLEL AC CIRCUIT

Consider an alternating source connected across an inductor L in parallel with a capacitance C

The resistance in series with the inductance is R and with the capacitor as zero



Let the instantaneous value of emf applied be V and the corresponding current is I, I_L and I_C . Then

$$I = I_L + I_C$$

Or

$$\frac{V}{Z} = \frac{V}{R + j\omega L} - \frac{V}{\frac{j}{\omega C}}$$

$$\frac{V}{Z} = \frac{V}{R + j\omega L} + j\omega CV \quad (\text{as } j^2 = -1)$$

$$\frac{1}{Z} = \frac{1}{R + j\omega L} + j\omega C$$

$\frac{1}{Z}$ is called admittance Y

$$\frac{1}{Z} = Y = \frac{1}{R + j\omega L} \frac{R - j\omega L}{R - j\omega L} + j\omega C$$

$$Y = \frac{R - j\omega L}{R^2 + \omega^2 L^2} + j\omega C$$

$$Y = \frac{R + j(\omega C R^2 + \omega^3 L^2 C - \omega L)}{R^2 + \omega^2 L^2}$$

Magnitude of admittance

$$|Y| = \frac{\sqrt{R^2 + (\omega CR^2 + \omega^3 L^2 C - \omega L)^2}}{R^2 + \omega^2 L^2}$$

The admittance will be minimum. When

$$\omega CR^2 + \omega^3 L^2 C - \omega L = 0$$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

It gives the condition of resonance and corresponding frequency

$$f = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

This is known as resonance frequency. At resonance frequency admittance is minimum or impedance is maximum. Thus the parallel circuit does not allow this frequency from source to pass in the circuit. Due to this reason the circuit with such frequency is known as rejecter circuit

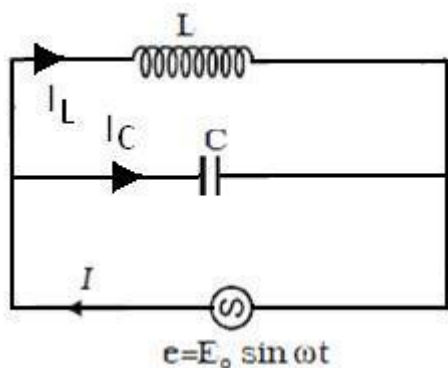
Note

If $R = 0$, resonance frequency

$f = \frac{1}{2\pi\sqrt{LC}}$ is same as resonance frequency in series circuit

Solved numerical

Q) For the circuit shown in figure. Current in inductance is 0.8A while in capacitance is 0.6A. What is the current drawn from the source



Solution:

In this circuit $E = E_0 \sin \omega t$ is applied across an inductance and capacitance in parallel, current in inductance will lag the applied voltage while across the capacitor will lead and so

$$i_L = \frac{V}{X_L} \sin \left(\omega t - \frac{\pi}{2} \right) = -0.8 \cos \omega t$$

$$i_C = \frac{V}{X_C} \sin \left(\omega t + \frac{\pi}{2} \right) = +0.6 \cos \omega t$$

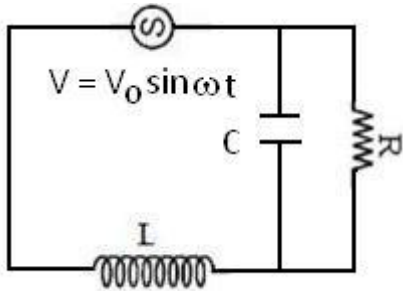
So current from the source

$$i = i_L + i_C = -0.2 \cos \omega t$$

$$|i_0| = 0.2A$$

Q) An emf $V_0 \sin \omega t$ is applied to a circuit which consists of self-inductance L of negligible resistance in series with a variable capacitor C . the capacitor is shunted by a variable resistance R . Find the value of C for which the amplitude of the current is independent of R

Solution



Solution:

First we will calculate impedance of the circuit

The complex impedance of the circuit

$$Z = j\omega L + Z'$$

Here Z' is complex impedance of parallel combination of Capacitor and R

$$\begin{aligned} \frac{1}{Z'} &= \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R} \\ Z' &= \frac{R}{1 + j\omega CR} = \frac{R(1 - j\omega CR)}{(1 + j\omega CR)(1 - j\omega CR)} = \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2} \\ Z &= j\omega L + \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2} \\ Z &= j\omega L + \frac{R}{1 + \omega^2 C^2 R^2} - \frac{j\omega CR^2}{1 + \omega^2 C^2 R^2} \\ Z &= j \left(\omega L - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2} \right) + \frac{R}{1 + \omega^2 C^2 R^2} \end{aligned}$$

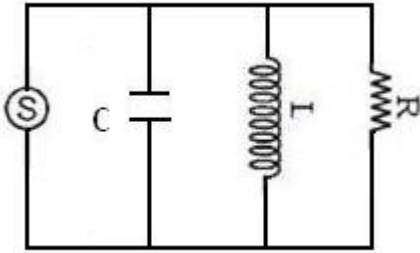
Magnitude of Z is given by

$$\begin{aligned} Z^2 &= \left(\frac{R}{1 + \omega^2 C^2 R^2} \right)^2 + \left(\omega L - \frac{\omega CR^2}{1 + \omega^2 C^2 R^2} \right)^2 \\ Z^2 &= \left(\frac{R}{1 + \omega^2 C^2 R^2} \right)^2 + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2} + \left(\frac{\omega CR^2}{1 + \omega^2 C^2 R^2} \right)^2 \\ Z^2 &= \left(\frac{R}{1 + \omega^2 C^2 R^2} \right)^2 (1 + \omega^2 C^2 R^2) + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2} \\ Z^2 &= \left(\frac{R^2}{1 + \omega^2 C^2 R^2} \right) + (\omega L)^2 - \frac{2\omega^2 LCR^2}{1 + \omega^2 C^2 R^2} \end{aligned}$$

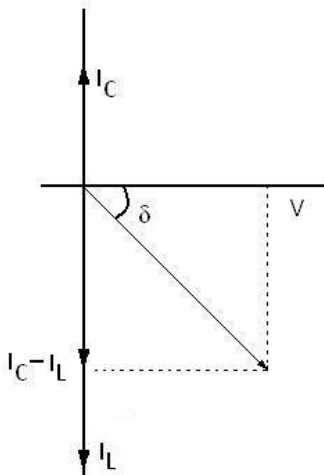
The value of current will be independent of R . It is possible when

$$\begin{aligned} R^2 - 2\omega^2 LCR^2 &= 0 \\ C &= \frac{1}{2} \omega^2 L \end{aligned}$$

Q) Derive the expression for the total current flowing in the circuit using phaser diagram

Solution:

The phasor diagram of the voltage and current is as shown in figure. In order to obtain the total current, we shall have to consider the addition of the currents. From the diagram we have



$$I = \sqrt{I_R^2 + (I_L - I_C)^2}$$

But

$$I_R = \frac{V}{R}; I_L = \frac{V}{X_L}; I_C = \frac{V}{X_C}$$

$$I = V \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_L} - \frac{1}{X_C} \right)^2}$$

From figure

$$\tan \delta = \frac{I_L - I_C}{I_R} = \frac{\frac{1}{X_L} - \frac{1}{X_C}}{\frac{1}{R}}$$

$$\tan \delta = R \left(\frac{1}{X_L} - \frac{1}{X_C} \right)$$

Q-factor

The selectivity or sharpness of a resonant circuit is measured by the quality factor or Q factor. In other words it refers to the sharpness of tuning at resonance. The Q factor of a

series resonant circuit is defined as the ratio of the voltage across a coil or capacitor to the applied voltage.

$$Q = \frac{\text{voltage across } L \text{ or } C}{\text{applied voltage}} \quad \text{---(1)}$$

$$\text{Voltage across } L = I \omega_0 L \quad \text{...(2)}$$

where ω_0 is the angular frequency of the a.c. at resonance. The applied voltage at resonance is the potential drop across R, because the potential drop across L is equal to the drop across C and they are 180° out of phase. Therefore they cancel out and only potential drop across R will exist.

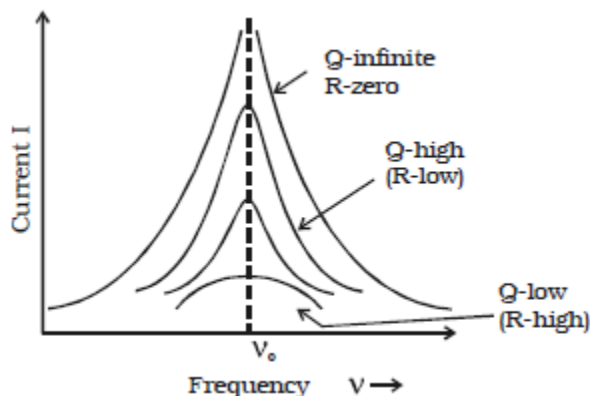
$$\text{Applied Voltage} = IR \quad \text{...(3)}$$

Substituting equations (2) and (3) in equation (1)

$$Q = \frac{I \omega_0 L}{IR} = \frac{\omega_0 L}{R}$$

$$Q = \frac{1}{\sqrt{RC}} \frac{L}{R}$$

Q is just a number having values between 10 to 100 for normal frequencies. Circuit with high Q values would respond to a very narrow frequency range and vice versa. Thus a circuit with a high Q value is sharply tuned while one with a low Q has a flat resonance. Q-factor can be increased by having a coil of large inductance but of small ohmic resistance. Current frequency curve is quite flat for large values of resistance and becomes more sharp as the value of resistance decreases. The curve shown in graph is also called the frequency response curve.



Sharpness of resonance

The amplitude of the current in the series LCR circuit is given by

$$i_{max} = \frac{v_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and is maximum when $\omega = \omega_0 = 1/\sqrt{LC}$ The maximum value is $i_{max} = V_{max}/R$

For values of ω other than ω_0 , the amplitude of the current is less than the maximum value.

Suppose we choose a value of ω for which the current amplitude is $1/\sqrt{2}$ times its maximum value. At this value, the power dissipated by the circuit becomes half. From the curve in Fig. ,we see that there are two such values of ω , say, ω_1 and ω_2 , one greater and the other smaller than ω_0 and symmetrical about ω_0 . We may

write

$$\omega_1 = \omega_0 + \Delta\omega$$

$$\omega_2 = \omega_0 - \Delta\omega$$

The difference $\omega_1 - \omega_2 = 2\Delta\omega$ is often called the *bandwidth* of the circuit. The quantity $(\omega_0 / 2\Delta\omega)$ is regarded as a measure of the sharpness of resonance. The smaller the $\Delta\omega$, the sharper or narrower is the resonance.

We see from Fig. that if the resonance is less sharp, not only is the maximum current less, the circuit is close to resonance for a larger range $\Delta\omega$ of frequencies and the tuning of the circuit will not be good. So, less sharp the resonance, less is the selectivity of the circuit or vice versa.

$$\text{Value of } \Delta\omega = \frac{R}{2L}$$

we see that if quality factor is large, i.e., R is low or L is large, the circuit is more selective.

POWER IN AN AC CIRCUIT

In case of steady current the rate of doing work is given by,

$$P = VI$$

In an alternatin circuit, current and voltage both vary with time, so the work done by the source in time intrval dt is given by

$$dw = Vidt$$

Suppose in an ac, the current is leading the voltage by an angle φ . Then we can write

$$V = V_m \sin \omega t \text{ and}$$

$$I = i_m \sin(\omega t + \varphi)$$

$$dw = V_m i_m \sin \omega t \sin(\omega t + \varphi) dt$$

$$dw = V_m i_m (\sin^2 \omega t \cos \varphi + \sin \omega t \cos \omega t \sin \varphi) dt$$

The total work done in a complete cycle is

$$\begin{aligned} W &= V_m i_m \cos \varphi \int_0^T \sin^2 \omega t dt + V_m i_m \sin \varphi \int_0^T \sin \omega t \cos \omega t dt \\ W &= \frac{1}{2} V_m i_m \cos \varphi \int_0^T (1 - \cos 2\omega t) dt + \frac{1}{2} V_m i_m \sin \varphi \int_0^T \sin 2\omega t dt \\ W &= \frac{1}{2} V_m i_m T \cos \varphi \end{aligned}$$

The average power delivered by the source is, therefore

$$P = W/T$$

$$P = \frac{1}{2} V_m i_m \cos \varphi$$

$$P = \frac{V_m}{\sqrt{2}} \frac{i_m}{\sqrt{2}} \cos \varphi$$

$$P = V_{rms} i_{rms} \cos \varphi$$

This can also be written as,

$$P = I^2 Z \cos \phi$$

Here, Z is impedance, the term $\cos \phi$ is known as power factor

It is said to be leading if current leads voltage, lagging if current lags voltage. Thus, a power factor of 0.5 lagging means current lags voltage by 60° (as $\cos^{-1}0.5 = 60^\circ$). The product of V_{rms} and i_{rms} gives the apparent power. While the true power is obtained by multiplying the apparent power by the power factor $\cos\phi$.

- (i) Resistive circuit: For $\phi=0^\circ$, the current and voltage are in phase. The power is thus, maximum.
- (ii) purely inductive or capacitive circuit: For $\phi=90^\circ$, the power is zero. The current is then stated as wattless. Such a case will arise when resistance in the circuit is zero. The circuit is purely inductive or capacitive
- (iii) *LCR series circuit*: In an LCR series circuit, power dissipated is given by $P = I^2 Z \cos\phi$ where

$$\phi = \tan^{-1} \left(\frac{X_C - X_L}{R} \right)$$

So, ϕ may be non-zero in a *RL* or *RC* or *RCL* circuit. Even in such cases, power is dissipated only in the resistor.

- (iv) *Power dissipated at resonance in LCR circuit*: At resonance $X_C - X_L = 0$, and $\phi = 0$. Therefore, $\cos\phi = 1$ and $P = I^2 Z = I^2 R$. That is, maximum power is dissipated in a circuit (through *R*) at resonance

Solved Numerical

Q) In an L-C-R A.C. series circuit $L = 5\text{H}$, $\omega = 100 \text{ rad s}^{-1}$, $R = 100\Omega$ and power factor is 0.5. Calculate the value of capacitance of the capacitor

Solution:

Power factor

$$\cos\delta = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Squaring on both side

$$\cos^2\delta = \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$\cos\delta = 0.5$

$$\begin{aligned} \frac{1}{4} &= \frac{R^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \\ R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 &= 4R^2 \\ \left(\omega L - \frac{1}{\omega C}\right)^2 &= 3R^2 \\ \omega L - \frac{1}{\omega C} &= \sqrt{3}R \\ \omega L - \sqrt{3}R &= \frac{1}{\omega C} \end{aligned}$$

$$C = \frac{1}{\omega} \left(\frac{1}{\omega L - \sqrt{3}R} \right)$$

$$C = \frac{1}{100} \left(\frac{1}{100 \times 5 - \sqrt{3} \times 100} \right)$$

$$C = \frac{10^{-2}}{500 - 173.2} = \frac{10^{-2}}{326.8} = 30.6 \times 10^{-6} F$$

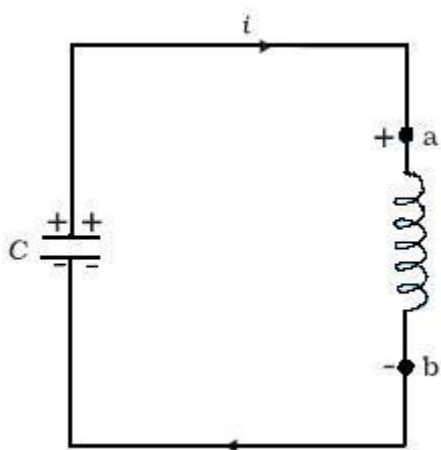
$$C = 30.6 \mu F$$

LC OSCILLATIONS

We know that a capacitor and an inductor can store electrical and magnetic energy, respectively.

When a capacitor (initially charged) is connected to an inductor, the charge on the capacitor and the current in the circuit exhibit the phenomenon of electrical oscillations similar to oscillations in mechanical systems.

Let a capacitor be charged q_m (at $t = 0$) and connected to an inductor as shown in Fig..



The moment the circuit is completed, the charge on the capacitor starts decreasing, giving rise to current in the circuit. Let q and i be the charge and current in the circuit at time t .

Since di/dt is positive, the induced emf in L will have polarity as shown, i.e., $v_b < v_a$.

According to Kirchhoff's loop rule,

$$\frac{q}{C} - L \frac{di}{dt} = 0$$

$i = -(dq/dt)$ in the present case (as q decreases, i increases).

Therefore, above equation becomes:

$$\frac{d^2q}{dt^2} - \frac{1}{LC} q = 0$$

Comparing above equation with standard equation for oscillation

$$\frac{d^2x}{dt^2} - \omega_0^2 x = 0$$

The charge, therefore, oscillates with a natural frequency

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and varies sinusoidally with time as $q = q_m \cos(\omega_0 t + \phi)$

where q_m is the maximum value of q and ϕ is a phase constant.

Since $q = q_m$ at $t = 0$, we have $\cos \phi = 1$ or $\phi = 0$. Therefore, in the present case,

$q = q_m \cos(\omega_0 t)$ current $i = i_m \sin(\omega_0 t)$ here $i_m = q_m \omega_0$

Initially capacitor is fully charged, it stores energy in the form of electric field

$$U_E = \frac{1}{2} CV^2$$

At $t = 0$, the switch is closed and the capacitor starts to discharge. As the current increases, it sets up a magnetic field in the inductor and thereby, some energy gets stored in the inductor in the form of magnetic energy:

$$U_B = \frac{1}{2} Li^2.$$

As the current reaches its maximum value i_m , (at $t = T/4$) all the energy is stored in the magnetic field:

$$U_B = \frac{1}{2} Li^2.$$

The capacitor now has no charge and hence no energy. The current now starts charging the capacitor. This process continues till the capacitor is fully charged (at $t = T/2$). But it is charged with a polarity opposite to its initial state. The whole process just described will now repeat itself till the system reverts to its original state. Thus, the energy in the system oscillates between the capacitor and the inductor.

Note that the above discussion of LC oscillations is not realistic for two reasons:

- (i) Every inductor has some resistance. The effect of this resistance is to introduce a damping effect on the charge and current in the circuit and the oscillations finally die away.
- (ii) Even if the resistance were zero, the total energy of the system would not remain constant. It is radiated away from the system in the form of electromagnetic waves (discussed in the next chapter). In fact, radio and TV transmitters depend on this radiation.

Solved Numerical

Q) A capacitor of capacitance $25\mu\text{F}$ is charged to 300V . It is then connected across a 10mH inductor. The resistance of the circuit is negligible

- (a) Find the frequency of oscillation of the circuit
- (b) Find the potential difference across capacitor and magnitude of circuit current 1.2ms after the inductor and capacitor are connected
- (c) Find the magnetic energy and electric energy at $t=0$ and $t = 1.2\text{ ms}$.

Solutions:

- (a) The frequency of oscillation of the circuit is

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Substituting the given values we have

$$f = \frac{1}{2\pi\sqrt{(10 \times 10^{-3})(25 \times 10^{-6})}} = \frac{10^3}{\pi} \text{ Hz}$$

(b) Charge across the capacitor at time t will be

$$q = q_0 \cos \omega_0 t \text{ and } i = -q\omega_0 \sin \omega_0 t$$

$$\text{Here } q_0 = CV_0 = (25 \times 10^{-6})(300) = 7.5 \times 10^{-3} \text{ C}$$

Now, charge in the capacitor after $t = 1.25 \times 10^{-3} \text{ s}$ is

$$q = (7.5 \times 10^{-3}) \cos(2\pi \times 318.3)(1.2 \times 10^{-3}) \text{ C} = 5.53 \times 10^{-3} \text{ C}$$

\therefore P.D across capacitor ,

$$V = \frac{|q|}{C} = \frac{5.53 \times 10^{-3}}{25 \times 10^{-6}} = 221.2 \text{ volt}$$

The magnitude of current in the circuit at $t = 1.2 \times 10^{-3} \text{ s}$ is

$$|i| = q\omega_0 \sin \omega_0 t$$

$$|i| = (7.5 \times 10^{-3})(2\pi)(318.3) \sin(2\pi \times 318.3)(1.2 \times 10^{-3}) \text{ A} = 10.13 \text{ A}$$

(c) At $t = 0$, Current in the circuit is zero. Hence $U_L = 0$

Charge on the capacitor is maximum

Hence

$$U_c = \frac{1}{2} \frac{q_0^2}{C}$$

$$U_c = \frac{1}{2} \frac{(7.5 \times 10^{-3})^2}{25 \times 10^{-6}} = 1.125 \text{ J}$$

At $t = 1.25 \text{ ms}$, $q = 5.53 \times 10^{-3} \text{ C}$

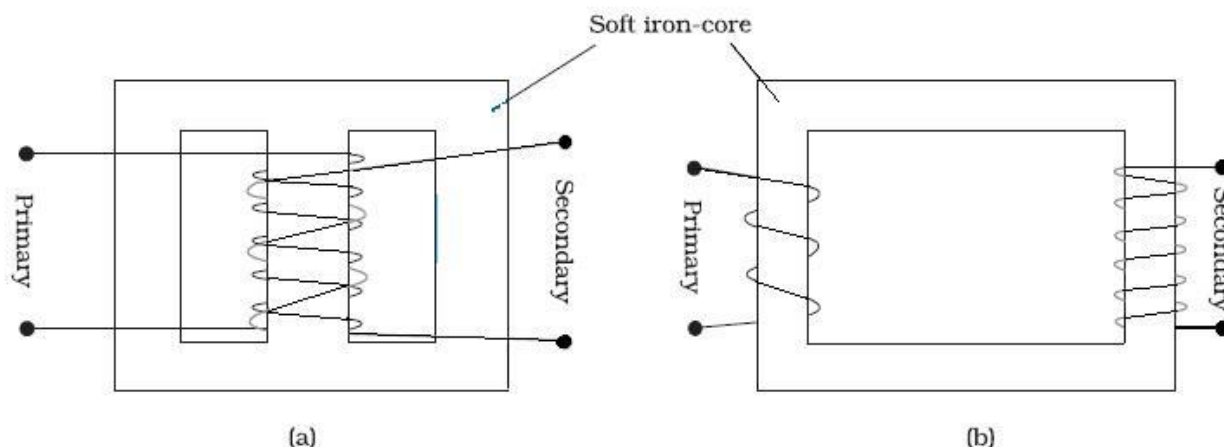
$$U_c = \frac{1}{2} \frac{q_0^2}{C}$$

$$U_c = \frac{1}{2} \frac{(5.53 \times 10^{-3})^2}{25 \times 10^{-6}} = 0.612 \text{ J}$$

TRANSFORMERS

For many purposes, it is necessary to change (or transform) an alternating voltage from one to another of greater or smaller value. This is done with a device called *transformer* using the principle of mutual induction. A transformer consists of two sets of coils, insulated from each other. They are wound on a soft-iron core, either one on top of the other as in Fig.a or on separate limbs of the core as in Fig. (b).

One of the coils called the *primary coil* has N_p turns. The other coil is called the *secondary coil*; it has N_s turns. Often the primary coil is the input coil and the secondary coil is the output coil of the transformer



When an alternating voltage is applied to the primary, the resulting current produces an alternating magnetic flux which links the secondary and induces an emf in it. The value of this emf depends on the number of turns in the secondary. We consider an ideal transformer in which the primary has negligible resistance and all the flux in the core links both primary and secondary windings. Let ϕ be the flux in each turn in the core at time t due to current in the primary when a voltage v_p is applied to it.

Then the induced emf or voltage E_s , in the secondary with N_s turns is

$$E_s = -N_s \frac{d\phi}{dt}$$

The alternating flux ϕ also induces an emf, called back emf in the primary. This is

$$E_p = -N_p \frac{d\phi}{dt}$$

But $E_p = V_p$. If this were not so, the primary current would be infinite since the primary has zero resistance (as assumed). If the secondary is an open circuit or the current taken from it is small, then to a good approximation

$$E_s = V_s$$

where V_s is the voltage across the secondary. Therefore, above equations can be written as

$$V_s = -N_s \frac{d\phi}{dt}$$

$$V_p = -N_p \frac{d\phi}{dt}$$

From above equations

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}$$

Note that the above relation has been obtained using three assumptions:

- (i) the primary resistance and current are small;
- (ii) the same flux links both the primary and the secondary as very little flux escapes from the core, and
- (iii) the secondary current is small.

If the transformer is assumed to be 100% efficient (no energy losses), the power input is equal to the power output, and since $p = iV$,

$$i_p V_p = i_s V_s$$

The equations obtained above apply to ideal transformers (without any energy losses). But in actual transformers, small energy losses do occur due to the following reasons:

- (i) *Flux Leakage*: There is always some flux leakage; that is, not all of the flux due to primary passes through the secondary due to poor design of the core or the air gaps in the core. It can be reduced by winding the primary and secondary coils one over the other.
- (ii) *Resistance of the windings*: The wire used for the windings has some resistance and so, energy is lost due to heat produced in the wire ($I^2 R$). In high current, low voltage windings, these are minimized by using thick wire.
- (iii) *Eddy currents*: The alternating magnetic flux induces eddy currents in the iron core and causes heating. The effect is reduced by having a laminated core.
- (iv) *Hysteresis*: The magnetization of the core is repeatedly reversed by the alternating magnetic field. The resulting expenditure of energy in the core appears as heat and is kept to a minimum by using a magnetic material which has a low hysteresis loss.

The large scale transmission and distribution of electrical energy over long distances is done with the use of transformers. The voltage output of the generator is stepped-up (so that current is reduced and consequently, the $I^2 R$ loss is cut down). It is then transmitted over long distances to an area sub-station near the consumers. There the voltage is stepped down. It is further stepped down at distributing sub-stations and utility poles before a power supply of 240 V reaches our homes.

Solved Numerical

Q) In an ideal step-up transformer input voltage is 110V and current flowing in the secondary is 10A. If transformation ratio is 10, calculate output voltage, current in primary, input and out put power

Solution:

Transformer ratio

$$r = \frac{N_s}{N_p} = 10$$

(i)

$$\frac{E_s}{E_p} = \frac{N_s}{N_p}$$

$$E_s = E_p \frac{N_s}{N_p} = 110 \times 10 = 1100 \text{ V}$$

$$\text{Output voltage } E_s = 1100 \text{ V}$$

(ii)

$$E_p I_p = E_s I_s$$

$$I_p = \frac{E_s}{E_p} I_s = 10 \times 10 = 100 \text{ A}$$

(iii)

Input power = Output power for ideal transformer

$$E_s I_s = E_p I_p = (1100)(10) = 11000 \text{ W}$$