Chapter 1
Relations & Functions

Consider the sets A = {1,2,3,4,5} and B= {3,4,5,6,7}. The Cartesian product of A and B is A × B = {(1,3), (1,4), (1,5), (1,6), (1,7), (2,3), ….., (5,6), (5,7)}.

A subset of A × B by introducing a relation R between the first element 'x' and the second element 'y' of each ordered pair (x, y) as

R= {(x,y): x is greater than y, x ∈ A, y ∈ B}. Then R = {(4,3), (5,3), 5,4)}.

Note1: Relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in A × B. The second element is called the image of the first element.

Note2: The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

Note3: The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R. Note that range ⊆ co-domain.

Tips ☞

1. A relation may be represented algebraically either by Roster method or by Set-builder method.
2. An arrow diagram is a visual representation of a relation.
3. The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B. If n(A) = p and n(B) = q, then n (A × B) = pq and the total number of relations is 2^{pq} .
4. A relation R from A to A is also stated as a relation on A.

Inverse relation: If $D=\{(a,b):a,b\in R\}$ is a relation from set A to a set B, then inverse of $R=R^{-1}=\{(b,a):a,b\in R\}$.

Note: $\text{Domain}(R)=\text{Range}(R^{-1})$ and $\text{Range}(R)=\text{Domain}\left(R^{-1}\right)$.

Types of relations

A relation R in a set A to itself is called:

1. **Universal relation: If each element of A is related to every element of A. i.e.,** $R=A\times A$
2. **An identity relation if** $R=\{(a,a):a\in A\}$
3. **An empty or void relation if no element of A is related to any element of A. i.e.,**
   Note: Empty relation and the universal relation are sometimes called trivial relations. $R=\phi\subset A\times A$
4. **A relation R in a set A is said to be**
   a) **Reflexive, if every element of A is related to itself.** $(a,a)\in R\ \forall\ a\in A$. i.e., ${}_aR_a\ \forall\ a\in A$.
   b) **Symmetric, if** $(a,b)\in R$ then $\forall\,(b,a)\in R$. i.e., ${}_aR_b={}_bR_a\ \forall\ a,b\in R$.
   c) **Transitive, if** $(a,b)\in R$ and $(b,c)\in R\Rightarrow(a,c)\in R\ \forall\ a,b,c\in R$. i.e., ${}_aR_b$ and ${}_bR_c\Rightarrow{}_aR_c$
5. **Equivalence Relation: A relation R in a set A is called an equivalent if**
   i) **R is reflexive, ii) R is symmetric and iii) R is transitive.**

Note: 1. If R and S are two relations on a set A, then $R\cap S$ is also an equivalence relation on A.

2. The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
3. The inverse of an equivalence relation is an equivalence relation.

Functions: Let A and B be two non-empty sets. A function *f* from A to B is a correspondence which associates elements of set A to element of set B such that

i. all elements of set A are associated to elements in set B.

ii. an element of set A is associated to a unique element in set B.

If *f* is a function from A to B and (a, b) ∈ *f*, then f(a)=b, where 'b' is called the image of 'a' under *f* and 'a' is called the pre-image of 'b' under *f*.

The function *f* from A to B is denoted by $f: A \to B$.

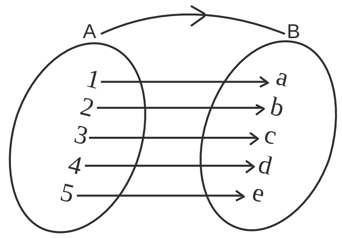
Types of Functions

One-one function (Injective): A function $f: A \to B$ is said to be an one-one function if different elements of A have different images in B.

No. of one-one functions from A to B $= \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n = 0 \end{cases}$

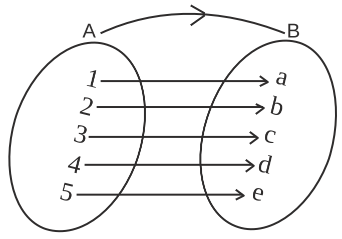
To check the injectivity of a function

i. Take two arbitrary elements x_1 and x_2 in the domain of *f*.
ii. Check whether $f(x_1) = f(x_2)$
iii. If $f(x_1) = f(x_2)$, which implies that $x_1 = x_2$ only then the function is a one-one function or injective function otherwise not.



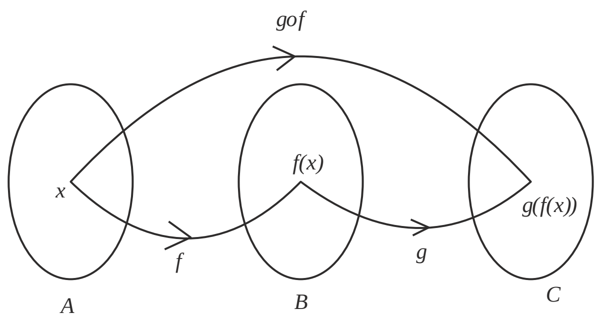
Onto function (surjective): A function $f: A \to B$ is said to be an onto function, if every element of B is the image of some element of A under *f*. i.e., for every element of $y \in B$, there exists an element $x \in A$ such that $f(x) = y$.

One-one onto function (bijective): A function $f: A \to B$ is said to be an one-one and onto, if it is both one-one and onto.



Composition of function

Let $f: A \to B$ and $g: A \to B$ be any two functions, the composition of f and g, denoted by gof is defined as the function $gof: A \to C$ given by $gof(x) = g[f(x)]\ \forall x \in A$



Invertible function

Let $f: A \to B$ and $g: A \to B$ be any two functions, the composition of f *and* g, denoted by gof is defined as the function $gof: A \to C$ given by $gof(x) = g\left[f(x)\right] \forall x \in A.$ For example, Let $f: R \to R$ be given by $f(x) = 4x + 3.$ Show that $f(x)$ is invertible. Also find the inverse of *f*.

$$f(x) = 4x + 3$$
$$f(x_1) = 4x_1 + 3$$
$$f(x_2) = 4x_2 + 3$$
$$f(x_1) = f(x_2) \Rightarrow 4x_1 + 3 = 4x_2 + 3 \Rightarrow 4x_1 = 4x_2 = x_1 = x_2.$$
$$\therefore f \text{ is one} - \text{one.}$$

Again, $y = 4x + 3$

$$y - 3 = 4x \Rightarrow x = \frac{y-3}{4}$$

$$f(x) = f\left(\frac{y-3}{4}\right) = 4 \times \frac{y-3}{4} + 3 = y - 3 + 3 = y \quad \Rightarrow f \text{ is onto.}$$

Hence, f is one-one, onto and therefore, invertible.

Now, $y = f(x) = y = 4x + 3 \Rightarrow x = \frac{y-3}{4} \Rightarrow f^{-1}(y) = \frac{y-3}{4}$

Binary operation

An operation $*$ on a non-empty set A, satisfying the closure property is known as a binary operation. For example, let * be the binary operation on N given by $a * b$ = LCM of a and b. Find

i. $4 * 3$ = LCM of 4 and 3= $4 \times 3 = 12$

ii. $16 * 24$ = LCM of 16 and 24 = $2 \times 2 \times 2 \times 2 \times 3 = 48$

Properties of binary operation:

1. **Commutative Property**: A binary operation $*$ on set A is said to be commutative, if $a * b = b * a,$ *for all* $a, b \in A.$

2. **Associative Property**: A binary operation $*$ on set A is said to be associative, if $(a * b) * c = a * (b * c),$ *for all* $a, b, c \in A.$

3. **Identity property**: A binary operation $*$ on set A is said to be identity, if an element $e \in A,$ if $a * e = a = e * a, \forall\, a \in A.$

   For example, find the identity element in Z for $*$ on Z, defined by $a * b = a + b + 1.$ Let e be the identity element in Z.

   $$a * e = a$$
   $$\therefore a * b = a * e = a + e + 1$$
   $$\Rightarrow e = -1$$

   $\Rightarrow -1 \in Z$ is the identity element for $*$.

Write the operation table of * on the set $\{1,2,3,4,5\}$ defined by $a*b=\min\{a,b\}$.

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Tips ☞

1. $(fog)(x)=f\left(g(x)\right)$
2. $(fof)(x)=f\left(f(x)\right)$
3. $(gof)(x)=g\left(f(x)\right)$
4. $(gog)(x)=g\left(g(x)\right)$
5. $\left(fof^{-1}\right)(x)=f\left(f^{-1}(x)\right)$, etc..