CONTINUITY – A QUICK REVIEW

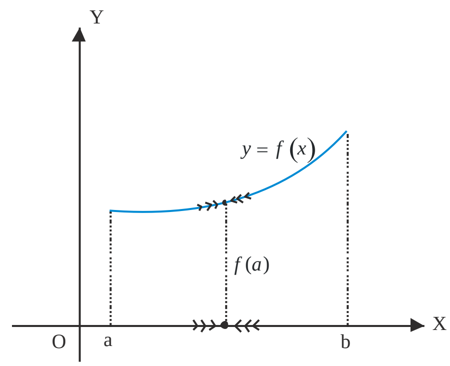
A real function *f* is said to be continuous at a real constant 'a' if

i) $f(a)$ is defined,

ii) $\lim\limits_{x \to a} f(x)$ exits, and

iii) $\lim\limits_{x \to a} f(x) = f(a)$.

Otherwise the function is said to be discontinuous function.



Everywhere continuous function: A function *f* is said to be everywhere continuous if it is continuous on the entire real line $-\infty$ to $+\infty$.

Note1 : A real function *f* is said to be continuous, if it continuous at each point of its domain.

Note2 : A function, which is not continuous, is known as discontinuous function.

Note3 : If a function consists of bracket function, modulus function and/or defined by more than one rule, then $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ are to be evaluated separately. If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$, then find $f(a)$.

If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = f(a)$, then *f* is continuous and if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) \neq f(a)$, then *f* is not continuous.

Fundamental theorems on continuous functions:

Let *f(x)* and *g(x)* be two continuous functions on their common domain D and let *k* be a real number.

i. kf is continuous
ii. $f + g$ is continuous
iii. $f - g$ is continuous
iv. fg is continuous
v. $\dfrac{f}{g}$ is continuous
vi. $\dfrac{1}{g}$ is continuous
vii. $f^n, n \in N$, is continuous

Common functions which are continuous in their domains:

a. Every constant function is continuous everywhere.
b. An identity function, f(x) = x, is continuous everywhere.
c. The modulus function is continuous everywhere.
d. The exponential function is continuous everywhere.
e. The logarithmic function is continuous everywhere.
f. The polynomial function is continuous everywhere.
g. The rational function is continuous everywhere.
h. The trigonometric function is continuous everywhere.
i. The inverse trigonometric function is continuous everywhere.
j. The composition of two function is continuous everywhere.

Discontinuous functions

A function f is said to be discontinuous at a point $x = a$ of its domain D if it is not continuous at the point a. The point $x = a$ is called the point of discontinuity. It may arise:

a. If $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ of both may not exist
b. If $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ may exist, but are unequal.
c. If $\lim_{x \to a^+} f(x)$ as well as $\lim_{x \to a^-} f(x)$ both may exist, but either of the two or both may not be equal to f(a).

or in simply we can say that:

A function is said to be discontinuous if:

Case (i) : $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

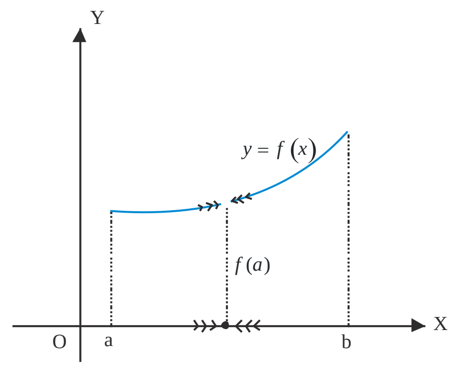
Case (ii) : $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ and is not equal to $\lim_{x \to a} f(x)$

Case (iii) : $\lim_{x \to a} f(x) \neq f(a)$

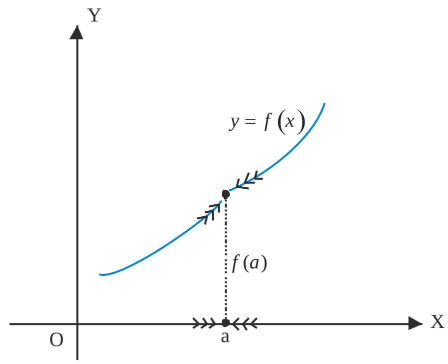
Removable discontinuity: A function f is said to be removable discontinuity

if $\lim_{x \to a} f(x)$ exists, i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ but it is not equal to $f(a)$.

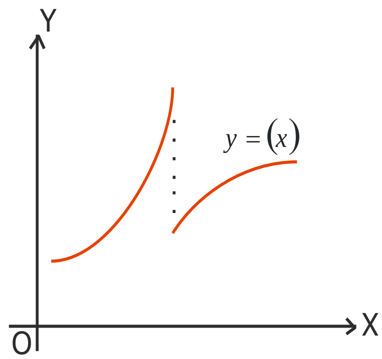
i.e., ., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$



Discontinuity of the first kind: A function f is said to be the discontinuity of the first kind at $x = a$ it $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal.



Discontinuity of the second kind: A function f is said to be the discontinuity of the second kind at $x = a$ if neither $\lim_{x \to a^-} f(x)$ nor $\lim_{x \to a^+} f(x)$ exist.



WORKING RULE

1. If the given function f(x) contains modulus function, bracket function and/or defined by more than one rule, then $\lim_{x \to a^-} f(x)$ nor $\lim_{x \to a^+} f(x)$ are to be evaluated separately, otherwise $\lim_{x \to a} f(x)$ is evaluated directly.
2. If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist and f is said to be a discontinuous function.
3. If $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ exists and is said to be continuous at x=a.

PRACTICE EXAMPLES:

1. If $f(x) = \begin{cases} \dfrac{x}{\sin 3x}, & x \neq 0 \\ k, & x - 0 \end{cases}$ is continuous at x=0, then write the value of k.

If f(x) is continuous at x=0, then $k = \lim_{x\to 0}\frac{x}{\sin 3x} = \lim_{x\to 0}\frac{3x}{\sin 3x}\times\frac{1}{3} = \frac{1}{3}\times\lim_{x\to 0}\frac{3x}{\sin 3x} = \frac{1}{3}\times 1 = \frac{1}{3}$

2. Show that $f(x) = x^3$ is continuous at $x = 2$.

If $f(x)$ *is continuous at* $x = 2$ *then* $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\lim_{x\to 2^-} f(x) = \lim_{h\to 0^+} f(2-h)^3 = \lim_{h\to 0^+}\left(8 - 12h + 6h^2 - h^3\right) = 8$

$\lim_{x\to 2^+} f(x) = \lim_{h\to 0^+} f(2+h)^3 = \lim_{h\to 0^+}\left(8 + 12h + 6h^2 + h^3\right) = 8$

$f(2) = 2^3 = 8$. *Hence* $f(x)$ *is continuousat* $x = 2$.

3. Determine the value of k for which the following function is continuous at $x = 3cm$.

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ k & , x = 3 \end{cases}.$$

Since f(x) is continuous at $x = 3cm$ $\lim_{x\to 3} f(x) = f(3)$

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$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$

$f(3) = k \Rightarrow k = 6$

4. Show that the function $f(x) = \begin{cases} \left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is discontinuous at $x = 0$.

If $f(x)$ *is continuousat* $x = 0$ *then* $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0^+} f(0+h) = \lim_{h\to 0^+} f(h) = \lim_{h\to 0^+}\left(\frac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}\right) = \lim_{h\to 0^+}\frac{e^{\frac{1}{h}}\left(1-e^{\frac{1}{h}}\right)}{e^{\frac{1}{h}}\left(1+e^{\frac{1}{h}}\right)} = \lim_{h\to 0^+}\frac{\left(1-e^{\frac{1}{h}}\right)}{\left(1+e^{\frac{1}{h}}\right)} = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0^+} f(0-h) = \lim_{h\to 0^+} f(-h) = \lim_{h\to 0^+}\left(\frac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1}\right) = \lim_{h\to 0^+}\frac{\left(\frac{1}{e^{\frac{1}{h}}}-1\right)}{\left(\frac{1}{e^{\frac{1}{h}}}+1\right)} = -1$$

Thus $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence $f(x)$ *is discontinuous at* $x = 0$.

5. Consider the function $f(x)=\begin{cases} \dfrac{k\cos x}{\pi-2x}, & if \quad x\neq\dfrac{\pi}{2} \\ 3, & if \quad x=\dfrac{\pi}{2}\end{cases}$ $at \quad x=\dfrac{\pi}{2}$. Find the value of k.

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{k\cos x}{\pi-2x}\right)$$

$$put \;\; x=\frac{\pi}{2}+h \quad As \;\; x\to\frac{\pi}{2},\; h\to 0$$

$$\therefore \lim_{x\to\frac{\pi}{2}}\left(\frac{k\cos x}{\pi-2x}\right)=\lim_{h\to 0}\left(\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}\right)=\lim_{h\to 0}\left(\frac{k\times-\sinh}{\pi-\pi-2h}\right)=\frac{-k}{-2}\times\lim_{h\to 0}\left(\frac{\sinh}{h}\right)=\frac{k}{2}$$

$$f\left(\frac{\pi}{2}\right)=3$$

Since the function is continuous at $x=\frac{\pi}{2}$

$$\frac{k}{2}=3\Rightarrow k=6$$