

CHAPTER 5

MARCH 2018

CONTINUITY AND DIFFERENTIABILITY

IMPROVEMENT 2018

1. a) Which of the following functions is not continuous at zero? (1)

i) $f(x) = \sin x$

ii) $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

iii) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

iv) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

- b) Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 10 & \text{if } x \leq 3 \\ ax + b & \text{if } 3 < x < 4 \\ 20 & \text{if } x \geq 4 \end{cases}$$
 is a

continuous function. (3)

2. a) Verify mean value theorem for the function $f(x) = x^2 - 4x - 3$ in the interval $[1, 4]$. (3)

- b) Consider the function

$$f(x) = \sin^{-1} 2x\sqrt{1-x^2}, -\frac{1}{2} \leq x \leq \frac{1}{\sqrt{2}}$$

i) Show that $f(x) = 2 \sin^{-1} x$ (2)

ii) Find $f'(x)$. (1)

3. a) $\frac{d}{dx}(a^x) = \dots\dots\dots$ (1)

i) a^x

ii) $\log(a^x)$

iii) $a^x \log a$

iv) $x a^{x-1}$

b) Find $\frac{dy}{dx}$ if $x^y = y^x$ (3)

4. a) Prove that the function defined by $f(x) = \cos(x^2)$ is a continuous function. (2)

b) i) If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, show that

$$\frac{dy}{dx} = -\frac{ae^{a \cos^{-1} x}}{\sqrt{1-x^2}} \quad (1)$$

ii) Hence prove that

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0 \quad (3)$$

SAY 2017

5. a) Examine whether the function defined by

$$f(x) = \begin{cases} x+5 & \text{if } x \leq 1 \\ x-5 & \text{if } x > 1 \end{cases}$$
 is continuous or not. (2)

b) If $x = a^{\sin^{-1} t}$; $y = a^{\cos^{-1} t}$, $a > 0$, show that

$$\frac{dy}{dx} = \frac{-y}{x} \quad (2)$$

c) If $y = \sin^{-1} x$, show that

$$(1-x^2) \frac{d^2 y}{dx^2} = x \frac{dy}{dx} \quad (2)$$

MARCH 2017

6. a) Find the values of a and b such that the function

$$f(x) = \begin{cases} 5a & x \leq 0 \\ a \sin x + \cos x & 0 < x < \frac{\pi}{2} \\ b - \frac{\pi}{2} & x \geq \frac{\pi}{2} \end{cases}$$

is continuous.

(3)

SAY 2016

7. a) Find $\frac{dy}{dx}$, if $x = a \cos^2 \theta, y = b \sin^2 \theta$ (3)

b) Find the second derivative of the function
 $y = e^x \sin x$ (3)

8. a) Find all points of discontinuity of f , where f is defined by $f(x) = \begin{cases} 2x+3 & x \leq 2 \\ 2x-3 & x > 2 \end{cases}$ (2)

b) If $e^{x-y} = x^y$, then prove that $\frac{dy}{dx} = \frac{\log x}{[\log ex]^2}$ (4)

SAY 2015

9. a) Find the relationship between a and b , if the function f defined by $f(x) = \begin{cases} ax+1, & x \leq 3 \\ bx+3, & x > 3 \end{cases}$ is continuous. (2)

b) Find $\frac{dy}{dx}$, if $y^x = x^y$. (2)

c) If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ (2)

MARCH 2015

10. a) Find a and b if the function

$$f(x) = \begin{cases} \frac{\sin x}{x}, & -2 \leq x < 0 \\ a \cdot 2^x, & 0 \leq x \leq 1 \\ b+x, & 1 < x \leq 2 \end{cases}$$

is a discontinuous function on $[-2, 2]$ (3)

b) How many functions $f(x) = |x|, g(x) = |x|^2$ and $h(x) = |x|^3$ are not differentiable at $g(x) = |x|^2$? (1)

(i) 0 (ii) 1 (iii) 2 (iv) 3

11. Find $\frac{dy}{dx}$ if

a) $x^3 + 2x^2y + 3xy^2 + 4y^3 = 5$ (2)

b) $x = 2 \cos^3 \theta; y = 2 \sin^3 \theta$ (2)

c) $y = \sin^{-1}(2x\sqrt{1-x^2}); -1 \leq x \leq 1$ (2)

SAY 2014

12. a) Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax^2 - 1, & x \leq 2 \\ bx + 3, & x > 2 \end{cases}$$
 is continuous. (2)

b) Given that $e^{x-y} = x^y$, then show that

$$\frac{dy}{dx} = \frac{\log x}{[\log ex]^2}$$
 (2)

c) Verify mean value theorem for the function $f(x) = x^2 - 4x - 3$ in the interval $[1, 4]$.

(2)

MARCH 2014

13. a) Find the value of k if the function:

$$f(x) = \begin{cases} kx+1 & \text{if } x \leq 5 \\ 3x-5 & \text{if } x > 5 \end{cases} \text{ is}$$

continuous at $x = 5$. (2)

- b) Find $\frac{dy}{dx}$, if $x = a(t - \sin t)$;
 $y = a(1 + \cos t)$ (2)
- c) Verify Rolle's theorem for the function
 $f(x) = x^2 + 2$ in the interval $[-2, 2]$ (2)

SAY 2013

14. a) Find $\frac{dy}{dx}$, if $y = \log x, x > 0$. (1)
- b) Is $f(x) = |x|$ differentiable at $x = 0$? (1)
- c) Find $\frac{dy}{dx}$, if $x = \sin \theta - \cos \theta$ and
 $y = \sin \theta + \cos \theta$ (1)

MARCH 2013

15. a) Consider $f(x) = \begin{cases} 3x-8, & x \leq 5 \\ 2k, & x > 5 \end{cases}$.
 Find the value of k if $f(x)$ is continuous
 at $x = 5$. (2)
- b) Find $\frac{dy}{dx}$, if $y = (\sin x)^{\log x}, \sin x > 0$. (2)
- c) If $y = (\sin^{-1} x)^2$, then show that
 $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2..$ (2)

SAY 2012

16. a) Determine the value of k so that the function
 $f(x) = \begin{cases} k(x^2 + x + 1), & x < 0 \\ \cos x, & x \geq 0 \end{cases}$ is continuous. (2)

b) If $y = \left[x + \sqrt{x^2 + 9} \right]^n$,
 show that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2 + 9}}$ (2)

c) Find $\frac{dy}{dx}$, if $x = \sqrt{a^{\sin^{-1} t}}$; $y = \sqrt{a^{\cos^{-1} t}}$ (2)

MARCH 2012

17. a) Find the value of k if $f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 2x, & \text{if } x > 2 \end{cases}$
 is continuous. (2)
- b) Find $\frac{dy}{dx}$, if $y = x^{\sin x}, x > 0$ (2)
- c) If $y = \sin^{-1} x$, show that
 $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 0$ (2)

SAY 2011

18. a) Check the continuity of the function f given by
 $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ (2)
19. a) Find the derivative of $x^a + a^x$ with respect
 to x . (1)
- b) Find $\frac{dy}{dx}$, if $x = a \cos^3 t$ and $y = a \sin^3 t$ (2)
- c) If $e^y(x+1) = 1$, show that
 $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx} \right)^2$ (3)

MARCH 2011

20. (a) The function $f(x) = \begin{cases} 5 & ; & x \leq 2 \\ ax+b & ; & 2 < x < 10 \\ 21 & ; & x \geq 10 \end{cases}$
 is continuous. Find a and b . (2)
- (b) (i) Find $\frac{dy}{dx}$ if $y = \sin x^{\sin x}$ (1)

(ii) If $y = ae^{mx} + be^{nx}$, show that

$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0 \quad (3)$$

SAY 2010

21. (a) Show that the function $f(x)$ defined by

$$f(x) = \sin(\cos x) \text{ is a continuous function.} \quad (2)$$

(b) If $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, show that $\frac{d^2y}{dx^2} = \frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$ (2)

MARCH 2010

22. (a) the value of $\lim_{x \rightarrow 0} \frac{1 - \cos 6x}{1 - \cos 4x}$ is

[First year portion] (1)

(b) Consider the function

$$f(x) = \begin{cases} 3ax + b & , \text{ if } x > 1 \\ " & , \text{ if } x = 1 \\ 5ax - 2b & , \text{ if } x < 1 \end{cases}$$

(i) Find $\lim_{x \rightarrow 1^-} f(x)$ and $\lim_{x \rightarrow 1^+} f(x)$ (1)

(ii) Find the constant a and b if $f(x)$ is continuous at $x = 1$. (2)

SAY 2009

23. a) Evaluate the limit $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 - 4}$

[First year portion] (2)

b) Prove that the function $f(x) = x - |x|$ is continuous at the origin. (2)

24. a) Differentiate e^x with respect to x from first principle. [First year portion] (2)

b) If $y = \sin(\log x)$,

prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$ (2)

MARCH 2009

25. Consider $f(x) = \begin{cases} \frac{x^2 - x - 6}{x + 2}, & x \neq -2 \\ -5, & x = 2 \end{cases}$

a) What is the value of $f(-2)$? (1)

b) Check whether the function is continuous at $x = -2$. (2)

c) Match the following: (2)

(a)	$\frac{d}{dx} \left[\frac{\sin^{-1}(3x - 4x^3) + \cos^{-1}(4x^3 - 3x)}{\cos^{-1}(4x^3 - 3x)} \right]$	$\frac{1}{2}$
(b)	$\frac{d}{dx} \left[\tan^{-1} \sqrt{\frac{1 - \cos x}{1 + \cos x}} \right]$	$\frac{1}{\sqrt{2}}$
		0

26. a) If $y = 3\cos(\log x) + 4\sin(\log x)$

i) Find $\frac{dy}{dx}$ (1)

ii) Prove that $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$ (2)

b) i) Find the derivative of e^{2x} from first principle. [First year portion] (2)

ii) Find the derivative of $e^{2x + \log x}$ w.r.t. x (1)

iii) Find $\frac{dy}{dx}$, if $x = a(\theta - \sin \theta)$;
 $y = a(1 - \cos \theta)$ (2)

MARCH 2008

27. (a) Let $y = \left(x + \sqrt{1+x^2}\right)^m$

i) Find $\frac{dy}{dx}$ (2)

(b) Let $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$

i) Express $1-\cos x$ and $\sin x$ in terms of

$\sin\left(\frac{x}{2}\right)$ and $\cos\left(\frac{x}{2}\right)$. (2)

ii) Hence find $\frac{dy}{dx}$ (2)

OR

(a) Differentiate $y = e^x$ with respect to x , from first principles. [First year portion] (4)

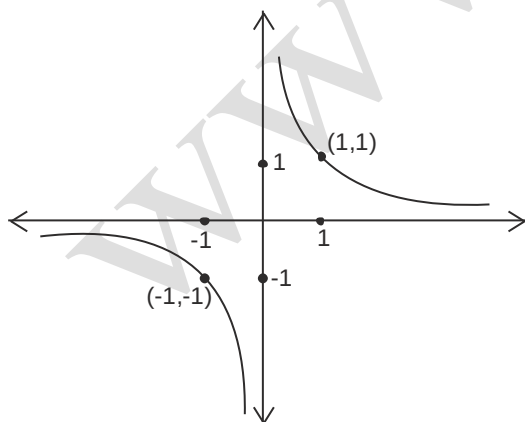
(b) If $x^y = e^{x-y}$

i) Express y in terms of x . (1)

ii) Find $\frac{dy}{dx}$ (2)

MARCH 2007

28. i) Choose the correct function from the bracket which represents the following graph



[$f(x) = \frac{1}{x}$; $f(x) = x^2$; $f(x) = e^x$;

$f(x) = \log_e x$] (1)

ii) [First year portion]

iii) Consider $f(x) = \begin{cases} 3x-8 & \text{if } x \leq 5 \\ 2k & \text{if } x > 5 \end{cases}$ find the value of K , if $f(x)$ is continuous at $x = 5$. (2)

29. i) Differentiate $x^{\sin x}$ w.r.t. x (2)

ii) If $x = at^2$, $y = 2t$, then find $\frac{d^2y}{dx^2}$. (2)

iii) If $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$, find $\frac{dy}{dx}$ (3)

MARCH 2006

30. If $y = a^x$, then $\frac{dy}{dx}$ is

a) a^x b) $a^x \log x$ c) xa^{x-1}
d) $a^x \log a$ (1)

31. If $\sin y = x \sin(a+y)$, prove that

$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ (2)

32. Prove that the function

$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$. (3)

33. If $x = a \cos^3 \theta$, $y = b \sin^3 \theta$, find $\frac{d^2y}{dx^2}$ (3)

34. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$, then prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$ (3)

35. Find $\frac{dy}{dx}$ for the function

$$(\cos x)^y + (\sin y)^x = k \quad (5)$$

MARCH 2005

36. Derivative of $\log_a x$ with respect to x is:

- a) $\frac{1}{x \log_a e}$ b) $\frac{1}{x \log_a x}$
 c) $\frac{1}{x \log_e a}$ d) $\frac{1}{x \log_x a}$ (1)

37. Derivative of $\tan^{-1}\left(\frac{\sin x}{1 - \cos x}\right)$ is

- a) $-\frac{1}{2}$ b) 1
 c) $\frac{1}{2}$ d) -1 (1)

38. If $x = a \cos^3 t$ and $y = a \sin^3 t$ find $\frac{dy}{dx}$ (2)

39. Determine k so that the function

$$f(x) = \begin{cases} k(x^2 - 2) & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases} \text{ is continuous at } x = 0 \quad (2)$$

40. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots}}}$, show that $\frac{dy}{dx} = \frac{1}{x(2y-1)}$ (2)

41. If $y = [x + \sqrt{x^2 + 1}]^m$, show that $(1 + x^2)y_2 + xy_1 - m^2y = 0$ (3)

SAY 2004

42. Find $\frac{dy}{dx}$, if $x = a \cos^3 \theta$, $y = a \sin^3 \theta$. (2)

43. Find $\frac{dy}{dx}$, if $y = \sin \sqrt{x} + \sqrt{\sin x}$. (2)

44. Examine the continuity of the function

$$f(x) = \begin{cases} 1+x; & x \leq 2 \\ 5-x; & x > 2 \end{cases} \text{ at the point } x=2. \quad (3)$$

45. Find $\frac{dy}{dx}$ if $y = x^x + a^x + x^a$ (3)

46. If $x = e^y$, then $\frac{dy}{dx}$ is

- a) x b) $\frac{1}{x}$
 c) e^x d) y (1)

47. If $y = \sec^{-1} 2x$, then $\frac{dy}{dx}$ is (1)

- a) $\frac{1}{2x\sqrt{4x^2-1}}$ b) $\frac{1}{x\sqrt{4x^2-1}}$
 c) $\frac{2}{x\sqrt{x^2-1}}$ d) $\frac{2}{\sqrt{4x^2-1}}$ (1)

48. The slope of the tangent of the curve $y = e^{-x^2}$ at $x=1$ is

- a) 0 b) $-\frac{2}{e}$
 c) $-2e$ d) $\frac{2}{e}$ (1)

49. If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$ (3)

50. Find $\frac{d^2y}{dx^2}$, given that $x = ct$, $y = \frac{c}{t}$ (3)

51. Differentiate $\sin^2 x$ w.r.t. $(\log x)^2$ (3)

52. If $y = (\sin^{-1} x)^2$,
 prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$ (5)

53. Differentiate $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}$ w.r.t $\tan^{-1} \frac{2x}{1-x^2}$ (5)

MARCH 2004

54. If $y = \log(\sec x - \tan x)$, then $\frac{dy}{dx} =$

- a) $-\sec x$ b) $\sec x + \tan x$

c) $\sec x$ d) $\sec x - \tan x$ (1)

55. If $x^3 + y^3 = 3axy$, find $\frac{dy}{dx}$. (2)

56. Find $\frac{dy}{dx}$, if $y = \sin(\tan^{-1} x)$ (3)

57. If $x^y = y^m$, find $\frac{dy}{dx}$ (3)

58. If $f(x) = \begin{cases} 1 & \text{for } x \leq 3 \\ ax+b & \text{for } 3 < x < 5 \\ 7 & \text{for } x \geq 5 \end{cases}$ is continuous, find a and b. (3)

59. If show that $(1+x^2)y_2 + (2x-1)y_1 = 0$ (5)

SAY 2003

60. If $x = \sec \theta$, $y = b \tan \theta$ then $\frac{dy}{dx}$ is:

a) $\frac{a \sin \theta}{b}$ b) $\frac{b}{a \sin \theta}$
c) $\frac{a \cos \theta}{b}$ d) $\frac{b}{a \cos \theta}$ (1)

61. If $y = \sqrt{3x^2 - 7}$ then $\frac{dy}{dx}$ is

a) $\frac{3}{2\sqrt{3x^2 - 7}}$ b) $\frac{3x}{\sqrt{3x^2 - 7}}$
c) $\frac{3x}{2\sqrt{3x^2 - 7}}$ d) $\frac{-3x}{2\sqrt{3x^2 - 7}}$ (1)

62. If $y = \sin^{-1} \sqrt{\frac{1+x^2}{2}}$,

show that $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^4}}$ (2)

63. Differentiate $(\sin x)^{\tan x}$ w.r.t x (2)

64. Find $\frac{dy}{dx}$ if $ax^2 + 2hxy + by^2 = 0$ (2)

65. Find the value of k such that the following function is continuous at $x = 2$

$$f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases} \quad (3)$$

66. If $y = (\cos^{-1} x)^2$,

prove that $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 2$ (5)

67. Differentiate $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$ w.r.t. $\tan^{-1} x$. (5)

MARCH 2003

68. If $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$, then $\frac{dy}{dx}$ is:

a) 1 (b) 0 (c) -1 (d) $\frac{\pi}{2}$ (1)

69. If $x = ct$; $y = \frac{c}{t}$, then $\frac{d^2 y}{dx^2}$ (3)

70. Find $\frac{dy}{dx}$ if $\sqrt{x} + \sqrt{y} = 4$. (2)

71. If $y = \frac{x}{x+4}$, prove that $x \frac{dy}{dx} = y(1-y)$. (2)

72. Examine the function

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

for continuity at $x = 0$. (3)

73. If $y = (\sin^{-1} x)^2$, prove that

$$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 2 = 0 \quad (3)$$