6 APPLICATION OF DERIVATIVES

MODEL QUESTIONS

Question :

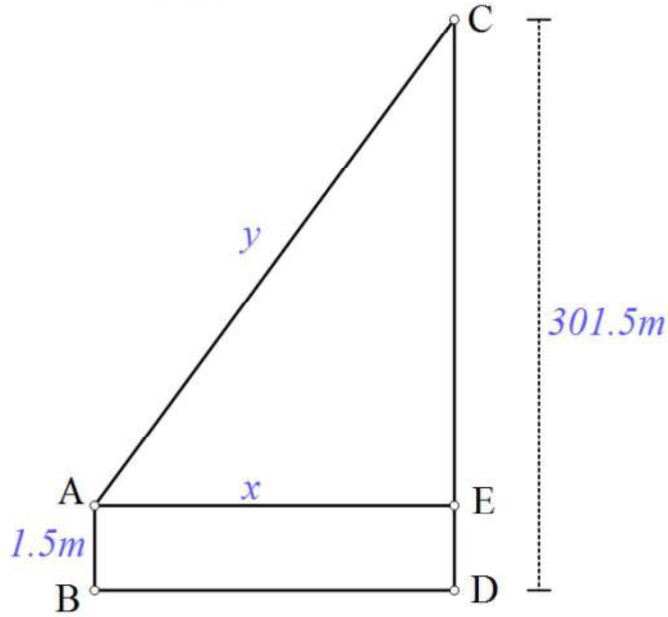
A kite is moving horizontally at a height of 301.5m. If the speed of kite is 20m/s, how fast is the string being led out; when the kite is 500m away from the boy who is flying the kite?The height of the boy 1.5m

Answer :

Given $h = 301.5\text{m}$,

$v = 20\text{m/s}$

$$\frac{dx}{dt} = 20$$



From figure

$$x^2 + 300^2 = y^2$$

Differentiating,we get

$$2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

When $y = 500\text{m}$

$$x^2 + 300^2 = 500^2$$

$$x^2 = 250000 - 90000 = 160000$$

or $x = 400\text{m}$

$$\left(\frac{dy}{dt}\right)_{y=500\text{m}} = \frac{400}{500}\times 20 = 16\ \text{ms}^{-1}$$

Question :

Find the intervals in which the function $(x+1)^3(x-3)^3$ are strictly increasing or decreasing

Answer :

We have $f(x)=(x+1)^3(x-3)^3$

$$\frac{dy}{dx}=3(x+1)^2(x-3)^3+3(x-3)^2(x+1)^3$$
$$=3(x+1)^2(x-3)^2[x-3+x+1]$$
$$=3(x+1)^2(x-3)^2[2x-2]$$
$$=6(x+1)^2(x-3)^2[x-1]$$

Now, $\frac{dy}{dx}=0$

$\Rightarrow \quad x=-1,3,1$

The points -1, 3, 1 divide the real line into four disjoint intervals: $(-\infty,-1),(-1,1),(1,3),(3,\infty)$

In intervals $(-\infty,-1)$ and $(-1,1)$, $\frac{dy}{dx}<0$

$\therefore$ f is strictly decreasing in intervals $(-\infty,-1)$ and $(-1,1)$.

Question :

Find the slope of the tangent to the curve $y=3x^2$ at the point on it whose x coordinate is 2

Answer :

Given $y=3x^2$

$$\frac{dy}{dx}=6x$$

At $x=2$,

$$\left(\frac{dy}{dx}\right)_{x=2}=6\times 2$$
$$=\underline{\underline{12}}$$

Question :

Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

Answer :

Given $x = a\sin^3\theta$

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$y = a\cos^3\theta$$

$$\frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\cot\theta$$

At $\theta = \frac{\pi}{4}$, $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -1$

At $\theta = \frac{\pi}{4}$, $x = a\left(\sin\frac{\pi}{4}\right)^3$, $y = a\left(\cos\frac{\pi}{4}\right)^3$

$$x = a\left(\frac{1}{\sqrt{2}}\right)^3 = a\left(\frac{1}{2}\right)^{3/2}, \quad y = a\left(\frac{1}{\sqrt{2}}\right)^3 = a\left(\frac{1}{2}\right)^{3/2}$$

Equation of tangent at the point $\left(\frac{a}{2^{3/2}}, \frac{a}{2^{3/2}}\right)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$Y - \frac{a}{2^{3/2}} = \frac{dy}{dx}\left(X - \frac{a}{2^{3/2}}\right)$$

$$Y - \frac{a}{2^{3/2}} = (-1)\left(X - \frac{a}{2^{3/2}}\right)$$

$$Y + X = \frac{2a}{2^{3/2}} = \frac{a}{\sqrt{2}}$$

$$X+Y-\frac{a}{\sqrt{2}}=0$$

Slope of the normal $=1$

$$Y-\frac{a}{2^{3/2}}=(1)\left(X-\frac{a}{2^{3/2}}\right)$$

$$X+Y=0$$

Question :

Find the points on the curve $y=x^3-11x+5$ at which equation of tangent is $y=x-11$.

Answer :

Given curve $y=x^3-11x+5$

Slope of the tangent to the curve is

$$\frac{dy}{dx}=3x^2-11$$

Also, slope of the tangent $y=x-1$ is 1

$$\frac{dy}{dx}=1$$

$$\therefore \quad 3x^2-11=1$$

$$3x=12$$

$$x=\pm 2$$

When $x=2, \quad y=2^3-11\times 2+5=-9$

When $x=-2, \quad y=-2^3+11\times 2+5=19$

Hence, the required points are $(2,-9)$ and $(-2,19)$

Question :

Find the equation of the normal to the curve $y^2=4x$ at the point $(1,2)$.

Answer :

Given $y^2=4x$

$$2y\frac{dy}{dx}=4$$

$$\frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1,2)}=\frac{2}{2}=1 \quad \text{or} \quad \text{slope}=12$$

Question :

Find the approximate value of $(1.999)^5$

Answer :

Let $x=2, \Delta x=-0.001$

Let $y=x^5$

On differentiating both sides,we get

$$\frac{dy}{dx}=5x^4$$

Now, $\Delta y=\frac{dy}{dx}\Delta x=5x^4\times\Delta x$

$$=5\times 2^4\times(-0.001)=-80\times 0.001=-0.080$$

$$\therefore \quad (1.999)^5 = y+\Delta y=2^5+(-0.080)$$

$$=32-0.080=31.920$$

Question :

Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 4cm and 4.0005 cm respectively.

Answer :

Volume of shell, $V=\frac{4}{3}\pi\left(4.0005^3-4^3\right)$

Let $(4.0005)^3=y+\Delta y$

Let $x=4, \Delta x=0.0005$

Let $y=x^3$

On differentiating both sides,we get

$$\frac{dy}{dx}=3x^2$$

Now, $\Delta y = \frac{dy}{dx}\Delta x = 3x^2 \times \Delta x$

$$= 3 \times 4^2 \times (0.0005)$$

$$= 48 \times 0.0005 = 0.024$$

$\therefore\ (4.0005)^3 = y + \Delta y = 4^3 + (0.024) = 64.024$

$$V = \frac{4}{3}\pi\left(4.0005^3 - 4^3\right) = \frac{4}{3}\pi(64.024 - 64)$$

$$= \frac{4}{3}\pi \times 0.024 = \underline{\underline{0.032\pi cm^3}}$$

Question :

Find the local maxima and minima of $x^3 - 6x^2 + 9x + 15$.Also find local maximum and minimum values.

Answer :

Given $f(x) = x^3 - 6x^2 + 9x + 15$

$$f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 0$$

$$0 = 3x^2 - 12x + 9$$

$$0 = 3(x-1)(x-3)$$

$\therefore\quad x = 1, 3$

Now,

$$f''(x) = 6x - 12$$

$$f''(1) = 6 \times 1 - 12 = -6 < 0$$

$$f''(3) = 6 \times 3 - 12 = 6 > 0$$

$\therefore$ By second derivative test, $x = 1$ is a point of local maxima.

The local maximum value of f at $x = 1$

$$f(1) = 1^3 - 6 \times 1^2 + 9 \times 1 + 15$$

$$= 1 - 6 + 9 + 15 = 19$$

And $x = 3$ is a point of local minima.

The local minimum value of f at $x = 3$

$$f(3) = 3^3 - 6\times 3^2 + 9\times 3 + 15$$
$$= 27 - 54 + 27 + 15 = 15$$

Question :

Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Answer :

Given curves $\quad xy = 4 \quad$ (i)

$$x^2 + y^2 = 8 \quad \text{(ii)}$$

For curve (ii), $\quad xy = 4 \quad$ or $\quad y = \frac{4}{x}$

$$x^2 + \left(\frac{4}{x}\right)^2 = 8$$
$$x^4 + 16 = 8x^2$$
$$x^4 - 8x^2 + 16 = 0$$
$$\left(x^2 - 4\right)^2 = 0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2 \quad \text{and} \quad y = \pm 2$$

$\therefore$ Points of intersection are $(2,2)$ and $(-2,-2)$

$\therefore$
$$y + x\left(\frac{dy}{dx}\right)_1 = 0$$
$$\left(\frac{dy}{dx}\right)_1 = -\frac{y}{x}$$

At point $(2,2)$ and $(-2,-2)$, $\left(\frac{dy}{dx}\right)_1 = -1$

For curve (ii), $\quad 2x + 2y\left(\frac{dy}{dx}\right)_2 = 0$

$$\left(\frac{dy}{dx}\right)_2 = -\frac{x}{y}$$

At point $(2,2)$ and $(-2,-2)$, $\left(\dfrac{dy}{dx}\right)_2 = -1$

Thus two curves touch each other.

Question :

At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ the tangents are parallel.

Answer :

Given equations of curve is $x^2 + y^2 - 2x - 4y + 1 = 0$

Differentiating, we get

$$2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{1-x}{y-2}$$

Since, the tangents are parallel to the y-axis,

$$\frac{dy}{dx} = \frac{1-x}{y-2} \to \infty$$

$$\therefore \quad y - 2 = 0 \quad \Rightarrow \quad y = 2$$

$$\therefore \quad x^2 + 2^2 - 2x - 4 \times 2 + 1 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

$$x = -1, \; x = 3$$

The required poits: $(-1,2)$ and $(3,2)$

HOME WORK QUESTIONS

Question : (March 2018)

$f(x)$ is a strictly increasing function, if $f'(x)$ is...........

Hint or Answer :

positive

Question : (March 2018)

(a) Find the slope of the tangent to the curve $y=(x-2)^2$ at $x=1$

(b) Find a point at which the tangent to the curve $y=(x-2)^2$ is parallel to the chord joining the points $A(2,0)$ and $B(4,4)$.

(c) Find the equation of the tangent to the above curve and parallel to the plane AB.

Hint or Answer :

(a) $\text{slope} = -2$

(b) $\text{slope} = 2, \quad \text{point is } (3,1)$

(c) $2x - y - 5 = 0$

Question : (Sept 2017)

(a) Slope of the tangent to the curve $y = 5 - 10x^2$ at the point $(-1,-5)$ is $(10, -10, 20, -20)$

(b) Show that of all rectangles inscribed in a fixed circle, the square has tha maxima area.

OR

(a) Maximum value of $f(x) = \log x$ in $[1, e]$ is $\left(1,\ e,\ \frac{1}{e},\ 0\right)$

(b) Using differentials, find the approximate value of $(255)^{1/4}$.

Hint or Answer :

(a) 20

(b) $x = \sqrt{2}r,\ y = \sqrt{2}r$

OR

(a) $\frac{1}{e}$

(b) 3.9961

Question : (March 2017)

(a) Slope of the normal to the curve $y = 4x$ at $(1,2)$ is

$$\left(1, \frac{1}{2},\ 2, -1\right)$$

(b) Find the interval in which $2x^3 + 9x^2 + 12x - 1$ is strictly increasing

OR

(a) The rate of change of volume of a sphere with respect to its radius when radius is 1unit. $\left(4\pi, 2\pi,\ \pi, \frac{\pi}{2}\right)$

(b) Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Hint or Answer :

(a) -1

(b) $(-\infty, -2), (-1, \infty)$

(a) 4π

(b) 8, 8

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