INDEFINITE INTEGRALS

The process of obtaining *f(x)* when its derivative $f'(x)$ is given, is known as integration. It is the reverse process of differentiation.

Anti-derivative or Primitive

Let *f(x)* and *g(x)* be any two functions such that $\frac{d}{dx}[f(x)] = g(x)$, then *f(x)* is called the integral of *g(x)*. It is symbolically written as $\int g(x)dx = f(x)$. The symbol $\int$ is read as integral and dx denotes that the function $g(x)$ is integrated with respect to x. The function being integrated is called integrand. Here *g(x)* is the integrand.

E.g.: If $\frac{d}{dx}\left(x^2\right) = 2x \Rightarrow \int 2x dx = \frac{x^2}{2}$

The indefinite integral and constant of integration

Suppose that $\frac{d}{dx}[f(x)] = g(x)$. Then by definition we have $\int g(x)dx = f(x)$. Also if $\frac{d}{dx}(f(x)+C) = g(x)+0 = g(x)$. $\therefore \int g(x)dx = f(x)$. Thus we arrive at two integrals for the same function $g(x)$. But the two integrals differ only by a constant C (is called **constant of integration**). Since the constant C is arbitrary*, we can assume different values so we get a number of functions all of which are integrals of $g(x)$, any one of them is called an indefinite integral. The function $f(x)+C$ is called the general integral.

Fundamental Theorems

1. $\int c\, f(x)dx = c\int f(x)dx$

E.g.: $\int 5\cos x\, dx = 5\int \cos x dx = 5(\sin x) + C = 5\sin x + C$

2. $\int [\, f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

E.g.: $\int \left(3\tan x - 5\sec^2 x\right)dx = 3\int \tan x dx - 5\int \sec^2 x dx = 3\log|\sec x| - 5\tan x + C$

* arbitrary constant – a constant, which can assume any values of R

Standard results:

Put a constant of integration C along with each integral.

$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log\lvert\sec x\rvert$
$\operatorname{cosec} x$	$\log\lvert\operatorname{cosec} x - \cot x\rvert$
$\sec x$	$\log\lvert\sec x + \tan x\rvert$
$\cot x$	$\log\lvert\sin x\rvert$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
x^n	$\frac{x^{n+1}}{n+1}$
e^x	e^x
e^{-x}	$-e^{-x}$
a^x	$\frac{a^x}{\log a}$
$\sqrt{x}$	$\frac{2}{3}x^{\frac{3}{2}}$
$\frac{1}{x}$	$\log\lvert x\rvert$

x	$\frac{x^2}{2}$
1	x
k	kx
$\frac{1}{x^2}$	$-\frac{1}{x}$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x}$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x \quad (or) \ -\cos^{=1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x \quad (or) \ -\cot^{-1} x$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x \ (or) \ -\cos ec^{-1} x$
$\frac{1}{x+a}$	$\log\lvert x+a\rvert$
$\frac{1}{x-a}$	$\log\lvert x-a\rvert$
$\frac{1}{a+x}$	$\log\lvert a+x\rvert$
$\frac{1}{a-x}$	$-\log\lvert a-x\rvert$
$\frac{1}{(x+a)^2}$	$-\frac{1}{(x+a)}$
$\frac{1}{(x-a)^2}$	$-\frac{1}{(x-a)}$

$\frac{1}{(a+x)^2}$	$-\frac{1}{(a+x)}$
$\frac{1}{(a-x)^2}$	$\frac{1}{(a-x)}$
$e^{ax}.\sin bx$	$\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
$e^{ax}.\cos bx$	$\frac{e^{ax}}{a^2+b^2}(a\cos bx - b\sin bx)$

Methods of integration

1. Integral of the product or quotient of two or more functions.
2. Integration by substitution
3. Integration by parts
4. Integration of rational algebraic functions by using partial fractions

Method 1

Integral of the product or equivalent of two functions. First multiply or divide by terms and then integrate it.

- $$\int x(x^2-3)dx = \int x^3dx - 3\int xdx = \frac{x^4}{4} - 3\frac{x^2}{2} = \frac{x^4}{4} - \frac{3x^2}{2} + C$$
- $$\int \frac{\cos 2x}{\sin^2 x\cos^2 x}dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x\cos^2 x}dx = \int \frac{\cos^2 x}{\sin^2 x\cos^2 x}dx - \int \frac{\sin^2 x}{\sin^2 x\cos^2 x}dx$$
$$= \int \operatorname{cosec}^2 x\,dx - \int \sec^2 xdx = -\cot x - \tan x + C = -(\cot x + \tan x) + C$$

Method2:

Type 1: Integral of functions of the form $I = \int f(ax+b)dx$

Let $I = \int f(ax+b)dx$

Put $ax+b = u$

$$(a.1+o)dx = du \Rightarrow adx = du \Rightarrow dx = \frac{1}{a}du$$

$\therefore I = \int f(u)\frac{1}{a}du = \frac{1}{a}\int f(u)du$, which can be evaluated.

Hint: Integral of the function of a function : let us assume the inside function be 'x' and then it becomes in the standard form. Find the integral of the standard function, replace 'x' by the inside function and divide the result by the derivative of the inside function.

$$I = \int f[\phi(x)]dx = \frac{F[\phi(x)]}{\phi'(x)} + C$$

E.g.:

- $\int \sin(2x+3)dx = \cos(2x+3) \times \frac{1}{\frac{d}{dx}(2x+3)} = \frac{\cos(2x+3)}{2\times1+0} = \frac{\cos(2x+3)}{2} + C$
- $\int \tan(3x-5)dx = \log|\sec(3x-5)| \times \frac{1}{\frac{d}{dx}(3x-5)} = \frac{\log|\sec(3x-5)|}{3\times1-0} = \frac{\log|\sec(3x-5)|}{3} + C$
- $\int \sqrt{3x-5}dx = \frac{2}{3}(3x-5)^{\frac{3}{2}} \times \frac{1}{\frac{d}{dx}(3x-5)} = \frac{2}{3}(3x-5)^{\frac{3}{2}} \times \frac{1}{3} + C = \frac{2}{9}(3x-5)^{\frac{3}{2}} + C$

Note: Second and third powers of sine and cosine functions can be integrated using this type after rewriting them as multiple angles.

Trigonometric functions are:

1. $\sin^2 x = \frac{1-\cos 2x}{2}$
2. $\cos^2 x = \frac{1+\cos 2x}{2}$
3. $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$
4. $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$
5. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
6. $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
7. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
8. $2\sin A\sin B = \cos(A-B) - \cos(A-B)$

E.g.: i) $I = \int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx == \frac{1}{2}\int dx - \frac{1}{2}\int \cos x dx = \frac{1}{2}x - \frac{1}{2}\sin x + C$

ii) $I = \int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] = \frac{1}{2}x + \frac{\sin 2x}{4} + C$

iii) $I=\int \sin^3 x dx=\int \frac{3\sin x-\sin 3x}{4}dx=\frac{3}{4}\int \sin x dx-\frac{1}{4}\int \sin 3x dx$

$=\frac{3}{4}(-\cos x)-\frac{1}{4}\left(-\frac{\cos 3x}{3}\right)+C=-\frac{3}{4}\cos x+\frac{1}{12}\cos 3x+C$

iv) $I=\int \cos^3 x dx=\int \frac{3\cos x+\cos 3x}{4}dx=\frac{3}{4}\int \cos x dx+\frac{1}{4}\int \cos 3x dx$

$=\frac{3}{4}\sin x+\frac{1}{4}\frac{\sin 3x}{3}+C=\frac{3}{4}\sin x+\frac{1}{12}\sin 3x+C$

v) $I=\int \sin 5x\cos 3x\, dx = \frac{1}{2}\int 2\sin 5x\cos 3x\, dx= \frac{1}{2}\int (\sin 8x+\sin 2x)\, dx$

$=\frac{1}{2}\left(\frac{-\cos 8x}{8}+\frac{-\cos 2x}{2}\right)+C = \frac{-1}{16}\cos 8x-\frac{1}{4}\cos 2x+C$

vi) $I=\int 2\sin 3x\cos 5x dx=\int [\sin(5x+3x)-\sin(5x-3x)]dx=\int (\sin 8x-\sin 2x)dx$

$=-\frac{\cos 8x}{8}+-\frac{\cos 2x}{2}+C=-\frac{\cos 8x}{8}-\frac{\cos 2x}{2}+C$

vii) $I=\int \cos 5x\cos 3x dx=\frac{1}{2}\int 2\cos 5x\cos 3x dx=\frac{1}{2}\int [\cos(5x+3x)+\cos(5x-3x)]dx$

$=\frac{1}{2}\int (\cos 8x+\cos 2x)dx=\frac{1}{2}\left(\frac{\sin 8x}{8}+\frac{\sin 2x}{2}\right)+C$

viii) $I=\int \sin 5x\sin 3x dx=\frac{1}{2}\int 2\sin 5x\sin 3x dx=\frac{1}{2}\int [\cos(5x-3x)-\cos(5x+3x)]dx$

$=\frac{1}{2}\int [\cos 2x-\cos 8x]dx=\frac{1}{2}\left(\frac{\sin 2x}{2}-\frac{\sin 8x}{8}\right)+C=\frac{\sin 2x}{4}-\frac{\sin 8x}{16}+C$

Type 2

Integral of the functions of the form $\int f(x^n)x^{n-1}dx$

Let $I=\int f(x^n)x^{n-1}dx$

Put $x^n=u$

$nx^{n-1}dx=du \Rightarrow x^{n-1}dx=\frac{1}{n}du$

$\therefore I=\int f(u)du$, can be evaluated.

E.g.: $I=\int x^2\sin(x^3)\, dx$

put $x^3 = u \Rightarrow 3x^2.dx = du \Rightarrow x^2.dx = \frac{1}{3}du$

$$\therefore \mathrm{I} = \int \sin u \,.\frac{1}{3}.du = \frac{1}{3}\int \sin u \,.du = \frac{1}{3}(-\cos u) + C = -\frac{1}{3}\cos(x^3) + C$$

Type 3

Integral of the functions of the form $\int [f(x)]^n . f'(x) dx$

Let $I = \int [f(x)]^n . f'(x) dx$

Put $f(x) = u \;\Rightarrow f'(x) dx = du$

$\therefore\ I = \int u^n\, du$, which can be evaluated.

E.g.: $I = \int \frac{\left(\tan^{-1} x\right)^2 dx}{(1+x^2)} = \int \left(\tan^{-1} x\right)^2 \frac{1}{(1+x^2)} dx$

Put $tan^{-1}x = u \Rightarrow \frac{1}{1+x^2} dx = du$

$\therefore I = \int u^2 du \;= \frac{u^3}{3} + C = \frac{\left(\tan^{-1} x\right)^3}{3} + C$

Corollary of type 3

Let $I = \int [f(x)]^n . f'(x) dx$

Put $n = -1$

Then $I = \int [f(x)]^{-n} . f'(x) dx = \int \frac{1}{f(x)} f'(x) dx$

Put $f(x) = u \;\Rightarrow f'(x) dx = du$

$\therefore\ I = \int \frac{1}{u} du \;\; log|u| + C = log|f(x)| + C$

i.e., if the numerator is the differential coefficient of the denominator, then integral of the function is logarithm of the denominator.

E.g.: i) $I = \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + C$

ii) $I = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\log|\cos x| = \log\left|\frac{1}{\cos x}\right| = \log|\sec x| + C$

$$\left[\because -\log x = \frac{1}{\log x}\right]$$

iii) $I = \int \sec x\,dx = \int \sec x.\frac{\sec x + \tan x}{\sec x + \tan x}dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$

$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}dx = \log\left|\sec x + \tan x\right| + C$

iv) $I = \int \cos ecx\,dx = \int \cos ecx.\frac{\cos ecx - \cot x}{\cos ecx - \cot x}dx = \int \frac{\cos ec^2 x - \cos ec \cot x}{\cos ecx - \cot x}dx$

$= \int \frac{-\cos ecx \cot x - -\cos ec^2 x}{\cos ecx - \cot x}dx = \log\left|\cos ecx - \cot x\right| + C$

Type 4

Integral of the functions of the form $\int \phi[f(x)]f'(x)dx$

Let $I = \int \phi[f(x)]f'(x)dx$

Put $f(x) = u \Rightarrow f'(x)dx = du$

$\therefore\ I = \int \phi(u).du$, can be evaluated.

E.g.:

1. $I = \int \frac{\sin(\log x)}{x}dx = \int \sin(\log x).\frac{1}{x}dx$

   Put $\log x = u \Rightarrow \frac{1}{x}dx = du$

   $\therefore I = \int \sin u\,du\ = \int \sin u\,du = -\cos u + C = -\cos(\log x) + C$

2. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int \sin\sqrt{x}.\frac{1}{2\sqrt{x}}dx$

   Put $\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}}dx = du \Rightarrow \frac{1}{\sqrt{x}}dx = 2du$

   $\therefore\ I = \int \sin u\,2du = 2\int \sin u\,du = 2[-\cos u] + C = -2\cos\sqrt{x} + C$

Type 5

Integral of functions of the form:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}+C\ (or)\ \frac{1}{2a}\ln\frac{x-a}{x+a}+C$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}+C\ (or)\ \frac{1}{2a}\ln\frac{a+x}{a-x}+C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+C\ (or)\ In\left|x+\sqrt{x^2+a^2}\right|+C\ (or)\ \sinh^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C\ (or)\ In\left|x+\sqrt{x^2-a^2}\right|+C\ (or)\ \cosh^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$$

E.g.:

1. $I=\int\frac{dx}{5^2+x^2}=\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$ (1) of Type 5
2. $I=\int\frac{dx}{x^2-3}=\frac{dx}{x^2-\left(\sqrt{3}\right)^2}=\frac{1}{2\times\sqrt{3}}\log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C=\frac{1}{2\sqrt{3}}\log\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C$ (2) of Type 5
3. $I=\int\frac{dx}{3^2-x^2}=\frac{1}{2\times 3}\log\left|\frac{3+x}{3-x}\right|+C=\frac{1}{6}\log\left|\frac{3+x}{3-x}\right|+C$ (3) of Type 5
4. $I=\int\frac{dx}{\sqrt{x^2+5^2}}=\log\left|x+\sqrt{x^2+5^2}\right|+C$ (4) of Type 5
5. $I=\int\frac{dx}{\sqrt{x^2-3}}=\frac{dx}{\sqrt{x^2-\left(\sqrt{3}\right)^2}}=\log\left|x+\sqrt{x^2-\left(\sqrt{3}\right)^2}\right|+C=\log\left|x+\sqrt{x^2-3}\right|+C$ (5) of Type 5
6. $I=\int\frac{dx}{\sqrt{3^2-x^2}}=\sin^{-1}\left(\frac{x}{3}\right)+C$ (6) of Type 5

Method to integral of functions of the form

(i). $\int \frac{dx}{ax^2+bx+c}$ and (ii). $\int \frac{dx}{\sqrt{ax^2+bx+c}}$. where, the denominator is a non-resolvable quadratic polynomial of x.

Then we can write the denominator $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right]$. Then depending on the sign of b^2-4ac, the function will take on the forms as in "Type V'.

E.g.:

1. $I = \int \frac{dx}{2x^2+3x+1}$

Here $2x^2+3x+1 = 2\left[\left(x+\frac{3}{2a}\right)^2 - \frac{3^2-4\times2\times1}{4\times2^2}\right] = 2\left[\left(x+\frac{3}{4}\right)^2 - \frac{9-8}{16}\right] = 2\left[\left(x+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right]$

$$\therefore\ I = 2\int \frac{dx}{\left(x+\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2} = 2\times\frac{1}{2\times\frac{1}{4}}\log\left|\frac{x+\frac{3}{4}-\frac{1}{4}}{x+\frac{3}{4}+\frac{1}{4}}\right| + C = 4\times\log\left|\frac{\frac{4x+3-1}{4}}{\frac{4x+3+1}{4}}\right| + C$$

$$= 4\times\log\left|\frac{4x+2}{4x+4}\right| + C = 4\log\left|\frac{2(2x+1)}{4(x+1)}\right| + C = 4\log\left|\frac{(2x+1)}{2(x+1)}\right| + C$$

2. $I = \int \frac{dx}{\sqrt{x^2+2x+3}}$

$$x^2+2x+3 = 1\left[\left(x+\frac{2}{2\times1}\right)^2 - \frac{2^2-4\times1\times3}{4\times1^2}\right] = (x+1)^2 - \frac{4-12}{4} = (x+1)^2 - \frac{-8}{4} = (x+1)^2+2 = (x+1)^2 + \left(\sqrt{2}\right)^2$$

$$\therefore\ I = \int \frac{dx}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}} = \log\left|x+\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}\right| + C = \log\left|x+\sqrt{x^2+2x+3}\right| + C$$

3. $I = \int \frac{dx}{\sqrt{3-4x-2x^2}}$

Let $I = \int \frac{dx}{\sqrt{3-4x-2x^2}}$

Let $3-4x-2x^2 = -\left(2x^2+4x-3\right) = -2\left\{\left[x+\frac{4}{2\times2}\right]^2 - \frac{4^2-4\times2\times-3}{4\times2^2}\right\}$

$$= -2\left\{(x+1)^2 - \frac{16+24}{16}\right\} = -2\left\{(x+1)^2 - \frac{40}{16}\right\} = -2\left\{(x+1)^2 - \frac{5}{2}\right\}$$

$$= -2\left\{(x+1)^2 - \left(\sqrt{\frac{5}{2}}\right)^2\right\} = 2\left[\left(\sqrt{\frac{5}{2}}\right)^2 - (x+1)^2\right]$$

$$I = \int \frac{dx}{\sqrt{2\left[\left(\sqrt{\frac{5}{2}}\right)^2 - (x+1)^2\right]}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\sqrt{\frac{5}{2}}\right)^2 - (x+1)^2} = \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{\frac{5}{2}}}\log\left(\frac{\sqrt{\frac{5}{2}} - (x+1)}{\sqrt{\frac{5}{2}} + (x+1)}\right) + C$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{2\sqrt{5}}\log\left(\frac{\sqrt{5} - 2(x+1)}{\sqrt{5} + 2(x+1)}\right) + C = \frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5} - 2x - 2}{\sqrt{5} + 2x + 2}\right) + C$$