7 INTEGRALS

INTEGRALS
INDEFINITE INTEGRAL

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int dx = x + C$

$\int \sin x dx = -\cos x + C$

$\int \cos x dx = \sin x + C$

$\int \sec^2 x dx = \tan x + C$

$\int \operatorname{cosec}^2 x dx = -\cot x + C$

$\int \sec x \tan x dx = \sec x + C$

$\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + C$

$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

$\int \frac{1}{1+x^2} dx = -\cot^{-1} x + C$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

$\int \frac{1}{x\sqrt{x^2-1}} dx = -\operatorname{cosec}^{-1} x + C$

$\int e^x dx = e^x + C$

$\int \frac{1}{x} dx = \log|x| + C$

$\int a^x dx = \frac{a^x}{\log a} + C$

INTEGRATION BY SUBSTITUTION

$$I = \int f(x) dx$$

$$\text{put } x = g(t)$$

$$I = \int f(g(t)) g'(t) dx$$

$\int \tan x dx = \log|\sec x| + C$

$\int \cot x dx = \log|\sin x| + C$

$\int \sec x dx = \log|\sec x + \tan x| + C$

$\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

INTEGRATION USING TRIGONOMETRIC IDENTITIES

$$\sin 2x = 2\sin x\cos x$$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$2\sin x\cos y = \sin(x+y) + \sin(x-y)$$

$$2\cos x\sin y = \sin(x+y) - \sin(x-y)$$

$$2\cos x\cos y = \cos(x+y) + \cos(x-y)$$

$$2\sin x\sin y = \cos(x-y) - \cos(x+y)$$

INTEGRALS OF SOME PARTICULAR FUNCTIONS

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2}\log\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

INTEGRATION BY PARTIAL FRACTIONS

$$\frac{px+q}{(x-a)(x-b)}=\frac{A}{(x-a)}+\frac{B}{(x-b)}$$

$$\frac{px+q}{(x-a)^2}=\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$$

$$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}=\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$$

$$\frac{px^2+qx+r}{(x-a)^2(x-b)}=\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$$

$$\frac{px^2+qx+r}{(x-a)\left(x^2+bx+c\right)}=\frac{A}{(x-a)}+\frac{Bx+C}{\left(x^2+bx+c\right)}$$

INTEGRATION BY PARTS

$$\int f(x)g(x)dx=f(x)\int g(x)dx-\int\left[f'(x)\int g(x)dx\right]dx$$

$$e^x\left[f(x)+f'(x)\ \right]=e^xf(x)+C$$

INTEGRALS OF SOME MORE TYPES

$$\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$$

$$\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

DEFINITE INTEGRAL

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

- $$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)....f(a+(n-1)h)\right]$$

$$h=\frac{b-a}{n}\to 0\ \text{as}\ n\to 0$$

SOME PROPERTIES OF DEFINITE INTEGRALS

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\int_a^b f(x)dx = -\int_b^a f(t)dt$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x)$$

$$\int_0^{2a} f(x)dx = 0, \text{ if } f(2a-x) = -f(x)$$

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x) = f(x)$$

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x) = -f(x)$$

MODEL QUESTIONS

Question 01 :

Find $\int\left(2x^2+4\sin x\right)\,dx.$

Solution :

$$I=\int\left(2x^2+4\sin x\right)\,dx.$$
$$=\int 2x^2dx+\int 4\sin x dx+C$$
$$=2\int x^2dx+4\int \sin x dx$$
$$=\underline{\frac{2}{3}x^3-4\cos x}$$

Question 02 :

Find $\int\left(5x^4+4x^3+3x^2+2\right)\,dx.$

Solution :

$$I=\int\left(5x^4+4x^3+3x^2+2\right)\,dx.$$
$$=\int 5x^4dx+\int 4x^3dx+\int 3x^2dx+\int 2dx$$
$$=5\int x^4dx+4\int x^3dx+3\int x^2dx+2\int dx$$
$$=5.\frac{x^5}{5}+4.\frac{x^4}{4}+3.\frac{x^3}{3}+3x$$
$$=\underline{x^5+x^4+x^3+3x+C}$$

Question 03 :

Find $\int\left(2\sin 2x+3\cos 3x+4\sec^2 4x+3\sec 3x\tan 3x\right)\,dx.$

Solution :

$$I=\int\left(2\sin 2x+3\cos 3x+4\sec^2 4x+3\sec 3x\tan 3x\right)\,dx.$$
$$=\int 2\sin 2x dx+\int 3\cos 3x dx+\int 4\sec^2 4x dx+\int 3\sec 3x\tan 3x\,dx.$$
$$=2\int \sin 2x dx+3\int \cos 3x dx+4\int \sec^2 4x dx+3\int \sec 3x\tan 3x\,dx.$$
$$=2\left(\frac{-\cos 2x}{2}\right)+3\frac{\sin 3x}{3}+4\frac{\tan 4x}{4}+3\frac{\sec 3x}{3}$$
$$=\underline{-\cos 2x+\sin 3x+\tan 4x+\sec 3x+C}$$

Question 04 :

Find $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx.$

Solution :

$$I = \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx$$

$$= \int \frac{1}{\cos^2 x}\, dx - \int \frac{1}{\sin^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \operatorname{cosec}^2 x\, dx$$

$$= \tan x - \cot x + C$$

Question 05 :

Find $\int (1-x)\sqrt{x}\, dx.$

Solution :

$$I = \int (1-x)\sqrt{x}\, dx$$

$$= \int \sqrt{x} - x\sqrt{x}\, dx$$

$$= \int x^{1/2} - x^1 x^{1/2}\, dx$$

$$= \int x^{1/2} dx - \int x^{3/2}\, dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{5/2}}{\frac{5}{2}} + C$$

$$= \frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{5} + C$$

Question 06 :

Find $\int \frac{\sin^6 x}{\cos^8 x}\, dx.$

Solution :

$$I = \int \frac{\sin^6 x}{\cos^8 x}\, dx$$

$$I = \int \frac{\sin^6 x}{\cos^6 x} \cdot \frac{1}{\cos^2 x}\, dx$$

$$I = \int \tan^6 x . \sec^2 x\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore\ I = \int t^6 dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C$$

Question 07 :

Find $\int \cos^{-1}(\sin x)\, dx.$

Solution :

$$I=\int \cos^{-1}(\sin x)\ dx$$

$$I=\int \cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)\ dx$$

$$I=\int\left(\frac{\pi}{2}-x\right)dx$$

$$I=\frac{\pi}{2}x-\frac{x^2}{2}+C$$

Question 08 :

Find $\int \frac{1}{\sin^2 x\cos^2 x}\ dx.$

Solution :

$$I=\int \frac{1}{\sin^2 x\cos^2 x}\ dx$$

$$=\int \frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}\cos^2 x}\ dx$$

$$=\int \frac{\sec^2 x}{\tan^2 x\cos^2 x}\ dx$$

$$=\int \frac{1+\tan^2 x}{\tan^2 x}\sec^2 x\ dx$$

Put $\tan x=t$

$$\Rightarrow \sec^2 x dx=dt$$

$$I=\int\left(\frac{1+t^2}{t^2}\right)\ dt$$

$$=\int\left(\frac{1}{t^2}+1\right)\ dt$$

$$=-\frac{1}{t}+t+C$$

$$=\tan x-\cot x+C$$

Question 09 :

Find $\int \frac{2}{1+\cos 2x}dx.$

Solution :

$$I=\int \frac{2}{1+\cos 2x}\ dx$$

$$=\int \frac{2}{2\cos^2 x}\ dx \qquad \because \cos 2x=2\cos^2 x-1$$

$$=\int \sec^2 x\ dx=\tan x+C$$

Question 10 :

Find $\int \frac{x+\cos 6x}{3x^2+\sin 6x}\ dx.$

Solution :

$$I=\int\frac{x+\cos 6x}{3x^2+\sin 6x}\,dx$$

Put $3x^2+\sin 6x=t$

$$\Rightarrow\ 6x+6\cos 6x=dt$$

$$=\frac{1}{6}\int\frac{(6x+6\cos 6x)dx}{3x^2+\sin 6x}$$

$$=\frac{1}{6}\int\frac{dt}{t}=\frac{1}{6}\log|t|+C$$

$$=\frac{1}{6}\log\left|3x^2+\sin 6x\right|+C$$

Question 11 :

Find $\int\frac{2-3\sin x}{\cos^2 x}\,dx.$

Solution :

$$I=\int\frac{2-3\sin x}{\cos^2 x}\,dx$$

$$=\int\frac{2}{\cos^2 x}\,dx-\int\frac{3\sin x}{\cos^2 x}\,dx$$

$$=2\int\sec^2 x dx-3\int\sec x\tan x\,dx=2\tan x-3\sec x+C$$

Question 12 :

Find $\int\frac{x^2}{1+x^3}\,dx.$

Solution :

$$I=\int\frac{x^2}{1+x^3}\,dx$$

Put $1+x^3=t\ \Rightarrow\ x^2dx=\frac{dt}{3}$

$$I=\frac{1}{3}\int\frac{1}{t}\,dt$$

$$=\frac{1}{3}\log t+C$$

$$=\frac{1}{3}\log\left(1+x^3\right)+C$$

Question 13 :

Find $\int\sin x\sin 2x\sin 3x\,dx.$

Solution :

$$I=\int\sin x\sin 2x\sin 3x\,dx$$

$$=\frac{1}{2}\int\sin x\left(2\sin 2x\sin 3x\right)\,dx$$

$$=\frac{1}{2}\int\sin x\left(\cos(2x-3x)-\cos(2x+3x)\right)\,dx$$

$$=\frac{1}{2}\int \sin x\left(\cos x-\cos 5x\right)\,dx$$

$$=\frac{1}{4}\int 2\sin x\cos x\,dx-\frac{1}{4}\int 2\sin x\cos 5x dx$$

$$=\frac{1}{4}\int \sin 2x dx-\frac{1}{4}\int \sin\left(x+5x\right)+\sin\left(x-5x\right)dx$$

$$=\frac{1}{4}\int \sin 2x dx-\frac{1}{4}\int \sin 6x-\sin 4x dx$$

$$=\frac{1}{4}\int \sin 2x dx-\frac{1}{4}\int \sin 6x dx-\frac{1}{4}\int \sin 4x dx$$

$$=-\frac{\cos 2x}{8}+\frac{\cos 6x}{24}-\frac{\cos 4x}{16}+C$$

Question 14 :

Find $\int e^{2x}\sin x\,dx.$

Solution :

$$I=\int e^{2x}\sin x\,dx$$

$$I=\int \sin x e^{2x}\,dx$$

$$I=\sin x\int e^{2x}\,dx-\int \frac{d}{dx}(\sin x)\int e^{2x}dx$$

$$I=\sin x\left[\frac{e^{2x}}{2}\right]-\int \cos x\left[\frac{e^{2x}}{2}\right]dx$$

$$I=\frac{e^{2x}.\sin x}{2}-\frac{1}{2}\int \cos x.e^{2x}dx$$

$$I=\frac{e^{2x}.\sin x}{2}-\frac{1}{2}I_1$$

$$I_1=\int \cos x.e^{2x}dx$$

$$=\cos x\int e^{2x}dx-\int \frac{d}{dx}(\cos x)\int e^{2x}dx$$

$$=\cos x\left[\frac{e^{2x}}{2}\right]+\int \sin x\left[\frac{e^{2x}}{2}\right]dx$$

$$=\frac{e^{2x}.\cos x}{2}+\frac{1}{2}\int \sin x.e^{2x}dx$$

$$=\frac{e^{2x}.\cos x}{2}+\frac{1}{2}I$$

$$I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left(\frac{e^{2x}\cos x}{2}+\frac{1}{2}I\right)$$

$$=\frac{e^{2x}.\sin x}{2}-\frac{e^{2x}.\cos x}{4}-\frac{1}{4}I$$

$$I+\frac{1}{4}I=\frac{e^{2x}.2\sin x}{4}-\frac{e^{2x}.\cos x}{4}$$

$$\frac{5}{4}I=e^{2x}\,\frac{2\sin x-\cos x}{4}$$

$$I=\frac{e^{2x}}{5}\left(2\sin x-\cos x\right)$$

Question 15 :

Find $\int \frac{x^2}{(x^2+4)(x^2+9)}\ dx.$

Solution :

$$I = \int \frac{x^2}{(x^2+4)(x^2+9)}\ dx$$

$$= \frac{1}{2}\int \frac{2x^2}{(x^2+4)(x^2+9)}\ dx = \frac{1}{2}\int \frac{2x^2+9-9+4-4}{(x^2+4)(x^2+9)}\ dx$$

$$= \frac{1}{2}\int \frac{x^2+4+x^2+9-9-4}{(x^2+4)(x^2+9)}\ dx = \frac{1}{2}\int \frac{(x^2+4)+(x^2+9)-13}{(x^2+4)(x^2+9)}\ dx$$

$$= \frac{1}{2}\int \frac{(x^2+4)dx}{(x^2+4)(x^2+9)} + \frac{1}{2}\int \frac{(x^2+9)dx}{(x^2+4)(x^2+9)} - \frac{1}{2}\int \frac{13dx}{(x^2+4)(x^2+9)}$$

$$= \frac{1}{2}\int \frac{dx}{(x^2+9)} + \frac{1}{2}\int \frac{dx}{(x^2+4)} - \frac{13}{2}\int \frac{dx}{(x^2+4)(x^2+9)}$$

$$= \frac{1}{2}.\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{2}.\frac{1}{5}\int\left(\frac{1}{(x^2+4)} - \frac{1}{(x^2+9)}\right)dx$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{20}\tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{30}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= \left(\frac{1}{6}+\frac{13}{30}\right)\tan^{-1}\left(\frac{x}{3}\right) + \left(\frac{1}{4}-\frac{13}{20}\right)\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \left(\frac{5}{30}+\frac{13}{30}\right)\tan^{-1}\left(\frac{x}{3}\right) + \left(\frac{5}{20}-\frac{13}{20}\right)\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \left(\frac{18}{30}\right)\tan^{-1}\left(\frac{x}{3}\right) - \left(\frac{8}{20}\right)\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \left(\frac{3}{5}\right)\tan^{-1}\left(\frac{x}{3}\right) - \left(\frac{2}{5}\right)\tan^{-1}\left(\frac{x}{2}\right) + C$$

Question 16 :

Find $\int (x-3)\sqrt{x^2+3x-18}\ dx.$

Solution :

$$I = \int (x-3)\sqrt{x^2+3x-18}\ dx$$

Here $\quad x-3 = A\frac{d}{dx}\left(x^2+3x-18\right)+B$

$$x-3 = A(2x+3)+B$$

$$x-3 = 2Ax+3A+B$$

On equating the coefficients of x and constant term

$$2A = 1 \quad \text{and} \quad 3A+B=-3$$

$$A = \frac{1}{2} \quad \text{and} \quad 3\left(\frac{1}{2}\right)+B=-3 \quad \Rightarrow \quad B = -\frac{9}{2}$$

$$I = \int \left\{\frac{1}{2}(2x+3)-\frac{9}{2}\right\}\sqrt{x^2+3x-18}\ dx$$

$$I = \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\ dx - \frac{9}{2}\int \sqrt{x^2+3x-18}\ dx$$

$$I = \frac{1}{2}I_1 - \frac{9}{2}I_2$$

$$I_1 = \int (2x+3)\sqrt{x^2+3x-18}\ dx$$

Put $x^2+3x-18=t \ \Rightarrow \ (2x+3)dx = dt$

$$I_1 = \int t^{1/2}\ dt = \frac{2}{3}t^{3/2}+C_1 = \frac{2}{3}\left(x^2+3x-18\right)^{3/2}+C_1$$

$$I_2 = \int \sqrt{x^2+3x-18}\ dx$$

$$= \int \sqrt{x^2+3x+\frac{9}{4}-18-\frac{9}{4}}\ dx$$

$$= \int \sqrt{\left(x^2+3x+\frac{9}{4}\right)-18-\frac{9}{4}}\ dx$$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2-\frac{81}{4}}\ dx = \int \sqrt{\left(x^2+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\ dx$$

$$=\frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x-18}-\frac{81}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|+C_2$$

$$=\frac{2x+3}{4}\sqrt{x^2+3x-18}-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|+C_2$$

$$\therefore \quad I=\frac{1}{2}I_1-\frac{9}{2}I_2$$

$$=\frac{1}{2}\left(\frac{2}{3}\left(x^2+3x-18\right)^{3/2}+C_1\right)$$

$$-\frac{9}{2}\left(\frac{2x+3}{4}\sqrt{x^2+3x-18}-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|+C_2\right)$$

$$=\frac{1}{3}\left(x^2+3x-18\right)^{3/2}+\frac{C_1}{2}-\frac{9(2x+3)}{8}\sqrt{x^2+3x-18}$$

$$+\frac{729}{16}\log\left|\left(\frac{2x+3}{2}\right)+\sqrt{x^2+3x-18}\right|-\frac{9}{2}C_2$$

$$=\frac{1}{3}\left(x^2+3x-18\right)^{3/2}-\frac{9(2x+3)}{8}\sqrt{x^2+3x-18}$$

$$+\frac{729}{16}\log\left|\left(\frac{2x+3}{2}\right)+\sqrt{x^2+3x-18}\right|+C$$

Question 17 :

Find $\int\frac{x^2+1}{\left(x^2+4\right)\left(x^2+25\right)}\,dx.$

Solution :

$$I=\int\frac{x^2+1}{\left(x^2+4\right)\left(x^2+25\right)}\,dx$$

$$\frac{x^2+1}{\left(x^2+4\right)\left(x^2+25\right)}=\frac{A}{x^2+4}+\frac{B}{x^2+25}$$

$$x^2+1=A\left(x^2+25\right)+B\left(x^2+4\right)$$

$$x^2+1=Ax^2+25A+Bx^2+4B$$

On equating the coefficients of x and constant term

$$A+B=1 \quad \text{and} \quad 25A+4B=1$$
$$B=1-A \quad \text{and} \quad 25A+4(1-A)=1$$
$$B=1-A \quad \text{and} \quad 21A=-3$$
$$B=1+\frac{1}{7}=\frac{8}{7} \quad \text{and} \quad A=-\frac{1}{7}$$

$$I=-\frac{1}{7}\int\frac{1}{x^2+4}dx+\frac{8}{7}\int\frac{1}{x^2+25}dx$$
$$=-\frac{1}{7}.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{7}.\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$
$$=-\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right)+C$$

Question 18 :

Find $\int\frac{3x+1}{(x+1)^2(x+3)}dx.$

Solution :

$$I=\int\frac{3x+1}{(x+1)^2(x+3)}dx$$
$$\frac{3x+1}{(x+1)^2(x+3)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$$
$$3x+1=A(x+1)(x+3)+B(x+3)+C(x+1)^2$$
$$3x+1=Ax^2+4Ax+3A+Bx+3B+Cx^2+2Cx+C$$

On equating the coefficients of x^2, x and constant term

$$A+C=0 \quad , \quad 4A+B+2C=3 \quad , \quad 3A+3B+C=1$$
$$C=-A \quad , \quad 4A+B-2A=3 \quad , \quad 3A+3B-A=1$$
$$C=-A \quad , \quad B=3-2A \quad , \quad 2A+3(3-2A)=1$$
$$C=-A \quad , \quad B=3-2A \quad , \quad -4A=-8 \quad , \quad A=2$$
$$C=-2 \quad , \quad B=-1 \quad , \quad A=2$$

$$I=\int\frac{2}{x+1}dx-\int\frac{1}{(x+1)^2}dx-\int\frac{2}{x+3}dx$$

$$= 2\int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx - 2\int \frac{1}{x+3}dx$$

$$= 2\int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx - 2\int \frac{1}{x+3}dx$$

$$= 2\log|x+1| - \frac{(x+1)^{-1}}{-1} - 2\log|x+3| + C$$

$$= 2\log|x+1| + \frac{1}{x+1} - 2\log|x+3| + C$$

$$= 2\log\left|\frac{x+1}{x+3}\right| + \frac{1}{x+1} + C$$

Question 19 :

Find $\int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\ dx$

Solution :

$$I = \int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\ dx \qquad ...(1)$$

$$I = \int_0^{\pi/2} \frac{\tan^7\left(\frac{\pi}{2} - x\right)}{\cot^7\left(\frac{\pi}{2} - x\right) + \tan^7\left(\frac{\pi}{2} - x\right)}\ dx$$

$$I = \int_0^{\pi/2} \frac{\cot^7 x}{\tan^7 x + \cot^7 x}\ dx \qquad ...(2)$$

$$2I = \int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\ dx + \int_0^{\pi/2} \frac{\cot^7 x}{\tan^7 x + \cot^7 x}\ dx$$

$$2I = \int_0^{\pi/2} \frac{\tan^7 x + \cot^7 x}{\cot^7 x + \tan^7 x}\ dx = \int_0^{\pi/2} 1dx = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

Question 20 :

Find $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}\ dx$

Solution :

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\ dx \qquad ...(1)$$

$$I = \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{(10-x)}+\sqrt{10-(10-x)}}\ dx$$

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}\ dx \qquad ...(2)$$

$$2I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\ dx + \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}\ dx$$

$$2I = \int_2^8 \frac{\sqrt{x}+\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\ dx = \int_2^8 1dx = [x]_2^8 = 8-2=6$$

$$I = 3$$

Question 21 :

Find $\int_0^{\pi/4} \sqrt{1+\sin 2x}\ dx$

Solution :

$$I = \int_0^{\pi/4} \sqrt{1+\sin 2x}\ dx$$

$$I = \int_0^{\pi/4} \sqrt{(\sin x+\cos x)^2}\ dx$$

$$I = \int_0^{\pi/4} \sin x+\cos x\ dx = [\sin x-\cos x]_0^{\pi/4} = 1$$

Question 22 :

Find $\int_2^3 x^2\ dx$

Solution :

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f\{a+(n-1)h\}\right]$$

$$h = \frac{b-a}{n} = \frac{3-2}{n} = \frac{1}{n}$$

$$\int_2^3 x^2 dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+...+f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[2^2 + \left(2+\frac{1}{n}\right)^2 + \ldots + \left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[2^2 + \left(2^2 + \left(\frac{1}{n}\right)^2 + 2.2.\frac{1}{n}\right) + \ldots + \left(2^2 + \frac{(n-1)^2}{n^2} + 2.2\frac{(n-1)}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\left(2^2 + .. + 2^2\right) + \left(\left(\frac{1}{n}\right)^2 + .. + \left(\frac{n-1}{n}\right)^2\right) + .. + 2.2\left(\frac{1}{n} + .. + \frac{n-1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{1}{n^2}\left\{1^2 + 2^2 + \ldots + (n-1)^2\right\} + \ldots + \frac{4}{n}\left\{1 + 2 + \ldots + (n-1)\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \ldots + \frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[4n + n\left\{\frac{(n-1)(2n-1)}{6}\right\} + \ldots + \left\{\frac{4(n-1)}{2}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \ldots + \frac{4n-4}{2}\right]$$

$$= \lim_{n\to\infty} \left[4 + \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \ldots + 2 - \frac{2}{n}\right]$$

$$= \lim_{n\to\infty} \left[4 + \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \ldots + 2 - \frac{2}{n}\right]$$

$$= 4 + \frac{2}{6} + 2 = \frac{19}{3}$$

Question 23 :

Find $\int \frac{x^2}{x^4 + x^2 + 1}\, dx.$

Solution :

$$I = \int \frac{x^2}{x^4 + x^2 + 1}\, dx.$$

$$= \frac{1}{2}\int \frac{2x^2}{x^4 + x^2 + 1}\, dx = \frac{1}{2}\int \frac{x^2 + 1 + x^2 - 1}{x^4 + x^2 + 1}\, dx$$

$$= \frac{1}{2}\int \frac{x^2 + 1}{x^4 + x^2 + 1}\, dx + \frac{1}{2}\int \frac{x^2 - 1}{x^4 + x^2 + 1}\, dx = \frac{1}{2} I_1 + \frac{1}{2} I_2$$

$$I_1=\int\frac{\frac{x^2+1}{x^2}}{\frac{x^4+x^2+1}{x^2}}\,dx=\int\frac{\left(1+\frac{1}{x^2}\right)}{x^2+1+\frac{1}{x^2}}\,dx$$

$$I_1=\int\frac{\left(1+\frac{1}{x^2}\right)}{x^2-2+\frac{1}{x^2}+1+2}\,dx=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+\left(\sqrt{3}\right)^2}\,dx$$

Put $x-\frac{1}{x}=t \quad\Rightarrow\quad \left(1+\frac{1}{x^2}\right)dx=dt$

$$I_1=\int\frac{dt}{t^2+\left(\sqrt{3}\right)^2}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}$$

$$I_1=\frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{3}}=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)$$

$$I_2=\int\frac{\frac{x^2-1}{x^2}}{\frac{x^4+x^2+1}{x^2}}\,dx=\int\frac{\left(1-\frac{1}{x^2}\right)}{x^2+1+\frac{1}{x^2}}\,dx$$

$$I_2=\int\frac{\left(1-\frac{1}{x^2}\right)}{x^2+2+\frac{1}{x^2}+1-2}\,dx=\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-(1)^2}\,dx$$

Put $x+\frac{1}{x}=t \quad\Rightarrow\quad \left(1-\frac{1}{x^2}\right)dx=dt$

$$I_2=\int\frac{dt}{t^2-(1)^2}=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|=\frac{1}{2}\log\left|\frac{\left(x+\frac{1}{x}\right)-1}{\left(x+\frac{1}{x}\right)+1}\right|=\frac{1}{2}\log\left|\frac{x^2+1-x}{x^2+1+x}\right|$$

$$I=\frac{1}{2}I_1+\frac{1}{2}I_2=\underline{\underline{\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)+\frac{1}{4}\log\left|\frac{x^2+1-x}{x^2+1+x}\right|}}$$

HOME WORK QUESTIONS

Question :(Imp2017)

Find the following

(a) $\int \frac{4x-10}{\sqrt{(x-2)(x-3)}}\,dx.$ (b) $\int \frac{1}{\sqrt{(x^2+1)(x^2+4)}}\,dx.$

Answer :

(a) $4\sqrt{x^2-5x+6}$ (b) $\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$

Question :(Imp2017)

Evaluate

$\int_0^{\pi/4} \log(1+\tan x)\,dx$ or $\int_0^1 e^x dx$ as the limit of a sum.

Answer :

$\frac{\pi}{8}\log 2$, $e-1$

Question :(March2017)

Find the following

(a) $\int \frac{1}{x(x^7+1)}\,dx.$ (b) $\int_1^4 |x-2|\,dx$

Answer :

(a) $\frac{1}{7}\left[\log x^7-\log(1+x^7)\right]+C$ (b) $\frac{5}{2}$

Question :(March2017)

Evaluate

$\int_0^{\pi/2} \log(\sin x)\,dx$ or $\int_0^4 x^2 dx$ as the limit of a sum.

Answer :

$-\frac{\pi}{2}\log 2$, $\frac{64}{3}$

Question : (Imp2016)

Find the following

(a) $\int \cot x \log(\sin x)\, dx$ (b) $\int \frac{1}{x^2+2x+2}\, dx$ (c) $\int xe^{9x} dx$

Answer :

(a) $\frac{(\log \sin x)^2}{2} + C$ (b) $\tan^{-1}(x+1) + C$ (c) $\frac{e^{9x}}{81}(9x-1) + C$

Question : (March2016)

Find the following

(a) Prove that $\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

(b) Find $\int \frac{1}{\sqrt{2x-x^2}}\, dx.$ (c) Find $\int x \cos x\, dx.$

Answer :

(b) $\sin^{-1}\frac{x}{a} + C$ (c) $x \sin x + \cos x + C$

Question : (March2016)

Evaluate

$\int_0^{\pi} \log(1+\cos x) dx.$ or $\int_0^5 (x+1) dx.$ as the limit of a sum

Answer :

$-\pi \log 2$, $\frac{35}{2}$

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