INDEFINITE INTEGRALS - PART 2

Method to integrate functions of the form $\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Here we take the $Nr. = A \times \frac{d}{dx}(Dr.) + B$, where A and B are constants. Find the values of A and B, **then** it becomes any one of the forms of $\int \frac{1}{ax^2+bx+c}dx$ **or** $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$ and hence can be integrated.

E.g.: $I = \int \frac{2x\ dx}{x^2+2x+2}$

$2x = A\frac{d}{dx}(x^2+2x+2)+B$

$2x = A(2x+2)+B$

Taking the coe. of x: $2 = 2A \Rightarrow 2A = 2 \Rightarrow A = 1$

Taking the constant terms: $0 = 2A + B \Rightarrow 2\times 1 + B = 0 \Rightarrow B + 2 = 0 \Rightarrow B = -2$

$\therefore 2x = 1(2x+2)+-2$

$$\therefore \int \frac{2x}{x^2+2x+2}\,dx = \int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{2}{x^2+2x+2}\,dx$$

$$= \log\left|x^2+2x+2\right| - 2\int \frac{dx}{x^2+2x+2} = \log\left|x^2+2x+2\right| - 2I_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $I_1 = \int \frac{dx}{x^2+2x+2}$

$$x^2+2x+2 = \left[\left(x+\frac{2}{2\times 1}\right)^2 - \frac{4-4\times 1\times 2}{4\times 1^2}\right] = \left[(x+1)^2 - \frac{-4}{4}\right] = (x+1)^2 + 1^2$$

$$\therefore I_1 = \int \frac{dx}{(x+1)^2+1^2} = \tfrac{1}{1}.\tan^{-1}\left(\frac{x+1}{1}\right) + C = \tan^{-1}(x+1) + C$$

In (1) we have $I = \log\left|x^2+2x+2\right| - 2\tan^{-1}(x+1) + C$

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Method to integrate functions of the form $\int \frac{P(x)}{ax^2+bx+c}dx$, where P(x) is a polynomial of degree 2 or more.

Here divide the numerator by the denominator until the degree of the numerator becomes less than that of the denominator and then the function is rewritten in the form $quotient + \frac{remainder}{divisor}$ and then integrate.

E.g.: $I = \int \frac{x^2}{2x^2+3x+4}dx$

$$x^2 \overline{)2x^2+3x+4}(2$$

$$\underline{(-)2x^2}$$

$$+3x+4$$

$$\therefore\ I = \int\left[2+\frac{3x+4}{2x^2+3x+4}\right]dx = 2x + \int\frac{3x+4}{2x^2+3x+4}dx = 2x + I_1 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Let $I_1 = \int \frac{3x+4}{2x^2+3x+4}dx$

Now $\quad 3x+4 = A\frac{d}{dx}(2x^2+3x+4)+B$

$3x+4 = A(4x+3)+B$

Taking the coe. of x: $\ 3 = 4A \Rightarrow 4A = 3 \Rightarrow A = \frac{3}{4}$

Taking the constant terms: $4 = 3A+B \Rightarrow 3\times\frac{3}{4}+B = 4 \Rightarrow \frac{9}{4}+B = 4 \Rightarrow 9+4B = 16$

$4B = 16-9 \Rightarrow 4B = 7 \Rightarrow B = \frac{7}{4}$

$$\therefore\ I_1 = \int\frac{3}{4}\frac{4x+3}{2x^2+3x+4}dx + \frac{7}{4}\int\frac{dx}{2x^2+3x+4} = \frac{3}{4}\log\left|2x^2+3x+4\right| + \frac{7}{4}I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now let $I_2 = \int\frac{dx}{2x^2+3x+4}$

$$2x^2+3x+4 = 2\left[\left(x+\frac{3}{2\times 2}\right)^2 - \frac{3^2-4\times 2\times 4}{4\times 2^2}\right] = 2\left[\left(x+\frac{3}{4}\right)^2 - \frac{9-32}{16}\right] = 2\left[\left(x+\frac{3}{4}\right)^2 - \frac{-23}{16}\right]$$

$$= 2\left[\left(x+\frac{3}{4}\right)^2 - \frac{-23}{16}\right] = 2\left[\left(x+\frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2\right]$$

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$$\therefore\ I_2 = \int \frac{dx}{2\left[\left(x+\frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2\right]} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2} = \frac{1}{2}\times\frac{1}{\frac{\sqrt{23}}{4}}\tan^{-1}\left(\frac{x+\frac{3}{4}}{\frac{\sqrt{23}}{4}}\right)+C$$

$$= \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{\frac{4x+3}{4}}{\frac{\sqrt{23}}{4}}\right)+C = \frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+C$$

In (2) we have

$$I_1 = \frac{3}{4}\log\left|2x^2+3x+4\right| + \frac{7}{4}\frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+C$$

$$= \frac{3}{4}\log\left|2x^2+3x+4\right| + \frac{7}{2\sqrt{23}}\tan^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+C$$

Substituting in (1) we have,

$$\therefore\ I = 2x + \frac{3}{4}\log\left|2x^2+3x+4\right| + \frac{7}{2\sqrt{23}}\tan^{-1}\left(\frac{4x+3}{\sqrt{23}}\right)+C$$

Integration by parts

If u and v are any two functions of x then $\int uv\,dx = u\int v\,dx - \int\left[\frac{d}{dx}(u).\int v\,dx\right]dx$

i.e., integral of the product of two functions $= 1^{st}$ *function* $\times$ *integral of the* 2^{nd} $-$
integral of (derivative of the first function $\times$ *integral of the* 2^{nd} *function)*

- If both functions are integrable, take that function as the first function, which can be finished by repeated differentiation and the other function as the second.
- If the integrand contains one unintegrable function (logarithmic, inverse t-function, etc.), take that function as the first function.
- If the integrand contains only one unintegrable function, then take the unintegrable function as the first function and 1 as the second function.
- If the integrand contains both functions integrable and none can be finished by repeated differentiation, then take any one as the first function and other as the second and repeat the rule of integration by parts.

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- If the integrand is the product of two functions, then their order is determined by the word ILATE.

  1. **I**nverse circular functions
  2. **L**ogarithmic functions
  3. **A**lgebraic function
  4. **T**rigonometric function
  5. **E**xponential function.

E.g.: .

1) $I \;=\int \log x\; dx=\int \log x.1dx = \log x \int 1dx-\int\left[\frac{d}{dx}(\log x).\int 1dx\right]dx$

$=\log x.x-\int \frac{1}{x}.x.\;dx = x\log x-\int 1dx = x\log x - x = x(\log x-1)+C$

2) $I \;=\int e^x \cos x\,dx \;=e^x\int \cos x\,dx-\int\left[\frac{d}{dx}\left(e^x\right).\int \cos x\,dx\right]dx$

$=e^x.\sin x-\int e^x.\sin x\;dx = e^x\sin x-\left[e^x.\int \sin x\,dx-\int\left\{\frac{d}{dx}\left(e^x\right).\int \sin x\,dx\right\}dx\right]$

$=e^x.\sin x-\left[e^x.-\cos x-\int e^x.-\cos x\,dx\right]=e^x\sin x+e^x\cos x-\int e^x\cos x\,dx$

$I \;=e^x\sin x+e^x\cos x-\mathrm{I} \qquad \Rightarrow \;\mathrm{I}+\mathrm{I} \;=e^x\sin x+e^x\cos x+C$

$\Rightarrow 2I \;=e^x\sin x+e^x\cos x+C \qquad \Rightarrow \therefore\;\mathrm{I}\;=\frac{1}{2}e^x(\sin x+\cos x)+C$

3) $I \;=\int\frac{\log x}{x^2}\,dx \;=\int \log x.x^{-2}dx \;=\log x.\int x^{-2}dx-\int[\frac{d}{dx}(\log x).\int x^{-2}dx]\;dx$

$=\log x.\frac{x^{-1}}{-1}-\int\frac{1}{x}.\frac{x^{-1}}{-1}\;dx=\log x.\frac{-1}{x}+\int\frac{1}{x}.\frac{1}{x}\;dx=-\frac{\log x}{x}+\int\frac{1}{x^2}.\;dx$

$=-\frac{\log x}{x}-\frac{1}{x}+C=-\frac{1}{x}(\log x+1)+C$

Integrals of the form $\int e^{ax}\sin bx\,dx$ (or) $\int e^{ax}\cos bx\,dx$

i) Let $I \;=\int e^{ax}\sin bx\,dx$

$=e^{ax}\int \sin bx\,dx-\int\left\{\frac{d}{dx}\left(e^{ax}\right)\int \sin bx\,dx\right\}dx$

$=e^{ax}\left(\frac{-\cos bx}{b}\right)-\int\left\{ae^{ax}\left(\frac{-\cos bx}{b}\right)\right\}dx=-\left(\frac{e^{ax}\cos bx}{b}\right)+\frac{a}{b}\int e^{ax}\cos bx\,dx$

$=-\left(\frac{e^{ax}\cos bx}{b}\right)+\frac{a}{b}\left[e^{ax}\int \cos bx\,dx-\int\left\{\frac{d}{dx}\left(e^{ax}\right)\int \cos bx\,dx\right\}dx\right]$

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$$
\begin{aligned}
&= -\left(\frac{e^{ax}\cos bx}{b}\right) + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int\left\{ae^{ax}.\frac{\sin bx}{b}\right\}dx\right] \\
&= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}e^{ax}\frac{\sin bx}{b} - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx \\
&= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}e^{ax}\frac{\sin bx}{b} - \frac{a^2}{b^2}I
\end{aligned}
$$

$$
\begin{aligned}
I + \frac{a^2}{b^2}I &= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx \\
b^2 I + a^2 I &= -be^{ax}\cos bx + ae^{ax}\sin bx \\
\left(b^2 + a^2\right)I &= e^{ax}\left(-b\cos bx + a\sin bx\right) \\
I &= \frac{1}{\left(a^2 + b^2\right)}e^{ax}\left(a\sin bx - b\cos bx\right) + C
\end{aligned}
$$

ii) Let $I = \int e^{ax}\cos bx\,dx$

$$
\begin{aligned}
&= e^{ax}\int\cos bx\,dx - \int\left\{\frac{d}{dx}\left(e^{ax}\right)\int\cos bx\,dx\right\}dx \\
&= e^{ax}\left(\frac{\sin bx}{b}\right) - \int\left\{ae^{ax}\left(\frac{\sin bx}{b}\right)\right\}dx = \left(\frac{e^{ax}\sin bx}{b}\right) - \frac{a}{b}\int e^{ax}\sin bx\,dx \\
&= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[e^{ax}\int\sin bx\,dx - \int\left\{\frac{d}{dx}\left(e^{ax}\right)\int\sin bx\,dx\right\}dx\right] \\
&= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[e^{ax}\left(\frac{-\cos bx}{b}\right) - \int\left\{ae^{ax}.\frac{-\cos bx}{b}\right\}dx\right] \\
&= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx \\
&= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I
\end{aligned}
$$

$$
\begin{aligned}
I + \frac{a^2}{b^2}I &= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx \\
\frac{b^2}{a^2 + b^2}I &= be^{ax}\sin bx + ae^{ax}\cos bx \\
\therefore I &= \left(\frac{a^2 + b^2}{b^2}\right)e^{ax}\left(b\sin bx + a\cos bx\right) + C
\end{aligned}
$$

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Integration by partial fractions

Partial fraction is the process of splitting off a single fraction into 2 or more simple fractions.

- The degree of Nr. Is less that of the Dr.

- If the Dr. is a non-repeated linear factors of the form

$$\frac{px+q}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)......(a_nx+b_n)^n}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+.........+\frac{A_n}{(a_nx+b_n)}$$

E.g.: i) $\frac{x}{(x-1)\,(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

- If the Dr. is a repeated linear factors of the form

$$\frac{px+q}{(ax+b)^r}=\frac{A_1}{(ax+b)^1}+\frac{A_2}{(ax+b)^2}+.........+\frac{A_n}{(ax+b)^n}$$

E.g.: i) $\frac{2x-1}{(x+1)\,(x-2)^2}=\frac{A}{(x+1)}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$

ii) $\frac{x+3}{x(x+1)^3}=\frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$

- If the Dr. is a non-repeated, non-resolvable quadratic polynomial of the form

$$\frac{px+q}{ax^2+bx+c}=\frac{Ax+B}{ax^2+bx+c}$$

E.g.: i) $\frac{2x+3}{(x+1)\,(x^2+2)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x+2)^2}$

ii) $\frac{1}{x^3+1}=\frac{1}{(x+1)(x^2-x+1)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2-x+1)}$

Problems

1. $I=\int\frac{x\,dx}{(x-1)\,(x-2)}$

Let $\frac{x}{(x-1)\,(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}$

$x=A(x-2)+B(x-1)$

$put\;\; x=2$

$2=A(0)+B(2-1) \qquad \Rightarrow\; 2=B\Rightarrow B=2$

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$$put\ x=1 \Rightarrow 1 = A(1-2)+B(0) \Rightarrow 1=-A \Rightarrow A=-1$$

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1}+\frac{2}{x-2}$$

$$\int \frac{x}{(x-1)(x-2)}dx \quad = -\int\frac{1}{x-1}dx + 2\int\frac{dx}{x-2} = -\log|x-1| + 2\log|x-2| = \log\left|(x-2)^2\right| - \log|x-1|$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

2. I $= \int \frac{x^2}{(x^2+2)(x^2-1)}dx$

Let $\frac{x^2}{(x^2+2)(x^2-1)} = \frac{u}{(u+2)(u-1)}$ $\qquad put\ x^2 = u$

$$\frac{u}{(u+2)(u-1)} = \frac{A}{u+2}+\frac{B}{u-1}$$

$$u = A(u-1)+B(u+2)$$

$$put\ u=1: \qquad 1 = A(1-1)+B(1+2) \Rightarrow 1 = 3B \Rightarrow B=\frac{1}{3}$$

$$put\ u=-2: \qquad -2 = A(-2-1)+B(-2+2) \Rightarrow -2=-3A \Rightarrow A = \frac{-2}{-3}=\frac{2}{3}$$

$$\therefore \frac{u}{(u+2)(u-1)} = \frac{\frac{-2}{3}}{u+2}+\frac{\frac{1}{3}}{u-1}$$

$$\therefore \int\frac{u}{(u+2)(u-1)}dx = \frac{-2}{3}\int\frac{1}{u+2}du + \frac{1}{3}\int\frac{1}{u-1}du = \frac{-2}{3}\log|u+2| + \frac{1}{3}\log|u-1| + C$$

$$= \frac{1}{3}\left[\log|u-1| - 2\log|u+2|\right] + C = \frac{1}{3}\left[\log|u-1| - \log\left|(u+2)^2\right|\right] + C = \frac{1}{3}\log\left|\frac{u-1}{(u+2)^2}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{x^2-1}{(x^2+2)^2}\right| + C \qquad \left[replacing\ u\ by\ x^2\right]$$

Integral of functions of the form *(only for ISC students)*

i. $\int\frac{dx}{a+b\sin x}$ ii. $ii.\ \int\frac{dx}{a+b\cos x}$ $iii.\int\frac{dx}{a+b\sin x + c\cos x}dx.$

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Here put $\tan\frac{x}{2} = t \Rightarrow \sec^2\left(\frac{x}{2}\right).\frac{1}{2} = \frac{dt}{dx} \Rightarrow dx = \frac{2\,dt}{\sec^2\left(\frac{x}{2}\right)} \Rightarrow dx = \frac{2\,dt}{1+\tan^2\left(\frac{x}{2}\right)} \Rightarrow dx = \frac{2dt}{1+t^2}$

and $\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$.

Now the given integral is any one of the previous forms and can be evaluated.

Eg:- $\int \frac{dx}{5+4\sin x}$, $\int \frac{dx}{3+2\cos x}$, $\int \frac{dx}{1+2\sin x+\cos x}$, etc

Evaluate:

1. $I = \int \frac{dx}{41+9\cos x}$

Put $\tan\frac{x}{2} = t$; $dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$\therefore I = \int \frac{\frac{2\,dt}{1+t^2}}{41+9.\frac{1-t^2}{1+t^2}} = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{41(1+t^2)+9(1-t^2)}{1+t^2}} = \int \frac{2\,dt}{41+41t^2+9-9t^2} = \int \frac{2\,dt}{32t^2+50}$$

$$= \int \frac{2dt}{32\left(t^2+\frac{50}{32}\right)} = \frac{1}{16}\int \frac{dt}{t^2+\left(\frac{5}{4}\right)^2} = \frac{1}{16}\times\frac{1}{\frac{5}{4}}\tan^{-1}\left(\frac{t}{\frac{5}{4}}\right)+C = \frac{1}{20}\tan^{-1}\left(\frac{4t}{5}\right)+C$$

(OR)

$I = \int \frac{dx}{41+9\cos x}$

$put\ \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\therefore I = \int \frac{dx}{41+9.\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} = \int \frac{dx}{\frac{41\left(1+\tan^2\frac{x}{2}\right)+9\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}$$

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$$=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{41\left(1+\tan^2\frac{x}{2}\right)+9\left(1-\tan^2\frac{x}{2}\right)}=\int\frac{\sec^2\frac{x}{2}}{41+41\tan^2\frac{x}{2}+9-9\tan^2\frac{x}{2}}dx=\int\frac{\sec^2\frac{x}{2}}{32\tan^2\frac{x}{2}+50}dx$$

$$put\ \tan\frac{x}{2}=t\Rightarrow\sec^2\left(\frac{x}{2}\right)\times\frac{1}{2}dx=dt\Rightarrow\sec^2\left(\frac{x}{2}\right)dx=2dt$$

$$=\int\frac{2dt}{32t^2+50}=\frac{2}{32}\int\frac{dt}{t^2+\frac{50}{32}}=\frac{1}{16}\int\frac{dt}{t^2+\left(\frac{5}{4}\right)^2}=\frac{1}{16}\times\frac{1}{\frac{5}{4}}\tan^{-1}\left(\frac{t}{\frac{5}{4}}\right)+C=\frac{1}{20}\tan^{-1}\left(\frac{4t}{5}\right)+C$$

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2. $I=\int\frac{dx}{4+5\sin x}$

Put $\tan\frac{x}{2}=t$; $dx=\frac{2dt}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$

$$=\int\frac{\frac{2dt}{1+t^2}}{4+5.\frac{2t}{1+t^2}}=\int\frac{\frac{2dt}{1+t^2}}{\frac{4\left(1+t^2\right)+10t}{1+t^2}}=\int\frac{2dt}{4+4t^2+10t}=\int\frac{2dt}{4t^2+10t+4}$$

$$=\int\frac{dt}{2t^2+5t+2}$$

$2t^2+5t+2 = 2\left[(t+\frac{5}{4})^2-\frac{25-16}{16}\right]$

$= 2\left[(t+\frac{5}{4})^2-(\frac{3}{4})^2\right]$

$$=\frac{1}{2}\int\frac{dt}{[(t+\frac{5}{4})^2-(\frac{3}{4})^2]}$$

$$=\frac{1}{2}\int\frac{dt}{[(t+\frac{5}{4})^2-(\frac{3}{4})^2]}=\frac{1}{2}\,\frac{1}{2\times\frac{3}{4}}\log\left|\frac{t+\frac{5}{4}-\frac{3}{4}}{t+\frac{5}{4}+\frac{3}{4}}\right|+C$$

$$=\frac{1}{3}\log\left|\frac{4t+5-3}{4t+5+3}\right|+C=\frac{1}{3}\log\left|\frac{4t+2}{4t+8}\right|+C=\frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}+4}\right|+C$$

3. $I=\int\frac{dx}{1+\sin x+\cos x}$

Put $\tan\frac{x}{2}=t$; $dx=\frac{2dt}{1+t^2}$ $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

$$=\int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{2dt}{1+t^2+2t+1-t^2}=\int\frac{2dt}{2+2t}=\int\frac{dt}{1+t}=\log|1+t|+C$$

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$$= \log\left|1+\tan\frac{x}{2}\right| + C$$

Integral of the function of the form:

$$I = \int e^x [f(x) + f'(x)]dx$$
$$= \int f(x)e^x dx + \int f'(x)e^x dx$$

To take the first function check which function is the derivative of the other function. Here $f'(x)$ is the derivative of $f(x)$. So $\int f(x)e^x dx$ is I_1.

$$\therefore I = I_1 + \int f'(x)e^x dx \;\ldots\ldots\ldots\ldots\ldots(1)$$

Let $I_1 = \int f(x)e^x dx$

$$= f(x)\int e^x dx - \int \frac{d}{dx}[f(x)]\int e^x dx = e^x.f(x) - \int f'(x)e^x dx$$

In (1) we have

$$\therefore I = e^x.f(x) - \int f'(x)e^x dx + \int f'(x)e^x dx = e^x f(x)$$

i.e., $\int e^x [f(x) + f'(x)]dx = e^x.f(x) + C$

E.g.:

1. $I = \int e^x(\sin x + \cos x)dx = \int e^x \sin x dx + \int e^x \cos x dx = \int \sin x.e^x dx + \int e^x \cos x dx$

$$= \sin x \int e^x dx - \int \frac{d}{dx}(\sin x)\int e^x dx + \int e^x \cos x dx = \sin x e^x - \int \cos x\, e^x dx + \int e^x \cos x dx$$
$$= \sin x e^x - \int e^x \cos x\, dx + \int e^x \cos x dx = e^x \sin x + C$$

2. $I = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx = \int \frac{1}{x}e^x dx - \int e^x.\frac{1}{x^2}dx = \frac{1}{x}\int e^x dx - \int \frac{d}{dx}\left(\frac{1}{x}\right)\int e^x dx - \int e^x.\frac{1}{x^2}dx$

$$= \frac{1}{x}e^x - \int -\frac{1}{x^2}e^x dx - \int e^x.\frac{1}{x^2}dx = \frac{1}{x}e^x + \int e^x \frac{1}{x^2}dx - \int e^x \frac{1}{x^2}dx = \frac{1}{x}e^x + C$$

Integral of functions of the form *(only for ISC students)*

$$I = \int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$$

Here we take the $Nr. = A \times Dr. + B \times \frac{d}{dx}(Dr.)$

$$a\sin x + b\cos x = A \times (c\sin x + d\cos x) + B \times \frac{d}{dx}(c\sin x + d\cos x)$$
$$= A(c\sin x + d\cos x) + B(c\cos x - d\sin x)$$

Taking the coefficient of $\sin x$ and $\cos x$, thus obtain the values of A and B and then integrate. The following example will illustrate the form.

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$$I = \int \frac{2\sin x + 3\cos x}{4\sin x + 5\cos x}\,dx$$

Here $2\sin x + 3\cos x = A \times (4\sin x + 5\cos x) + B \times \frac{d}{dx}(4\sin x + 5\cos x)$

$$2\sin x + 3\cos x = A(4\sin x + 5\cos x) + B(4\cos x - 5\sin x)$$

Taking the coe. of $\sin x$ and $\cos x$

$$4A - 5B = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$
$$5A + 4B = 3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (2)$$

$(1)\times 4 + (2)\times 5 \Rightarrow$

$$16A - 20B = 8$$
$$25A + 20B = 15 \qquad \Rightarrow 41A = 23 \Rightarrow A = \frac{23}{41}$$

Substituting in (2) $\quad 4B = 3 - 5A \Rightarrow B = \frac{1}{4}(3 - 5A) \Rightarrow B = \frac{1}{4}\left(3 - 5 \times \frac{23}{41}\right) \Rightarrow B = \frac{1}{4}\left(\frac{3\times 41 - 5\times 23}{41}\right)$

$$\Rightarrow B = \frac{1}{4}\left(\frac{123 - 115}{41}\right) = \frac{1}{4}\left(\frac{8}{41}\right) = \frac{2}{41}$$

$$2\sin x + 3\cos x = \frac{23}{41}(4\sin x + 5\cos x) + \frac{2}{41}(4\sin x + 5\cos x)$$

$$\int \frac{2\sin x + 3\cos x}{4\sin x + 5\cos x}dx \quad = \frac{23}{41}\int \frac{4\sin x + 5\cos x}{4\sin x + 5\cos x}dx + \frac{2}{41}\int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx$$

$$= \frac{23}{41}\int dx + \frac{2}{41}\int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx$$

$$= \frac{23}{41}x + \frac{2}{41}\int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx$$

$$= \frac{23}{41}x + \frac{2}{41}\log|4\sin x + 5\cos x| + C \qquad [\because \textit{Corrolary of Type } 3]$$

Integral of functions of the form:

i. $\int \sqrt{a^2 - x^2}\,dx \; = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$

ii. $\int \sqrt{a^2 + x^2}\,dx \; = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left|x + \sqrt{a^2 + x^2}\right| + C$

iii. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

iv. $\int \sqrt{ax^2 + bx + c}\;dx$

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Then we can write the denominator as

$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$. Now the function becomes one the above three forms and hence can be evaluated.

E.g.:

$$I=\int\sqrt{3-2x-2x^2}\ dx$$

$$3-2x-2x^2=-\left(2x^2+2x-3\right)=-2\left[\left(x+\frac{2}{2\times 2}\right)^2-\frac{2^2-4\times 2\times -3}{4\times 2^2}\right]=-2\left[\left(x+\frac{1}{2}\right)^2-\frac{4+24}{16}\right]$$

$$=-2\left[\left(x+\frac{1}{2}\right)^2-\frac{28}{16}\right]=-2\left[\left(x+\frac{1}{2}\right)^2-\frac{7}{4}\right]=-2\left[\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{7}}{2}\right)^2\right]$$

$$=2\left[\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2\right]$$

$$\therefore I=\int\sqrt{2\left[\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2\right]}\ dx=\sqrt{2}\int\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\ dx$$

$$=\sqrt{2}\left\{\frac{x+\frac{1}{2}}{2}\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}+\frac{\left(\frac{\sqrt{7}}{2}\right)^2}{2}\sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)\right\}+C$$

$$=\sqrt{2}\left\{\frac{2x+1}{2}\sqrt{\frac{7}{4}-\left(\frac{2x+1}{2}\right)^2}+\frac{7}{2\times 4}\sin^{-1}\left(\frac{\frac{2x+1}{2}}{\frac{\sqrt{7}}{2}}\right)\right\}+C$$

$$=\sqrt{2}\left\{\frac{2x+1}{2}\sqrt{\frac{7-(2x+1)^2}{4}}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right\}+C$$

$$=\sqrt{2}\left\{\frac{2x+1}{2}\frac{\sqrt{7-4x^2-4x-1}}{2}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right\}+C$$

$$=\sqrt{2}\left\{\frac{(2x+1)}{2}\frac{\sqrt{6-4x^2-4x}}{4}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right\}+C$$

$$=\sqrt{2}\left\{\frac{(2x+1)}{2}\frac{\sqrt{2}\sqrt{3-2x-2x^2}}{4}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right\}+C$$

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$$=\frac{(2x+1)}{2}\frac{2\sqrt{3-2x-2x^2}}{4}+\frac{7\sqrt{2}}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C$$

$$\frac{(2x+1)\sqrt{3-2x-2x^2}}{4}+\frac{7\sqrt{2}}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C$$

(v) To Find $\int(\mathrm{px}+\mathrm{q})\sqrt{\mathrm{ax}^2+\mathrm{bx}+\mathrm{c}}\ \mathrm{dx}$

Here $px+q=A\frac{d}{dx}(ax^2+bx+c)+B.$ Now the function becomes one the above forms and hence can be evaluated.

E.g.:

Evaluate $\int(x+1)\sqrt{x^2-x+1}\ dx$

Let $I=\int(x+1)\sqrt{x^2-x+1}\ dx$

Here $x+1=A\frac{d}{dx}(x^2-x+1)+B\Rightarrow x+1=A(2x-1)+B$

coefficient of x: $1=2A\Rightarrow A=\frac{1}{2}$

constant terms: $1=-A+B\Rightarrow B=A+1\Rightarrow B=\frac{1}{2}+1=\frac{3}{2}$

$\therefore\ x+1=\frac{1}{2}(2x-1)+\frac{3}{2}$

$$\therefore\int(x+1)\sqrt{x^2-x+1}\ dx=\int\left[\frac{1}{2}(2x-1)+\frac{3}{2}\right]\sqrt{x^2-x+1}\,dx$$

$$=\int\left[\frac{1}{2}(2x-1)\sqrt{x^2-x+1}\ dx+\frac{3}{2}\int\sqrt{x^2-x+1}\right]dx=I_1+I_2$$

Let $I_1=\frac{1}{2}\int(2x-1)\sqrt{x^2-x+1}\ dx$ put $x^2-x+1=u\Rightarrow(2x-1)dx=du$

$$\therefore I_1=\frac{1}{2}\int\sqrt{u}\ du=\frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}}+C_1=\frac{1}{2}\int\sqrt{x^2-x+1}+C_1$$

$$=\frac{1}{2}\times\frac{2}{3}u^{\frac{3}{2}}+C_1=\frac{1}{3}(x^2-x+1)^{\frac{3}{2}}+C_1$$

Let $I_2\qquad=\frac{3}{2}\int\int\sqrt{x^2-x+1}dx$

Here $x^2-x+1=1\left[\left(x+\frac{-1}{2\times1}\right)^2-\frac{(-1)^2-4\times1\times1}{4\times1^2}\right]=\left[\left(x-\frac{1}{2}\right)^2-\frac{-3}{4}\right]=\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$

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$$\therefore I_2 = \frac{3}{2}\int\left[\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]dx$$

$$= \frac{3}{2}\left[\frac{x-\frac{1}{2}}{2}\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\frac{3}{4}}{2}\log\left|x-\frac{1}{2}+\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]+C_2$$

$$= \frac{3}{2}\left[\frac{\frac{2x-1}{2}}{2}\sqrt{x^2-2\times x\times\frac{1}{2}+\frac{1}{4}+\frac{3}{4}}+\frac{3}{8}\log\left|\frac{2x-1}{2}+\sqrt{x^2-2\times x\times\frac{1}{2}+\frac{1}{4}+\frac{3}{4}}\right|\right]+C_2$$

$$= \frac{3}{2}\left[\frac{2x-1}{4}\sqrt{x^2-x+1}+\frac{3}{8}\log\left|\frac{2x-1}{2}+\sqrt{x^2-x+1}\right|\right]+C_2$$

$$= \frac{3}{8}(2x-1)\sqrt{x^2-x+1}+\frac{9}{16}\log\left|\frac{2x-1}{2}+\sqrt{x^2-x+1}\right|+C_2$$

$$\therefore I = \frac{1}{3}\left(x^2-x+1\right)^{\frac{3}{2}}+C_1+\frac{3}{8}(2x-1)\sqrt{x^2-x+1}+\frac{9}{16}\log\left|\frac{2x-1}{2}+\sqrt{x^2-x+1}\right|+C_2$$

$$= \frac{1}{3}\left(x^2-x+1\right)^{\frac{3}{2}}+\frac{3}{8}(2x-1)\sqrt{x^2-x+1}+\frac{9}{16}\log\left|\frac{2x-1}{2}+\sqrt{x^2-x+1}\right|+C,$$

where $\frac{1}{2}C_1$ *and* $\frac{3}{2}C_2 = C$.

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