# DEFINITE INTEGRALS

Let F(x) be the anti-derivative or primitive of a function f(x) defined on the closed interval [a , b], then integral of the function f(x) over [a ,b] is denoted by $\int_a^b f(x)dx$ and is defined as $F(b)-F(a)$. i.e., $\int_a^b f(x)dx = F(b)-F(a)$. Here *a* and *b* are called limits of integration; *a* is called lower limit and *b* is called upper limit. The intervals [a,b] is called the interval of integration.

Note1: $F(b)-F(a)=[F(x)]_a^b$

Note2: $\int_a^b f(x)dx$ read as "integral *a* to *b*, $f(x)$ or "the integral of $f(x)$ from *a* to *b*.

Note3: To evaluate definite integral, there is no need to keep the constant of integration.

e.g.: $\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -\cos\pi - -\cos 0 = -(-1)+1 = 1+1 = 2$

Definite integral as the limit of a sum.

Let $f(x)$ is a continuous function of x in the closed interval [a,b]. Then the definite integral of $f(x)$ between a and b is defined as:

$$\int_a^b f(x)dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+...+f(a+\overline{n-1}h)\right], \text{ where } h=\frac{b-a}{n} \Rightarrow nh=b-a$$

E.g. : $\int_1^2 (x+1)\,dx$

Here $a=1, b=2$ and $h=\frac{2-1}{n}=\frac{1}{n} \Rightarrow nh=1$

$$\int_1^2 (x+1)dx = \lim_{h\to 0} h\left[2+(2+h)+(2+2h)+....+(2+\overline{n-1}\,h)\right]$$

$$= \lim_{h\to 0} h\left[(2+2+2+...\text{ to } n \text{ terms}+\left(h+2h+...+\overline{n-1}h\right)\right]$$

$$= \lim_{h\to 0} h\left[(2+2+2+...\text{ to } n \text{ terms}+h\left(1+2+...+\overline{n-1}\right)\right]$$

$$= \lim_{h\to 0} h\left[2n + h\times\frac{(n-1)n}{2}\right] = \lim_{h\to 0} h\left[2n + \frac{(n-1)nh}{2}\right]$$

$$= \lim_{h\to 0} h\left[2n + \frac{(n-1)\times 1}{2}\right] = \lim_{h\to 0}\left[2nh + \frac{(nh-h)}{2}\right] = \lim_{h\to 0}\left[2\times 1 + \frac{(1-h)}{2}\right] = 2 + \frac{1-0}{2} = 2 + \frac{1}{2} = \frac{5}{2}$$

## Definite Integrals by Substitution

Integral of the function of the forms $f(u)$, *where* $u = \phi(x)$.

- Substitute and reduce the given integral to a known form.
- Change the limits of integration.
- Find the integral of the new function w.r.t the new variable.
- Evaluate the function with new limits

E.g.: $I = \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\, dx$

$\sin x = u$

$\cos x\, dx = du$

*when* $x = 0$, $u = \sin 0 = 0$

*when* $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

$\therefore I = \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\, dx$

$$= \int_0^1 \frac{du}{1+u^2}\, dx = \left[\tan^{-1} u\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

## Properties of Definite Integral.

**Property 0 or** $[P_0]$

A definite integral does not depend on the variable of integration. i.e., $\int_a^b f(x)dx = \int_a^b f(y)dy$

Proof: LHS $= \int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$ ………… (1)

RHS $= \int_a^b f(y)\,dx = \left[F(y)\right]_a^b = F(b) - F(a)$ ……………. (2)

From (1) & (2), hence verified.

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**Property 1 or $\left[P_1\right]$:**

The sign of a definite integral is changed when its limits are interchanged.

i.e., $\boldsymbol{\int_a^b f(x)dx = -\int_b^a f(x)dx}$

LHS$= \int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$ ………… (1)

RHS $= -\int_b^a f(x)\,dx = -\left[F(x)\right]_b^a = -\left[F(a) - F(b)\right]$ ………… (2)

From (1) and (2) hence verified.

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**Property 2 or $\left[P_2\right]$:**

$\boldsymbol{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx}$

LHS $= \int_a^b f(x)\,dx = F(b) - f(a)$

Now, $\int_a^c f(x)\,dx = F(c) - f(a)$ and $\int_c^b f(x)\,dx = F(b) - f(c)$

RHS $= \int_a^c f(x)\,dx + \int_c^b F(x)\,dx = F(c) - F(a) + F(b) - F(c) = F(b) - F(a) = \int_a^b f\left(x\right)dx$

$\therefore$ LHS=RHS

$$I=\int_0^3 |x-2|\,dx$$

$$put\ x-2=0\Rightarrow x=2$$

$$\therefore I=\int_0^2 |x-2|\,dx+\int_2^3 |x-2|\,dx \qquad [P_2]$$

$$=\int_0^2 -(x-2)\,dx+\int_2^3 (x-2)\,dx$$

$$=\left[-\frac{x^2}{2}+2x\right]_0^2+\left[\frac{x^2}{2}-2x\right]_2^3$$

$$=\left(-\frac{2^2}{2}+2\times 2\right)-\left(-\frac{0^2}{2}+2\times 0\right)+\left(\frac{3^2}{2}-2\times 3\right)-\left(\frac{2^2}{2}-2\times 2\right)$$

$$=(-2+4)-0+\left(\frac{9}{2}-6\right)-(2-4)=2+\frac{9}{2}-6-(-2)$$

E.g.: $$=2+\frac{9}{2}-6+2=\frac{9}{2}-2=\frac{9-4}{2}=\frac{5}{2}$$

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**Property 3 or $[P_3]$:**

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

RHS $=\int_a^b f(a+b-x)\,dx$

$$Put\ \ a+b-x=t\Rightarrow -dx=dt\Rightarrow dx=-dt$$
$$When\ \ x=a,\ t=a+b-a=b$$
$$When\ \ x=b,\ t=a+b-b=a$$

$$\therefore RHS=\int_b^a f(t)\,(-dt)=\int_a^b f(t)\,dt=\int_a^b f(x)\,dx=LHS \qquad [P_0]$$

E.g.:

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\,dx \qquad .............(1)$$

$$We\ have\ \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}}\,dx \quad \text{.............(2)}$$

$(1)+(2)$, *we have*

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}}\,dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{1}{1+\sqrt{\cot x}}\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{1}{1+\frac{1}{\sqrt{\tan x}}}\right)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{1}{\frac{\sqrt{\tan x}+1}{\sqrt{\tan x}}}\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\,dx = \left[x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore I = \frac{\pi}{12}$$

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**Property 4 or $\left[P_4\right]$:**

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$RHS = \int_0^a f(a-x)\,dx$$

*Put* $a - x = t \Rightarrow -dx = dt \Rightarrow dx = -dt$

*When* $x = 0,\ t = a - 0 = a$

*When* $x = a,\ t = a - a = 0$

RHS $= \int_0^a f(a-x)\,dx = \int_a^0 f(t)(-dt) = -\int_a^0 f(t)\,du = \int_0^a f(t)\,du$ [Property II]

$= \int_0^a f(x)\,dx$ [Property I]

= LHS.

*(This property is important and questions related to this property may be asked). There are 3 types of problems.*

**Type1:**

E.g. $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ ………. (1)

We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{…………..} \quad (2)$$

(1) + (2)

$$I + I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$2I = \int_0^{\pi/2} \left[ \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} \right] dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \int_0^{\pi/2} dx = (x)_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore\ I = \frac{\pi}{4}$$

**Try yourself:**

1. $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx$ [ Say 2002]
2. $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\, dx$
3. $\int_0^{\pi/2} \frac{1}{1+\tan x}\, dx$
4. $\int_0^{\pi/2} \frac{1}{1+\cot x}\, dx$
5. $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx$
6. $\int_0^{\pi/2} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}\, dx$
7. $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\, dx$
8. $\int_0^{5} \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}\, dx$
9. $\int_0^{2\pi} \frac{\sin 2x}{a-b\cos x}\, dx$
10. $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}\, dx$
11. $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$
12. $\int_0^{\pi/2} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}\, dx$

13. $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx$

14. $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\, dx$

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**Type 2:**

$I = \int_0^{\pi/2} \log(\tan x)\, dx$ …………….. (1)

We know that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi/2} \log\left(\tan x\left(\frac{\pi}{2}-x\right)\right) dx = \int_0^{\pi/2} \log\cot x\, dx$ …………….. (2)

(1) + (2) $\Rightarrow 2I = \int_0^{\pi/2} \log(\tan x)\, dx + \int_0^{\pi/2} \log(\cot x)\, dx = \int_0^{\pi/2} \left[\log(\tan x)+\log(\cot x)\right] dx$

$2I = \int_0^{\pi/2} \log(\tan x.\cot x)dx = \int_0^{\pi/2} \log(1)dx = \log(1)\times\int_0^{\pi/2} dx = 0\times\int_0^{\pi/2} dx = 0$

i.e., $2I = 0 \Rightarrow I = 0$

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**Try yourself:**

1. $\int_0^{\pi/2} \log(\cot x)\, dx$

2. $\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)dx$

3. $\int_0^{\pi/4} \log(1+\tan x)\, dx$

4. $\int_0^{\pi/2} \log(\tan x+\cot x)\, dx$

5. $\int_0^{1} \log\left(\sqrt{1-x}+\sqrt{1+x}\right) dx$

6. $\int_0^1 \log\left(\frac{1}{x}-1\right) dx$

**Type 3**

1. $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x} dx$ [M 2000]

2. $\int_0^{\pi/2} \log(\tan x)\, dx$ [M 2002] [M2003] [Say 03]

3. $\int_0^{\pi} \frac{x\,dx}{1+\sin x}$ [Say 2002]

4. $\int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x} dx$ [M 2003]

5. $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x} dx$

**Property 5** $[P_5]$

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$$

$$RHS = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

$Put\ \ 2a-x=t \Rightarrow -dx=dt$

$if\ \ x=0,\ t=2a-0=2a$

$if\ \ x=a,\ t=2a-a=a$

$$RHS = \int_0^{a} f(x)\,dx + \int_{2a}^{a} f(t)(-dt) = \int_0^{a} f(x)\,dx - \int_{2a}^{a} f(t)\,dt = \int_0^{a} f(x)\,dx + \int_a^{2a} f(t)\,dt \qquad [P_1]$$

$$= \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad [P_0]$$

$$=\int_0^{2a} f(x)\,dx = LHS \qquad [P_2]$$

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**Property 6** $[P_6]$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if\ \ f(2a-x)=f(x) \\ 0, & if\ \ f(2a-x)=-f(x) \end{cases}$$

We have, $\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ................(1)

If $f(2a-x)=f(x)$, (1) becomes

$$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)dx$$

Again, if $f(2a-x)=-f(x)$, (1) becomes

$$\int_0^{2a} f(x)dx = \int_0^a f(x)\,dx + \int_0^a -f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

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**Property 7** $[P_7]$

$$\int_{-a}^{+a} f(x)dx = 2\int_0^a f(x)$$ if f(x) is an even function.

Let $\int_{-a}^{+a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^{+a} f(x)\,dx$ [Prop. III]

In the 1st integral of RHS put $x=-y$

Then $dx=-dy$

Also $x=-a,\ -y=-a \Rightarrow y=a$

Also $x=0, -y=0 \Rightarrow y=0$

$$\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y).-dy$$

$$= -\int_{a}^{0} f(-y)\,dy$$

$$= \int_{0}^{a} f(-y)\,dy$$

$$= \int_{0}^{a} f(y)\,dy$$ [given f(x) is an even function ∴f(y) is also an even function]

$$= \int_{0}^{a} f(x)\,dx$$ [$P_0$]

$$\therefore \text{RHS} = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx = \text{LHS}$$

Hence proved.

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**Property 8** $[P_8]$

$$\int_{-a}^{+a} f(x)dx = 0$$ if f(x) is an odd function.

Let $\int_{-a}^{+a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{+a} f(x)\,dx$ [$P_2$]

In the 1st integral of RHS put $x=-y$

Then $dx=-dy$

Also $x=-a, -y=-a \Rightarrow y=a$

Also $x=0, -y=0 \Rightarrow y=0$

$$\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y).-dy$$

$= -\int_a^0 f(-y)\, dy$

$= \int_0^a f(-y)\, dy$

$= \int_0^a -f(y)\, dy$ [given f(x) is an odd function $\therefore$ f(y) is also an odd function]

$= -\int_0^a f(x)\, dx$ [$P_0$]

$\therefore$ RHS $= \int_0^a f(x)\,dx + -\int_0^a f(x)\, dx$

$= 0$ = LHS

Hence proved.

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E.g.:

1. $I = \int_{-\pi/2}^{\pi/2} \sin x\, dx$

Since, sin x is an odd function, $\int_{-\pi/2}^{\pi/2} \sin x\, dx = 0$

2. $I = \int_{-\pi/2}^{\pi/2} \cos x\, dx$ $= 2\int_0^{\pi/2} \cos x\, dx$ [$\because$ cosx is an even function]

$= 2\,[\sin x]_0^{\pi/2}$

$= 2\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= 2\,(1-0)$

$= 2$

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**Property based questions:**

1. $\int_0^{\pi/2} \log(\sin x)\, dx$

2. $\int_0^{\pi/2} \log(\cos x)\, dx$

3. $\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\, dx$

4. $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

5. $\int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x}$