Chapter - 8

Gravitation

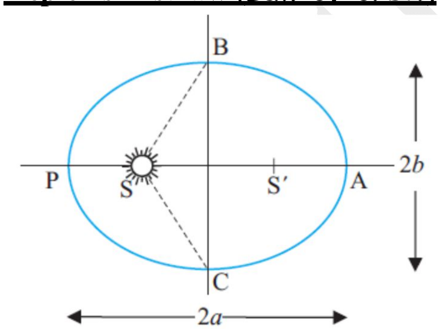
Ptolemy about 2000 years ago introduced the '**geocentric**' model to explain the motion of planets.

Later **Copernicus** (1473 – 1543) put forward the '**heliocentric theory**' in which the sun is at the centre of the planetary system.

Johannes Kepler (1571–1630) was a scientist of German origin. He formulated the three laws of planetary motion based on the painstaking observations of Tycho Brahe and coworkers. Kepler himself was an assistant to Brahe and it took him sixteen long years to arrive at the three planetary laws. He is also known as the founder of geometrical optics, being the first to describe what happens to light after it enters a telescope.

Kepler's laws of planetary motion

Kepler's first law (Law of orbit)



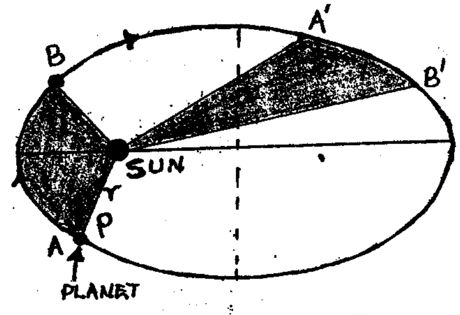
"Every planet revolves round the sun in an elliptical orbit with sun at one of its foci".

Kepler's Second Law (Law of Area)

"The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time"

i.e., **the areal velocity of the planet around the sun is constant**.

Proof:



Let a planet P is at 'A' at t = 0 and travels to 'B' in a time interval Δt. Let ΔA be the area covered during this time.

∴ From figure,

$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$

Dividing by Δt on both sides,

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t}$$

Taking limits as $\Delta t \to 0$ on both sides,

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega$$

$$= \frac{m r^2 \omega}{2m}$$

$$= \frac{L}{2m}$$

i.e., $$\frac{dA}{dt} = \frac{L}{2m}$$

The gravitational force is acting along the line joining the sun and the planet (or it is a central force). So the torque on the planet about the sun is zero.

Since $\tau = 0$, L = Constant

$$\therefore \frac{dA}{dt} = \frac{L}{2m} = \text{Constant}$$

i.e., **areal velocity of the planet is constant**.

Kepler's Third Law (Law of Period)

"**The square of the time period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the elliptical orbit**"

$$T^2 \alpha a^3$$

Pr. 1: (i) A comet orbits round the sun in a highly elliptical orbit. Does the comet have a constant

a) Linear speed
b) Angular speed
c) Angular momentum
d) Kinetic energy
e) Potential energy
f) Total energy throughout the orbit?

(ii) What are the consequences if the angular momentum is conserved?

Soln:

Pr.2: Earth revolves around the sun in elliptical orbit. The closest approach of earth with the sun is called perihelion. When earth approaches the perihelion, its speed increases. Explain

Ans:

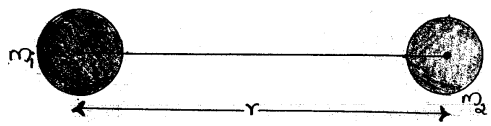
Pr.3: A Saturn year is **29.5** times the earth year. How far is the saturn from the sun if the earth is **1.50×10^8 km** away from the sun?

Soln:

Newton's Universal Law of Gravitation

According to Newton's law of gravitation, "**everybody in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.**"

HSSLiVE.IN



$F \propto m_1 m_2$

$F \propto \frac{1}{r^2}$

$\therefore F \propto \frac{m_1 m_2}{r^2}$ Hence $\boxed{F = \frac{G\, m_1 m_2}{r^2}}$

Where **G** is a constant called universal gravitational constant

The value of G is $\mathbf{6.67 \times 10^{-11}\ Nm^2/Kg^2}$

Dimensional formula for G

$$[G] = \frac{[MLT^{-2}][L^2]}{[M][M]} = [M^{-1} L^3 T^{-2}]$$

$$F = \frac{Gm_1m_2}{r^2}$$
$$\therefore G = \frac{Fr^2}{m_1m_2}$$

Definition of G

We have

$$F = \frac{Gm_1m_2}{r^2}$$

If $m_1 = m_2 = 1$ Kg and $r = 1$ m

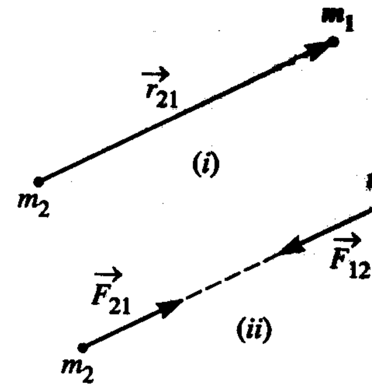
$$F = \frac{G \times 1 \times 1}{1^2} = G$$

"Universal gravitational constant is numerically equal to the force of attraction between two unit masses kept at a distance of 1 m apart".

Note:- The value of G is independent of the (i) size and shape of the bodies and

(ii) The nature of the medium in which they are kept.

Vector form of Newton's Law of Gravitation



We have,

$\vec{r}_{21} = -\vec{r}_{12}$ And

$|\vec{r}_{12}| = |\vec{r}_{21}|$ or $r_{12} = r_{21}$

The gravitational force on m_1 exerted by m_2 is

$$\vec{F}_{12} = \frac{G\, m_1 m_2}{r_{12}^2}\ \hat{r}_{12}$$

The gravitational force on m_2 exerted by m_1 is,

$$\vec{F}_{21} = \frac{G\, m_1 m_2}{r_{21}^2}\ \hat{r}_{21}$$
$$= \frac{G\, m_1 m_2}{r_{12}^2}(-\hat{r}_{12})$$
$$= \frac{-G\, m_1 m_2}{r_{12}^2}\ \hat{r}_{12} = -\vec{F}_{12}$$

ie, $\vec{F}_{21} = -\vec{F}_{12}$

i.e., Newton's Law of Gravitation obeys Newton's third Law of motion.

Characteristics of Gravitational force

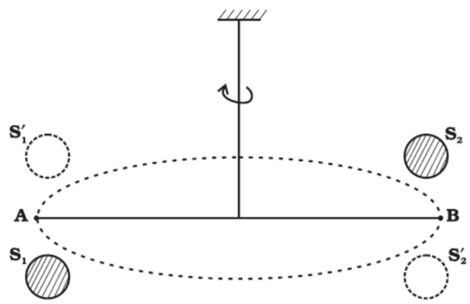
1. It is always attractive.
2. It is independent of the intervening medium.

HSSLiVE.IN

3. It is independent of the presence or absence of other bodies.
4. It is a central force.
5. It is a conservative force.
6. It obeys the principle of superposition.
7. It is the weakest force in nature.
8. It is a long range force.

Determination of G – Cavendish method

Cavendish calculated the value of **G** using the given arrangement called torsional balance. It consist of two small identical lead balls of mass '**m**' each, suspended by a wire. Two equal heavy lead spheres of mass '**M**' each kept at a distance of'**d**' from each '**m**' mass, on opposite sides.



The gravitational force on each pair of masses (M, m) is

$$F = \frac{G\ Mm}{d^2}$$

The two equal and opposite forces at the two ends create a deflecting torque which is given by

$$\tau = \frac{G\ Mm}{d^2} \times \ell$$

Due to this torque, the suspension wire twists through an angle 'θ'. The restoring couple developed in the suspension wire = C θ,
Where C is the couple per unit twist.

In equilibrium, $\frac{G\ Mm}{d^2} \times \ell = C\,\theta$

$$\therefore\ G = \frac{C\,\theta\ d^2}{M\ m\ \ell}$$

Substituting all values in the above equation, Cavendish calculated the value of G as 6.75×10^{-11} N m^2/ Kg^2

Pr.4: A rocket is fired from the earth towards the sun. At what distance from the earth's centre is the gravitational force on the rocket is zero? Mass of the sun= 2×10^{30} kg, mass of the earth=6×10^{24} kg. Neglect the effect of other planets etc. (Orbital radius of earth=1.5×10^{11}m)

Soln:

Acceleration due to Gravity of Earth

The acceleration with which a body falls towards the surface of earth

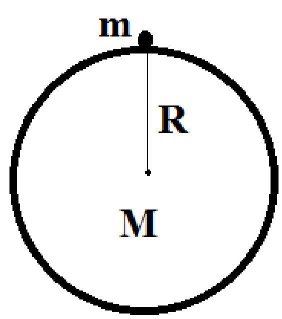
HSSLIVE.IN

when dropped from a height is called acceleration due to gravity.

Or, **it is the acceleration of a freely falling body**.

Acceleration due to gravity near the surface of earth, $\mathbf{g = 9.8\ m/s^2}$

Expression for acceleration due to gravity



Consider a body of mass 'm' placed on the surface of earth.

The gravitational force on the body is given by

$$F = \frac{GMm}{R^2} \cdots\cdots\cdots\cdots\cdots (1)$$

Again we have, $F = mg \cdots\cdots\cdots\cdots\cdots (2)$

$\therefore$ From eqns (1) and (2),

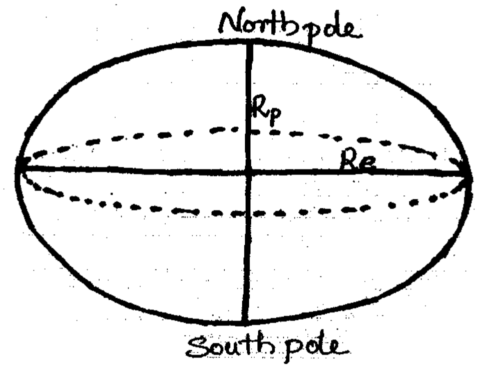
$$\cancel{m}g = \frac{GM\cancel{m}}{R^2}$$

$$\boxed{g = \frac{GM}{R^2}}$$

Variation of Acceleration due to gravity

Acceleration due to gravity varies with the **shape of earth**, **altitude**, **depth** and **rotation of earth**.

1. Variation of 'g' due to the shape of earth



We have,

$$g = \frac{GM}{R^2}$$

$$\Rightarrow g \propto \frac{1}{R^2}$$

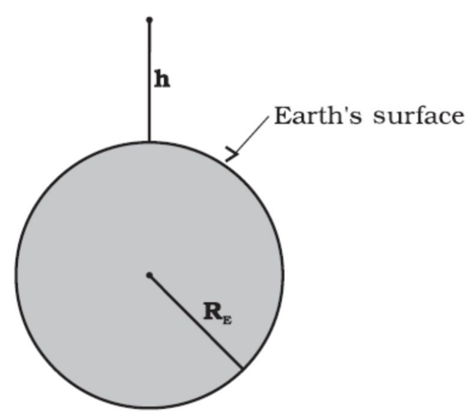
The equatorial radius of earth is greater than the polar radius. That is

$\mathbf{R_p < R_e}$, Therefore $\mathbf{g_p > g_e}$,

Thus the acceleration due to gravity at the pole is greater than that at the equator.

Note: - When a body is taken from equator to the pole, its weight increases.

2. Variation of 'g' with altitude (Height)



The value of 'g' on the surface of earth is

$$g = \frac{GM}{R^2} \cdots\cdots\cdots\cdots (1)$$

Suppose the body is taken to a height '**h**' above the surface of earth, the value of acceleration due to gravity is

$$g_{(h)} = \frac{GM}{(R+h)^2} \cdots\cdots\cdots\cdots (2)$$

$$\frac{(2)}{(1)} \rightarrow \frac{g_{(h)}}{g} = \frac{\cancel{GM}}{(R+h)^2} \times \frac{R^2}{\cancel{GM}}$$

$$\frac{g_{(h)}}{g} = \frac{R^2}{(R+h)^2}$$

$$= \frac{R^2}{\left[R\left(1+\frac{h}{R}\right)\right]^2}$$

$$= \frac{\cancel{R}^2}{\cancel{R}^2\left(1+\frac{h}{R}\right)^2}$$

$$\frac{g_{(h)}}{g} = \left(1+\frac{h}{R}\right)^{-2}$$

If $h << R$, then $\frac{h}{R}$ is very small compared to 1. Expanding the RHS of the above equation by Binomial theorem and neglecting the higher powers of $\frac{h}{R}$, we get ,

$$\frac{g_{(h)}}{g} = \left(1 - \frac{2h}{R}\right)$$

$$\Rightarrow \boxed{g_{(h)} = g\left(1 - \frac{2h}{R}\right)}$$

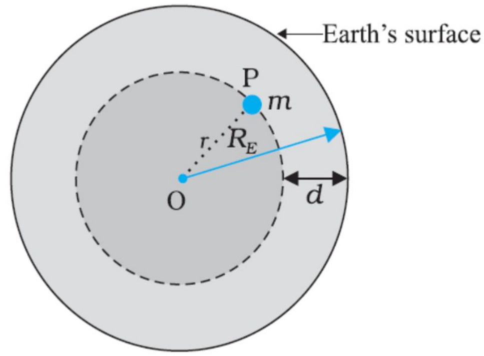
The above equation shows that the value of acceleration due to gravity decreases with height.

3. <u>Variation of 'g' with depth</u>

For a body of mass 'm' placed on the surface of earth, we have

$$\cancel{m}g = \frac{GM\cancel{m}}{R^2}$$

$$\Rightarrow g = \frac{GM}{R^2} \cdots\cdots\cdots\cdots\cdots (1)$$



If 'ρ' is the mean density of earth, then mass of earth,

M = Volume × density

$$= \frac{4}{3}\pi R^3 \times \rho$$

$$\therefore \text{eqn}(1) \rightarrow g = \frac{G\frac{4}{3}\pi R^3 \rho}{R^2}$$

$$g = \frac{4}{3}\pi G R \rho \cdots\cdots\cdots (2)$$

If the body is kept at a depth of 'd' from the surface of earth, then the mass of radius R–d will only be effective for the gravitational pull towards the centre.

$$g_{(d)} = \frac{4}{3}\pi G(R-d)\rho \cdots\cdots\cdots\cdots (3)$$

$$\frac{(3)}{(2)} \rightarrow \frac{g_{(d)}}{g} = \frac{R-d}{R} = 1 - \frac{d}{R}$$

$$\Rightarrow \boxed{g_{(d)} = g\left(1 - \frac{d}{R}\right)}$$

The above eqn. shows that the value of **g** decreases with depth.

At the centre of earth, d = R

$$g_{(d)} = g\left(1 - \frac{R}{R}\right) = 0$$

<u>Note:</u> - The weight of a body of mass '**m**' at the centre of earth is zero.

<u>Pr.5:</u> A body of mass 'm' falls freely under gravity, near the surface of earth.

(a) Will the acceleration of the body change if a part of the mass is thrown away from it?

(b) What will be the free fall acceleration if it is falling from a height equal to R, the radius of earth?

Soln: -

(a) No. The acceleration due to gravity does not depend on the mass of the falling body.

(b)

Pr.6: Find the value of acceleration due to gravity at a height of **400km** above the surface of the earth. Given radius of earth=6400 km and acceleration due to gravity at the surface of earth is **9.8 ms^{-2}**.

Soln:

Pr.7: A body weighs **63N** on the surface of the earth. What is the gravitational force on it due to the earth at a height equal to the radius of earth?

Soln:

Pr.8: Assuming the earth to be a sphere of uniform mass density, how much would be a body weigh half way down to the centre of earth if it weighed 250N on the surface?

Soln:

HSSLiVE.IN

Pr.9: Find the height at which the acceleration due to gravity is reduced to half.

Soln:

Pr.10: Find the percentage decrease in the weight of a body when taken **16km** below the surface of earth. Radius of earth=6400 km.

Soln:

Pr.11: How far above the earth's surface does the value of **g** become the **16%** of its value on the surface? Assume radius of earth to be **6400km**.

Soln:

Pr.12: At what height above earth's surface, value of **g** is the same as in a **100km** deep?

Ans:

HSSLiVE.IN

Pr.13: Keeping the mass of earth the same if the diameter is shrinked by 1%, what will be the percentage change in acceleration due to gravity on the surface?

Soln:

Pr.14: Acceleration due to gravity '**g**' depends on the distance '**r**' from the centre of the earth. Draw a graph showing the variation of 'g' with 'r'.

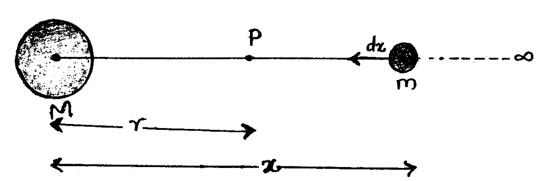
Ans:

Pr.15: Imagine the motion of a body from the centre of earth to the surface of moon, what changes will you observe in the weight of the body during that motion? (Neglect the effect of all other bodies)

Ans:

Gravitational Potential Energy

Gravitational potential energy is defined as "**the work done in bringing a mass from infinity to a point in the gravitational field of another body.**"



Let the body of mass '**m**' be at a distance **x** from the mass '**M**'.

Now the gravitational force on 'm' is

$$F = \frac{GMm}{x^2}$$

The work done to displace the body through a distance **dx** is

$$dW = F \cdot dx = \frac{GMm}{x^2} dx$$

Then the total work done to bring the body from infinity to the point **P** is given by,

$$W = \int_{\infty}^{r} dw = \frac{GMm}{x^2} dx$$

$$= GMm \int_{\infty}^{r} \frac{1}{x^2} dx$$

$$= GMm \left[\frac{-1}{x}\right]_{\infty}^{r}$$

$$= -GMm \left[\frac{1}{x}\right]_{\infty}^{r}$$

$$= -GMm \left[\frac{1}{r} - \frac{1}{\infty}\right]$$

$$= \frac{-GMm}{r}$$

This work done is stored in the body as its gravitational potential energy U.

$$\therefore \boxed{U = \frac{-GMm}{r}}$$

The work done to bring the body from r_1 to r_2 (the change in potential energy of the body) is given by ,

$$\Delta U = -GMm\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

i.e., $\boxed{\Delta U = GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]}$ ---(1)

Gravitational potential energy near the surface of earth

If a body of mass '**m**' is taken from the surface of earth to a height '**h**', then we can substitute $\mathbf{r_1 = R}$ and $\mathbf{r_2 = R + h}$ in

$$\Delta U = GMm\left[\frac{1}{r_1} - \frac{1}{r_2}\right], \text{ We get}$$

$$\Delta U = GMm\left[\frac{1}{R} - \frac{1}{R+h}\right]$$

$$= GMm\left[\frac{\cancel{R} + h - \cancel{R}}{R(R+h)}\right]$$

If $h << R,\ R + h \approx R$

$$\therefore \Delta U \approx GMm\left[\frac{h}{R \cdot R}\right]$$

$$= GMm\left[\frac{h}{R^2}\right]$$

$$= \left(\frac{GM}{R^2}\right) mh = gmh$$

$$\boxed{\Delta U = mgh}$$

Gravitational Potential

Gravitational potential at a point in a gravitational field is defined as "**the work done in bringing a body of unit mass from infinity to that point**".

$$\boxed{V = \frac{-GM}{r}} \qquad V = \frac{W}{m}$$

It is a Scalar quantity.

The S.I unit of V is J/Kg

Dimensional formula is

$$[V] = \frac{[W]}{[m]} = \frac{[ML^2 T^{-2}]}{[M]}$$

$$= [M^0 L^2 T^{-2}]$$

Escape Velocity

Escape velocity is the **minimum velocity with which a body must be projected so that it escapes from the gravitational attraction of earth permanently**.

Expression for escape velocity

The total work done to move a body of mass '**m**' from the surface of earth ($r = R$) to infinity ($r = \infty$) is given by,

$$W = \frac{GMm}{R}$$

Let v be velocity of the body, then K.E of the body when projected is

$$K = \frac{1}{2} m v^2$$

The body can escape from the gravitational pull of earth only if its KE is greater than or equal to the work done in overcoming the gravity.

$$\frac{1}{2} \cancel{m} v^2 \geq \frac{GM\cancel{m}}{R}$$

$$\frac{1}{2} \cancel{m} v_e^2 = \frac{GM\cancel{m}}{R}$$

$$v_e^2 = \frac{2GM}{R}$$

$$\boxed{v_e = \sqrt{\frac{2GM}{R}}}$$

But $g = \frac{GM}{R^2}$ or $GM = gR^2$

$$v_e = \sqrt{\frac{2g\cancel{R}^2}{\cancel{R}}}$$

$$\boxed{v_e = \sqrt{2gR}}$$

Substituting the values,

$R = 6.4 \times 10^6$ m, $g = 9.8$ m/s^2

$$v_e = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$= 11.2 \times 10^3 \text{ m/s} = 11.2 \text{ Km/s}$$

Note:- (1) Escape velocity is independent of the mass of the body.

(2) Escape velocity of a body from moon is 2.38 km/s.

Question: Moon has no atmosphere. Why?

Ans: - Escape velocity of a body from the surface of moon is **2.38km/s**. The root mean square velocity of gas molecule on the surface of moon is (≈ **2.5 km/s**) more than escape velocity from the moon. Therefore gas molecules escape from the moon and hence moon has no atmosphere.

Earth Satellites

A body revolving around a planet in a fixed orbit is called a satellite.

The natural satellite of earth is moon.

Examples for artificial (manmade) satellites are Sputnik, Aryabatta, INSAT etc.

Orbital Velocity

Orbital velocity of a satellite is the velocity with which it revolves round a planet in its fixed orbit.

Expression for orbital velocity

Consider a satellite of mass 'm' that revolves round the earth in an orbit of radius R + h, with velocity v_o. **The centripetal force for the revolution of the satellite is provided by the gravitational force between satellite and earth.**

$$\frac{\cancel{m} v_o^2}{R+h} = \frac{GM\cancel{m}}{(R+h)^2}$$

$$v_o^2 = \frac{GM}{(R+h)}$$

$$\Rightarrow \boxed{v_o = \sqrt{\frac{GM}{R+h}}} \qquad g = \frac{GM}{R^2}, \quad GM = gR^2$$

But $GM = gR^2$

$$\boxed{v_o = \sqrt{\frac{gR^2}{R+h}}}$$

If the satellite is very close to earth, then $R + h \approx R$

HSSLiVE.IN

$$\therefore\ v_o = \sqrt{\frac{g R_E^{\cancel{2}}}{\cancel{R_E}}} \Rightarrow \boxed{v_o = \sqrt{g R_E}}$$

v_o = **7.92** km/s

The nearest orbit of a satellite is called minimum orbit and the corresponding velocity is called first cosmic velocity.

Relation between escape velocity and first cosmic velocity

$$\text{Escape velocity } v_e = \sqrt{2 g R_E} = \sqrt{2}\ \sqrt{g R_E} = \sqrt{2}\ v_0$$

Escape velocity is called second cosmic velocity.

Pr.17: The radius of the earth is reduced by **4%** of its initial value. The mass of earth remains unchanged. What will be the percentage change in the escape velocity?

Soln:

Pr.18: A rat and a horse are to be projected from earth to space. State whether the velocity is the same or different in projecting each animal. Justify

Ans:

The Time Period of a satellite (T)

"The time taken by a satellite to complete one orbital motion around a planet is called time period of a satellite".

Consider a satellite revolving around the earth in an orbit of radius 'R + h' with a velocity v_o.

Time period= Distance moved for one revolution / Orbital velocity

$$T = \frac{2\pi (R+h)}{v_o}$$

$$\text{But } v_0 = \sqrt{\frac{g R^2}{R+h}}$$

$$\therefore T = 2\pi (R+h) \times \sqrt{\frac{R+h}{g R^2}}$$

$$= 2\pi \sqrt{\frac{(R+h)^2 (R+h)}{g R^2}}$$

$$= 2\pi \sqrt{\frac{(R+h)^3}{g R^2}}$$

$$\boxed{T = 2\pi \sqrt{\frac{(R+h)^3}{g R^2}}}$$

For minimum orbit, h=0

$$T = 2\pi \sqrt{\frac{(R+0)^3}{g R^2}}$$

$$\Rightarrow T = 2\pi \sqrt{\frac{R}{g}}$$

Substituting the values of R and g, **T = 84.6 minutes**.

Energy of an orbiting Satellite

HSSLIVE.IN

The K.E of the satellite in a circular orbit with orbital velocity 'v_o' is

$$K = \frac{1}{2} m v_o^2$$

But $$v_o = \sqrt{\frac{GM}{R+h}}$$

$$\therefore K = \frac{1}{2} m \frac{GM}{R+h}$$

$$= \frac{GMm}{2(R+h)}$$

The gravitational P.E at a distance (R + h) from the centre of earth is

$$U = \frac{-GMm}{(R+h)}$$

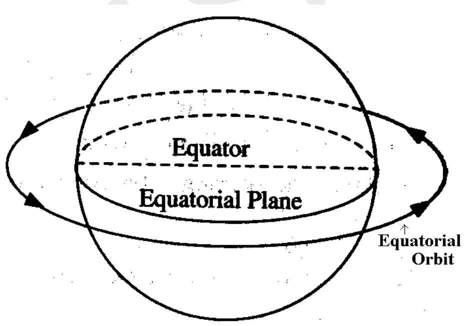
Thus the total energy,

$$E = K + U$$

$$= \frac{GMm}{2(R+h)} - \frac{GMm}{R+h}$$

$$E = \frac{-GMm}{2(R+h)}$$

Geostationary and polar Satellites



Geostationary Satellites

A satellite which appears stationary with respect to earth is called geo-stationary satellite.

The orbit of such a satellite is called geosynchronous orbit.

Geostationary satellite revolves the earth in the equatorial plane.

The time period is 24 hours.

The direction of motion of the satellite is same as that of earth, i.e., from west to east.

The height of the orbit of geostationary satellite is **36,000km.**

Uses

These satellites are mainly used for communication purpose like T V and radio broadcasting and weather forecasting.

Polar Satellites

A satellite which revolves in polar orbit is called a polar satellite. The polar orbit is in the north-south direction, while earth spins below it in the west-east direction. Thus a polar satellite can scan the entire surface of earth.

Uses

The satellites low lying polar orbits (500 – 800 km) are used for studying the weather, environment and spying. Polar satellites are extremely useful for remote sensing, meteorology as well as for environmental studies of the earth.

Weightlessness

Consider an astronaut (or space-man) of mass 'm' is present in the artificial satellite. When the satellite is orbiting around the earth, the man in the satellite experiences a centrifugal force whose direction is away from the centre.

The gravitational force,

$$F_g = \frac{G M m}{r^2}, \text{ directed towards the centre of earth.}$$

Centrifugal force,

$$F_c = \frac{m v_0^2}{r^2}, \text{ directed opposite to the force of gravity.}$$

$$\therefore \text{ Net force, } F = F_g - F_c$$

$$= \frac{G M m}{r^2} - \frac{m v_0^2}{r^2}$$

$$\text{But } v_0 = \frac{G M}{r}, \text{where } r = R + h$$

$$\therefore F = \frac{G M m}{r^2} - \frac{m}{r}\left(\frac{G M}{r}\right)$$

$$= \frac{G M m}{r^2} - \frac{G M m}{r^2} = 0$$

Hence an astronaut feels weightlessness in an artificial satellite.

Pr.19: If a satellite is put into an orbit at height where it has no sufficient velocity for revolution. How will the motion of satellite be affected? Draw the path of it.

Ans:

Pr. 20: A person in an artificial satellite experiences weightlessness. The moon is a natural satellite of the earth.

a) Can a person on the moon experience weight? Why?

b) A satellite is revolving very close to earth. What is the percentage increase in the velocity needed to make it escape from the gravitational field of earth?

HSSLiVE.IN