8. BINOMIAL THEOREM

PASCAL'S TRIANGLE

Pascal's triangle is a triangle with 1at the top vertex and running down the two slanting sides. It was introduced by a French Mathematician Blaise Pascal. It is also known as Meru Prastara or Pingla.

Index	Coefficients	Expanded form
0	1	1
1	1 1	$a+b$
2	1 2 1	$a^2+2ab+b^2$
3	1 3 3 1	$a^3+3a^2b+3ab^2+b^3$
4	1 4 6 4 1	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$
5	1 5 10 10 5 1	$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
6	1 6 15 20 15 6 1	$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$

Binomial Theorem

For any positive integer 'n', we have,

1. $(a+b)^n = a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + {}^nC_3\,a^{n-3}b^3 + \ldots + {}^nC_r\,a^{n-r}b^r + \ldots + b^r$
2. $(a-b)^n = a^n - {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 - {}^nC_3\,a^{n-3}b^3 + \ldots + (-1)^r \times {}^nC_r\,a^{n-r}b^r + \ldots + (-1)^n\,b^r$

Properties:

i. The expansion has $(n+1)$ terms

ii. The index of 'a' in the first term is same as the index of $(a+b)$. In succeeding terms, the index of 'a' decreases by 1 and in the term, it is '0'.

iii. The index of 'b' in the first term is '0'. In succeeding terms, the index of 'b' increases by 1 and in the term, it is same as the index of $(a+b)$.

iv. The term ${}^nC_r\,a^{n-r}b^r$ is known as $(r+1)^{th}$ term or general term in the expansion and is denoted by t_{r+1}.

v. The sum of the indices of a and b in each term is same as n, the index of $(a+b)$.

vi. Since ${}^nC_r = {}^nC_{n-r}$, the coefficients equidistant from either end are equal.
E.g.: ${}^nC_0 = {}^nC_n;\ {}^nC_1 = {}^nC_{n-1};\ {}^nC_2 = {}^nC_{n-2}$, *etc.*

Questions and Answers:

1. Expand $\left(2x+y\right)^5$

   Here $a=2x,\ b=y,\ n=5$

   $$\left(2x+y\right)^5=\left(2x\right)^5+{}^5C_1\left(2x\right)^{5-1}y+{}^5C_2\left(2x\right)^{5-2}y^2+{}^5C_3\left(2x\right)^{5-3}y^3+{}^5C_4\left(2x\right)^{5-4}y^4+y^5$$
   $$=2^5x^5+5\times2^4x^4y+10\times2^3x^3y^2+10\times2^2x^2y^3+5\times2^1x^1y^4+y^5$$
   $$=32x^5+80x^4y+80x^3y^2+40x^2y^3+10xy^4+y^5$$

2. Expand $\left(2x-y\right)^5$

   $a=2x,\ b=y,\ n=5$

   $$\left(2x-y\right)^5=\left(2x\right)^5-{}^5C_1\left(2x\right)^{5-1}y+{}^5C_2\left(2x\right)^{5-2}y^2-{}^5C_3\left(2x\right)^{5-3}y^3+{}^5C_4\left(2x\right)^{5-4}y^4-y^5$$
   $$=2^5x^5-5\times2^4x^4y+10\times2^3x^3y^2-10\times2^2x^2y^3+5\times2^1x^1y^4-y^5$$
   $$=32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$$

3. Expand $\left(1-2x\right)^5$

   $$\left(1-x\right)^n=1-nC_1x+nC_2\ x^2-n\,C_3\,x^3+....$$
   $$\therefore\left(1-2x\right)^5=1-5C_1\left(2x\right)+5C_2\left(2x\right)^2-5C_3\left(2x\right)^3+5C_4\left(2x\right)^4-5C_5\left(2x\right)^5$$
   $$5C_1=5\quad\Rightarrow5C_2=\frac{5\times4}{1\times2}=10\quad\Rightarrow5C_3=5C_2=10\quad\Rightarrow5C_4=5C_1=5$$
   $$\therefore\left(1-2x\right)^5=1-5\left(2x\right)+10\left(4x^2\right)-10\left(8x^3\right)+5\left(6x^4\right)-1\times2^5x^5$$
   $$=1-10x+40x^2-80x^3+80x^4-32x^5$$

4. Using binomial theorem, indicate which number is larger $\left(1.1\right)^{10000}$ or 1000.

   $$\left(1.1\right)^{10000}=\left(1+0.1\right)^{10000}$$
   $$=1^{10000}+10000\mathrm{C}_1\left(0.1\right)+10000\mathrm{C}_2\left(0.1\right)^2+\text{Positive terms}$$
   $$=1+10000\left(0.1\right)+\text{Positive terms}$$
   $$=1+1000+\text{Positive terms}$$
   $$=1001+\text{positive terms}$$
   $$>1000$$
   $$\therefore\left(1.1\right)^{10000}>1000.$$

5. Expand 102^6

   $$=\left(100-2\right)^6=100^6-{}^6C_1\times100^5\times2+{}^6C_2\times100^4\times2^2-{}^6C_3\times100^3\times2^3+{}^6C_4\times100^2\times2^4-{}^6C_5\times100^1\times2^5+2^6$$
   $$=1000000000000-6\times10000000000\times2+15\times100000000\times4-20\times1000000\times8+15\times10000\times16-600\times32+64$$
   $$=1000000000000-120000000000+6000000000-160000000+2400000-19200+64$$
   $$=1006002400060-120160019200$$
   $$=885842380864$$

6. 98^6

$$=(100+2)^6=100^6+{}^6C_1\times 100^5\times 2+{}^6C_2\times 100^4\times 2^2+{}^6C_3\times 100^3\times 2^3+{}^6C_4\times 100^2\times 2^4+{}^6C_5\times 100^1\times 2^5+2^6$$
$$=1000000000000+6\times 10000000000\times 2+15\times 100000000\times 4+20\times 1000000\times 8+15\times 10000\times 16+600\times 32+64$$
$$=1000000000000+120000000000+6000000000+160000000+2400000+19200+64$$
$$=1126162419260$$

7. Find $(a+b)^4-(a-b)^4$. Hence evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4-\left(\sqrt{3}-\sqrt{2}\right)^4$

$$(a+b)^4-(a-b)^4$$

$$\begin{array}{lcllllll}
(a+b)^4 & = & a^4 & +{}^4C_1\times a^3\times b & +{}^4C_2\times a^2\times b & +{}^4C_3\times a\times b^3 & +b^4 \\
(a-b)^4 & = & a^4 & -{}^4C_1\times a^3\times b & +{}^4C_2\times a^2\times b & -{}^4C_3\times a\times b^3 & +b^4 \\
(-) & & (-) & (+) & (-) & (+) & (-)
\end{array}$$

$$(a+b)^4-(a-b)^4=2\times{}^4C_1\times a^3b+2\times{}^4C_3\times a\times b^3=8a^3b+4ab^3=8ab\left(a^2+b^2\right)$$

$$\therefore\ \left(\sqrt{3}+\sqrt{2}\right)^4-\left(\sqrt{3}-\sqrt{2}\right)^4=8\sqrt{3}\sqrt{2}\left[\left(\sqrt{3}\right)^2+\left(\sqrt{2}\right)^2\right]=8\sqrt{6}\,(3+2)=40\sqrt{6}$$

Note:

i) $(1+x)^n=1+{}^nC_1\,x+{}^nC_2\,x^2+\ldots+{}^nC_nx^n$

ii) $(1-x)^n=1-{}^nC_1\,x+{}^nC_2\,x^2-\ldots+(-1)^n\,.\,{}^nC_n\,x^n$

8. Show that $9^{n+1}-8n-9$ is divisible by 64, whenever n is a positive integer.

$$9^{n+1}=9^n9=9(1+8)^n=9\left[1+{}^nC_1(8)+{}^nC_2\left(8^2\right)+{}^nC_3\left(8^3\right)+\ldots\right]$$
$$\therefore\ 9^{n+1}-8n-9=9\left[1+8n+{}^nC_2\left(8^2\right)+{}^nC_3\left(8^3\right)+\ldots\right]-8n-9$$
$$=9+72n+9\times{}^nC_2\left(8^2\right)+9\times{}^nC_3\left(8^3\right)+\ldots-8n-9$$
$$=64n+9\times{}^nC_2\left(8^2\right)+9\times{}^nC_3\left(8^3\right)+\ldots$$
$$=64\left[n+9\times{}^nC_2+9\times{}^nC_3(8)+\ldots\right]$$
$=64d$, divisible by 64.

Hence proved.

9. Prove that $\sum_{r=0}^{n} 3^r.{}^nC_r = 4^n$

$$LHS = \sum_{r=0}^{n} 3^r.{}^nC_r = 3^0.{}^nC_0 + 3^1.{}^nC_1 + 3^2.{}^nC_2 + ... + 3^n.{}^nC_n$$

$$= 1 + {}^nC_1.3^1 + {}^nC_2.3^2 + ... + {}^nC_n.3^n \qquad \| \; 3^0 = 1,\; {}^nC_0 = {}^nC_n = 1$$

$$= (1+3)^n$$

$$= 4^n = RHS$$

General term in the expansion of $(a+b)^n$

The general term, $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

Middle term(s) in the expansion of $(a+b)^n$

Since the binomial expansion of $(a+b)^n$ contains $(n+1)$ terms, so

i) If n is even, then $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term (only one middle term).

ii) If n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ are the middle terms (two middle terms).

NOTE: *The following example will help you to clear the idea.*

Find the general term, 4^{th} term, coefficient of x^6 and the term independent in the expansion of $\left(x - \frac{3}{x^2}\right)^{12}$.

General term:

$$Here\; a = x,\; b = -\frac{3}{x^2},\; n = 12$$

$$t_{r+1} = {}^nC_r\, a^{n-r} b^r$$

$$= {}^{12}C_r\, x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r\,(-3)^r\, x^{12-r}\left(\frac{1}{x^2}\right)^r$$

$$= {}^{12}C_r\,(-3)^r\, x^{12-r} x^{-2r} = {}^{12}C_r\,(-3)^r\, x^{12-r-2r}$$

$$t_{r+1} = {}^{12}C_r\,(-3)^r\, x^{12-3r} \quad ....(1)$$

To find the 4^{th} term:

$$r + 1 = 4 \Rightarrow r = 4 - 1 = 3$$

$$\therefore t_{3+1} = {}^{12}C_3\,(-3)^3\, x^{12-3(3)}$$

$$t_4 = \frac{12\times11\times10}{1\times2\times3}\times(-27)x^3 = -220\times27x^3 = -5940x^3$$

To find the coefficient of x^6:

From (1), $12-3r=6 \Rightarrow 12-6=3r \Rightarrow 6=\frac{6}{3}=2$

$$\therefore t_{2+1} = {}^{12}C_2(-3)^2 x^{12-3(2)}$$

$$t_3 = \frac{12\times11}{1\times2}\times9x^6 = 594x^6$$

The term independent of x:

To get the term independent of $x, 12-3r=0 \Rightarrow 12=3r \Rightarrow r=4$

(1) Becomes:

$$\therefore t_{4+1} = {}^{12}C_4(-3)^4 x^{12-3(4)}$$

$$t_5 = \frac{12\times11\times10\times9}{1\times2\times3\times4}\times81x^0 = 495\times81 = 40095$$

The middle term:

$Here\ n=12(even)$. Hence there is only one middle term.

Middle term $=\left(\frac{12}{2}+1\right)^{th} term = 7^{th}\ term$

$r+1=7 \Rightarrow r=7-1=6$

$$\therefore t_{6+1} = {}^{12}C_6(-3)^6 x^{12-3(6)}$$

$$t_7 = \frac{12\times11\times10\times9\times8\times7}{1\times2\times3\times4\times5\times6}\times(729)x^{-6}$$

$$= 924\times729x^{-6} = \frac{673596}{x^6}$$

Problems:

1. Find the coe. of x^5 in $(x+3)^8$

$a=x, b=3, n=8$

$t_{r+1} = {}^nC_r\ a^{n-r}.b^r$

$= {}^8C_r\ x^{8-r}.3^r$

$t_{r+1} = {}^8C_r\ .3^r\ .x^{8-r}$ ....(1)

To get coe. of x^5, put $r=3$
in (1), we have,

$$t_{3+1} = {}^8C_3 .3^3 .x^{8-3}$$
$$= \frac{8\times7\times6}{1\times2\times3}\times27\times x^5$$
$$= 56\times27x^5$$
$$= 1512x^5$$

$\therefore$ coe. of $x^5 = 1512$

2. Find the coe. of a^5b^7 in $(a-2b)^{12}$

$a=a, b=-2b; n=12$

$$t_{r+1} = {}^nC_r \ a^{12-r}(-2b)^r$$
$$= {}^{12}C_r.a^{12-r}(-2)^r.b^r$$
$$t_{r+1} = {}^{12}C_r(-2)^r.a^{12-r}.b^r \quad ....(1)$$

To get the coe. of a^5b^7, put $r=7$
in $(1) \Rightarrow$

$$t_{7+1} = {}^{12}C_7(-2)^7.a^{12-7}.b^7$$
$$= {}^{12}C_7(-2)^7.a^5.b^7$$
$$= 792\times-128\ a^5b^7$$

$\therefore$ coe. of $a^5b^7 = -101376$ $\qquad \| 12C_7 = 12C_5 = \frac{12\times11\times10\times9\times8}{1\times2\times3\times4\times5} = 11\times72 = 792$

3. Write the general term in the expansion of $(x^2-y)^6$

$a=x^2, b=-y, n=6$

$$The\ general\ term = t_{r+1} = {}^nC_r\ a^{n-r}.b^r$$
$$= {}^6C_r.(x^2)^{n-r}(-y)^r$$
$$= {}^6Cr(-1)^r.x^{12-2r}.y^r$$

4. Write the general term in the expansion of $(x^2-y^2)^{12}, x\neq0$

$a=x^2, b=-yx, n=12$

$The\ general\ term, t_{r+1} = {}^nCr\ a^{n-r}.b^r$

$$t_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} (-yx)^r = {}^{12}C_r\ x^{24-2r} (-1)^r\ y^r x^r$$
$$= {}^{12}C_r (-1)^r\ x^{24-r} y^r = {}^{12}C_r (-1)^r\ x^{24-r} y^r$$

5. *Find the* 4^{th} *term in the expantion of* $(x-2y)^{12}$

$a = x, \quad b = -2y, n = 12$

$$t_{r+1} = {}^nC_r\ a^{n-r}\ b^r$$
$$t_{r+1} = {}^{12}C_3\ x^{12-3}.(-2y)^3$$
$$= {}^{12}C_3.x^9.(-2)^3.y^3$$
$$= 220 \times x^9 (-8).y^3 = -1760\ x^9\ y^3 \qquad \| \ {}^{12}C_3 = \frac{12\times 11\times 10}{1\times 2\times 3} = 2\times 11\times 10 = 220$$

6. *Find the* 13^{th} *term in the expansion of* $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0?$

$a = 9x\ , b = -\frac{1}{3\sqrt{x}}, n = 18$

$t_{r+1} = {}^nCr\ \ a^{n-r}\ \ b^r$

$$t_{12+1} = {}^{18}C_{12}.(9x)^{18-12}\left(\frac{-1}{3\sqrt{x}}\right)^{12}$$

$$t_{13} = {}^{18}C_6.\ (9x)^6.\frac{(-1)^{12}}{3^{12}\left(\sqrt{x}\right)^{12}}$$

$$= {}^{18}C_6 \times 9^6 \times x^6.\frac{1}{9^6.x^6}$$

$\therefore t_{13} = 18564$

$$18C_6 = \frac{18\times 17\times 16\times 15\times 14\times 13}{1\times 2\times 3\times 4\times 5\times 6}$$

$(-1)^{12} = 1$

$$\left(\sqrt{x}\right)^{12} = \left(x^{\frac{1}{2}}\right)^{12} = x^{\frac{1}{2}\times 12} = x^6$$

$$3^{12} = \left(3^2\right)^6 = 9^6$$

7. *Find the middle terms in the expantion of* $\left(3 - \frac{x^3}{6}\right)^7$

Here $n = 7$ *(odd). Hence, there are 2 middle terms. They are* $\left(\frac{7+1}{2}\right)^{th}$ *term and*

$\left(\frac{7+1}{2}+1\right)^{th}$ *terms. i.e.,* 4^{th} *term and* 5^{th} *term.*

$a=3\,;b=-\frac{x3}{6};\,n=7$

$t_{r+1}={}^{n}Cr\ a^{n-r}.b^{r}$

$$t_{3+1}={}^{7}C_{3}\times 3^{7-3}\times\left(-\frac{x^{3}}{6}\right)^{3}$$

$$=35\times 3^{4}.\frac{-x^{9}}{6^{3}}$$

$$=\frac{35\times 81\times -x^{9}}{216}$$

$$=-\frac{105}{8}x^{9}$$

Again

$$t_{4+1}={}^{7}C_{4}\times 3^{7-4}\times\left(-\frac{x^{3}}{6}\right)^{4}$$

$$={}^{7}C_{3}\times 3^{3}\times\frac{\left(-x^{3}\right)^{4}}{6^{4}} \qquad \| 7C_{4}=7C_{7-4}=7C_{3}=\frac{7\times 6\times 5}{1\times 2\times 3}=35$$

$$={}^{7}C_{3}\times 3^{3}\times\frac{x^{12}}{3^{4}\times 2^{4}}=\frac{35}{3\times 16}x^{12}=\frac{35}{48}x^{12}$$

8. *Find the middle terms in the expantion of* $\left(\frac{x}{3}+9y\right)^{10}$

Here $n=10(even)$, *there is only are middle term and middle term* $=\left(\frac{10}{2}+1\right)th=(5+1)th=6th$ *term.*

$a=\frac{x}{3},\,b=9y\quad,n=10$

$t_{r+1}={}^{n}C_{r}\ a^{n-r}.\,b^{r}$

$$t_{5+1}={}^{10}C_{5}\times\left(\frac{x}{3}\right)^{10-5}\times(9y)^{5}$$

$$=252\times\frac{x^{5}}{3^{5}}\times 9^{5}\times y^{5}$$

$$=252\times\frac{x^{5}}{3^{5}}.\,3^{10}\times\ y^{5}=252\times 3^{5}\times x^{5}y^{5}$$

$$=252\times 243\ x^{5}y^{5}=61236\ \ x^{5}y^{5}$$

9. $In\ the\ expantion\ of\ (1+a)^{m+n}$, $using\ binomial\ therom, prove\ that\ coefficient\ of\ a^m\ and\ a^n\ are\ equal.$

We have, $(1+a)^{m+n} = {}^{m+n}C_0 + {}^{m+n}C_1\, a + {}^{m+n}C_2\, a^2 + ... + {}^{m+n}C_r\, a^r + ... + a^{m+n}$

$$Coe.\ of\ a^m = {}^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!\ n!}$$

$$Coe.of\ a^n = {}^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!\ m!} = \frac{(m+n)!}{m!\ n!}$$

$\therefore\ Coe.\ of\ a^m\ and\ coe.of\ a^n\ are\ equal.$

10. $The\ coefficient\ of\ the\ (r-1)^{th}, r^{th}\ and\ (r+1)^{th}\ terms\ of\ (x+1)^n\ are\ in\ the\ ratio\ 1:3:5.\ Find\ both\ n\ and\ r$

$Let\ the\ coe.of\ (r-1)^{th}, r^{th}\ and\ (r-1)^{th}\ terms\ be\ {}^nC_{r-2}\ and\ {}^nC_{r-2}, nC_{r-1}\ and\ {}^nC_r\ in\ that$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3} \quad and \quad \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

$$\Rightarrow \frac{\frac{n!}{(r-2)![n-(r-2)]!}}{\frac{n!}{(r-1)![n-(r-1)]!}} = \frac{1}{3} \quad and \quad \frac{\frac{n!}{(r-1)![n-(r-1)]!}}{\frac{n!}{r!(n-r)!}} = \frac{3}{5}$$

$$\frac{\frac{n!}{(r-2)!(n-r+2)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{1}{3} \quad and \quad \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{3}{5}$$

$$\frac{(r-1)!(n-r+1)!}{(r-2)!(n-r+2)!} = \frac{1}{3} \quad and \quad \frac{r!(n-r)!}{(r-1)!(n-r+1)!} = \frac{3}{5}$$

$$\frac{(r-2)!(r-1)(n-r+1)!}{(r-2)!(n-r+1)!(n-r+2)} = \frac{1}{3} \quad and \quad \frac{(r-1)!r(n-r)!}{(r-1)!(n-r)!(n-r+1)} = \frac{3}{5}$$

$$\frac{(r-1)}{(n-r+2)}=\frac{1}{3} \quad \text{and} \quad \frac{r}{(n-r+1)}=\frac{3}{5}$$

$$3(r-1)=1(n-r+2) \quad \text{and} \quad 5r=3(n-r+1)$$

$$3r-3=n-r+2 \quad \text{and} \quad 5r=3n-3r+3$$

$$3r-3-n+r-2=0 \quad \text{and} \quad 8r-3n-3=0 \ ....(2)$$

$$4r-n-5=0 \ ....(1)$$

$$(1)\times 3-(4)$$

$$\begin{array}{r} 12r-3n-15=0 \\ 8r-\ 3n-\ 3=0 \\ (-)\quad (+)\quad (+) \\ \hline 4r \qquad -12=0 \end{array}$$

$$4r=12$$

$$r=\frac{12}{4}=3$$

$$in(4)$$

$$8\times 3-3n=3$$

$$-3n=3-24 \Rightarrow -3n=-21 \Rightarrow n=7$$

You should practice example questions as well as miscellaneous for the preparation of entrance examinations.