Standard IX

MATHEMATICS

PART- 2

Government of Kerala
Department of General Education

State Council of Educational Research and Training (SCERT), Kerala
2019

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THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear children,

Man invented various types of numbers to understand the world through measurements and the relations between measures. You have already seen how natural numbers and fractions evolved like this and how their operations were defined based on the physical contexts in which they were used. In this book, you can get acquainted with measures which cannot be indicated by natural numbers or fractions and the new kind of numbers used to represent them.

The study of geometry also continues in this book. We discuss the relations between parallel lines, triangles and circles. We have explained how new geometric theorems and applications arise from the recognition of such relations. We have also described how the program **GeoGebra** can be used to present geometry in a dynamic manner. More material are made available through the Samagra portal and QR codes.

With love and regards

Dr. J. Prasad
Director, SCERT

CONSTITUTION OF INDIA

Part IV A

FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

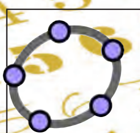
(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

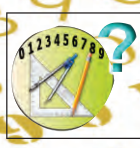
Contents

Certain icons are used in this textbook for convenience


Computer Work

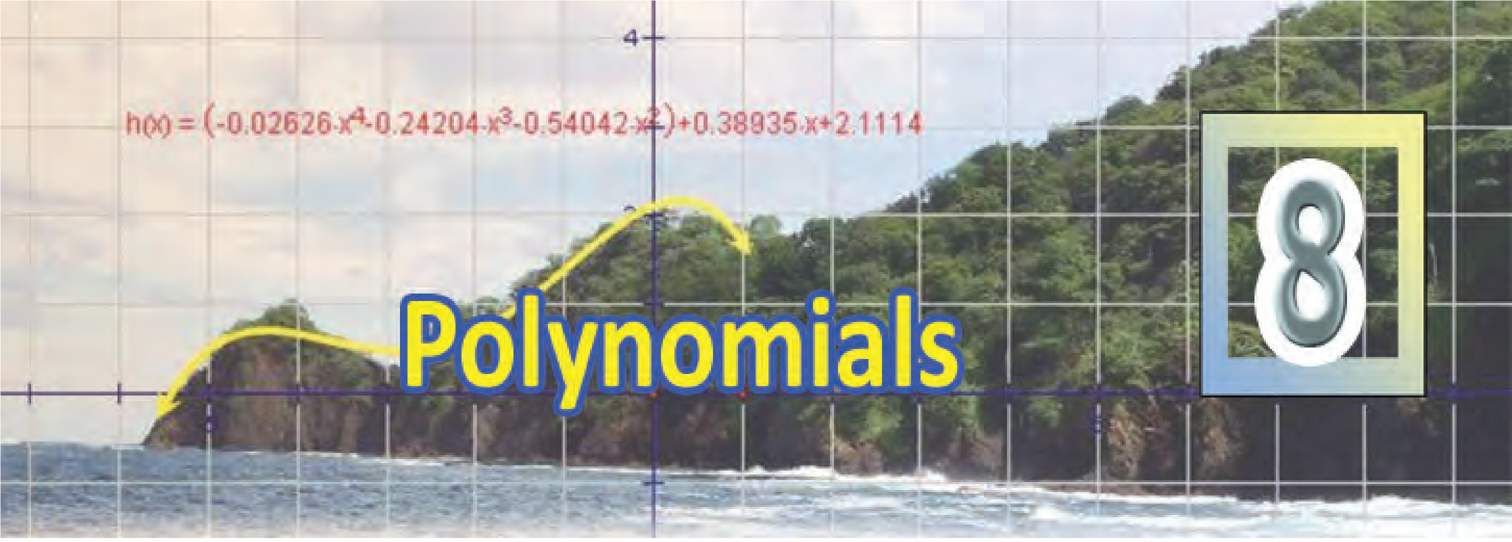

Additional Problems


Project



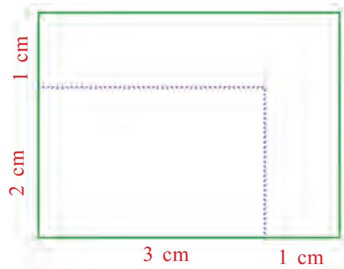
For Discussion


NSQF



Algebra of Measures

The sides of a rectangle are 2 and 3 centimetres and they are extended by 1 centimetre to make a new rectangle:



What is the perimeter of the new rectangle?

Its sides are 3 and 4 centimetres and so perimeter is 14 centimetres.

We can do this in a different way:

The perimeter of the original rectangle is 10 centimetres. All four sides are increased by 1 centimetre; so the total increase is 4 centimetres. The new perimeter is $10 + 4 = 14$ centimetres.

What if each side is extended by 2 centimetres? Using the second line of thought, each side is increased by 2 centimetres, the total increase is 8 centimetres and so the new perimeter is $10 + 8 = 18$ centimetres.

This computation is quicker, isn't it? If each side is extended by $2\frac{1}{2}$ centimetres, the perimeter of the new rectangle is.

$$\left(4 \times 2\frac{1}{2}\right) + 10 = 20 \text{ centimetres}$$

Thus we can see that, the perimeter of the new rectangle is 10 added to 4 times the extension of sides.

Let's put this in algebra; if the extension of each side is taken as x centimetres and the perimeter of the new rectangle as p centimetres, then

$$p = 4x + 10$$

Now we can quickly find the new perimeter on extending by different lengths:

If extension is 3 centimetres, perimeter is 22 centimetres.

If extension is $3\frac{1}{2}$ centimetres, perimeter is 24 centimetres.

If extension is $3\frac{3}{4}$ centimetres, perimeter is 25 centimetres.

We can shorten this using algebra:

$$\text{If } x = 3, \quad \text{then } p = 22$$

$$\text{If } x = 3\frac{1}{2}, \quad \text{then } p = 24$$

$$\text{If } x = 3\frac{3}{4}, \quad \text{then } p = 25$$

There is an algebraic way to shorten it further.

$$p(3) = 22$$

$$p\left(3\frac{1}{2}\right) = 24$$

$$p\left(3\frac{3}{4}\right) = 25$$

In general, we write:

$$p(x) = 4x + 10$$

Let's look at this shorthand in a little more detail. First, we write our finding in ordinary language:

If each side of a rectangle of sides 2 and 3 centimetres is extended by the same length to make a larger rectangle, then the perimeter of this new rectangle is four times the extension added to ten. For example, if each side is extended by one and a half centimetres, the perimeter is sixteen centimetres.

We can shorten this using algebra:

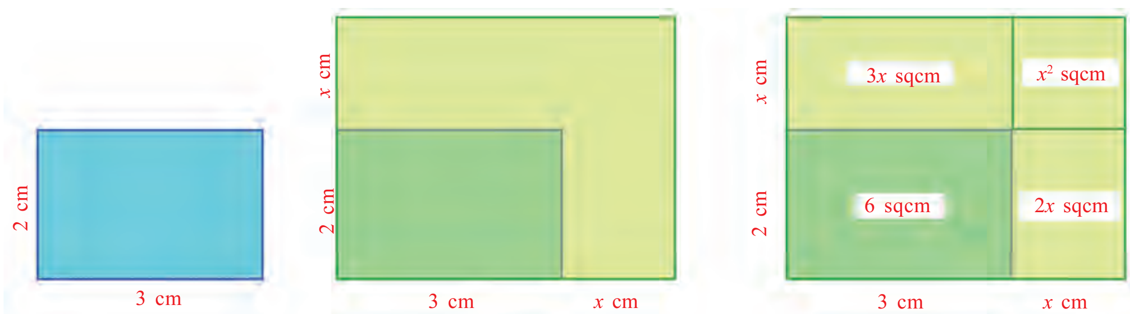
If each side of a rectangle of sides 2 and 3 centimetres is extended by x centimetres to make a larger rectangle, and the perimeter of the new rectangle is p centimetres, then $p = 4x + 10$.

For example, if $x = 1\frac{1}{2}$, then $p = 16$.

To emphasise the fact that p changes according to x in this, we write $p(x)$ instead of just p. So we can rewrite our second statement like this:

If each side of a rectangle of sides 2 and 3 centimetres is extended by x centimetres, then the perimeter of the new rectangle is given by $p(x) = 4x + 10$. For example, $p\left(1\frac{1}{2}\right) = 16$.

Now let's see how the area changes in this. Instead of noting how the area changes for different extensions and then generalising, let's straightaway start by taking the extension of each side as x centimetres.



From the pictures, the new area is

$$6 + 2x + 3x + x^2 = 6 + 5x + x^2$$

(Recall the lesson, **Identities** of Class 8).

As in the case of perimeters, we write $a(x)$ to denote the area on extending each side by x centimetres. Then

$$a(x) = x^2 + 5x + 6$$

From this, we find

$$a(1) = 1 + 5 + 6 = 12$$

$$a\left(1\frac{1}{2}\right) = 2\frac{1}{4} + 7\frac{1}{2} + 6 = 15\frac{3}{4}$$

$$a(2) = 4 + 10 + 6 = 20$$

These can be written in usual language like this:

If each side is extended by 1 centimetre, the area is 12 square centimetres.

If each side is extended by $1\frac{1}{2}$ centimetres, the area is $15\frac{3}{4}$ square centimetres.

If each side is extended by 2 centimetres, the area is 20 square centimetres.

As another example, let's see how the volume of a rectangular block of sides 1, 2, 3 centimetres changes when each side is extended by the same length to make a larger block. Taking the extension as x centimetres, the volume of the larger block is $(x + 1)(x + 2)(x + 3)$ cubic centimetres. To expand this, we first write as before,

$$(x + 2)\ (x + 3) = x^2 + 5x + 6$$

Now this is to be multiplied by $x + 1$. For this, we have to multiply each of the three numbers of the first sum by each of the two numbers in the second sum and add:

$$(x + 1)\ (x^2 + 5x + 6) = x^3 + 5x^2 + 6x + x^2 + 5x + 6 = x^3 + 6x^2 + 11x + 6$$

Thus we can write what we have found as follows:

If each side of a rectangular block of sides 1, 2, 3 centimetres is increased by x centimetres and if the volume of the larger block is written as $v(x)$ cubic centimetres, then $v(x) = x^3 + 6x^2 + 11x + 6$.

Let's look at a different situation : We have seen that if an object is thrown straight up with a speed of 49 metres/second, then its speed decreases at the rate of 9.8 metres/second, every second and that the speed becomes 0 at 5 seconds and thereafter it falls down with the speed increasing at the rate of 9.8 metres/second (The section **Negative speed** of the lesson, **Negative Numbers** in Class 8). We also know the equation giving the relation between time and speed. If we write the speed at x second as $s(x)$ metres/second, then

$$s(x) = 49 - 9.8x$$

We can compute the speed at different times using this:

time x	0	1	2	3	4	5	6	7	8	9	10
speed $s\ (x)$	49	39.2	29.4	19.6	9.8	0	–9.8	–19.6	–29.4	–39.2	–49

8JBZIS

Can you explain mathematically why in the second row, the negatives of the same numbers occur on either side of 0? What is the physical explanation?

(1) In rectangles with one side 1 centimetre shorter than the other, take the length of the shorter side as x centimetres.

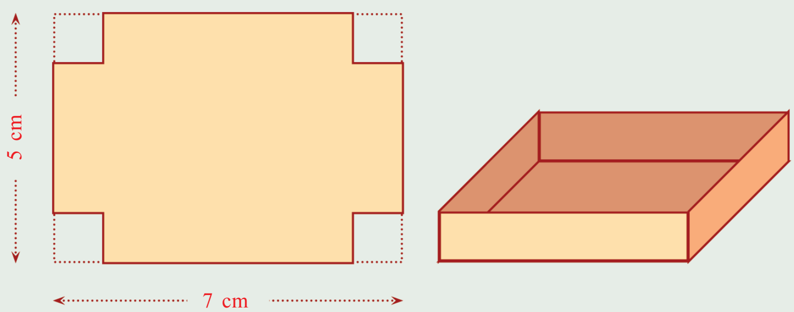
i) Taking their perimeters as $p\ (x)$ centimetres, write the relation between $p\ (x)$ and x as an equation.

ii) Taking their areas as $a\ (x)$ square centimetres, write the relation between $a\ (x)$ and x as an equation.

iii) Calculate $p\ (1)$, $p\ (2)$, $p\ (3)$, $p\ (4)$, $p\ (5)$. Do you see any pattern?

iv) Calculate $a\ (1)$, $a\ (2)$, $a\ (3)$, $a\ (4)$, $a\ (5)$. Do you see any pattern?

(2) From the four corners of a rectangle, small squares are cut off and the sides are folded up to make a box, as shown below:



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i) Taking a side of the square as x centimetres, write the dimensions of the box in terms of x.

ii) Taking the volume of the box as $v\ (x)$ cubic centimetres, write the relation between $v\ (x)$ and x as an equation.

iii) Calculate $v\left(\frac{1}{2}\right)$, $v\ (1)$, $v\left(1\frac{1}{2}\right)$.

(3) Consider all rectangles that can be made with a 1 metre long rope. Take one of its sides as x centimetres and the area enclosed as $a(x)$ square centimetres.

i) Write the relation between $a(x)$ and x as an equation.

ii) Why are the numbers $a(10)$ and $a(40)$ equal?

iii) To get the same number as $a(x)$, for two different numbers x, what must be the relation between the numbers?

Special expressions

We have seen many examples of the relation between different quantities written as algebraic equations. These can also be seen as operations on pure numbers. For example, in our first rectangle problem, the relation between the extension of sides and the new perimeter was

$$p(x) = 4x + 10$$

More than a method of calculating perimeters of rectangles, this can be seen as the operation of multiplying numbers by 4 and adding 10. Similarly, we can examine some of the relations found earlier:

- $a(x) = x^2 + 5x + 6$
- $v(x) = x^3 + 6x^2 + 11x + 6$
- $s(x) = 49 - 9.8x$

Looking at them as just operations on numbers, we note some common features. In all these, the only mathematical operations involved are multiplying different powers of the number x by various numbers and adding or subtracting them. A definite number other than x is also sometimes added or subtracted. An algebraic expression involving only such operations is called a *polynomial*.

There are instances where we do operations other than these on numbers. For example, let's consider the diagonals of rectangles of one side 1 centimetre longer than the other.

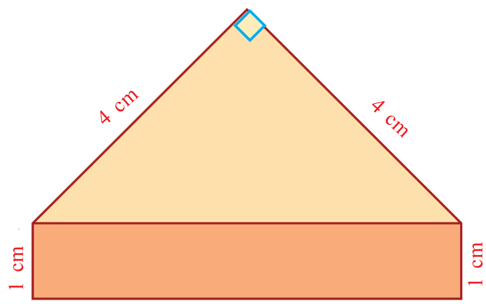
Let's see how we can show in GeoGebra the box made by cutting squares off the corners of a rectangle. Make a slider a with **Min = 0** and **Max = 2.5**. Draw a rectangle of sides 7–2a and 5–2a, Next open the **3D Graphics window** (View → 3D Graphics), we can see our rectangle in it. Choose **Extrude to Prism or Cylinder** and click on the rectangle. Give the height of the box as a . The volume can be marked using **Volume**. Move the slider to change a and see how the box and its volume change.

Taking the shorter side as x centimetres, the diagonal is

$$\sqrt{x^2 + (x+1)^2} = \sqrt{2x^2 + 2x + 1} \text{ centimetres.}$$

Since this involves the operation of extracting square roots, it is not a polynomial, according to our definition.

See this figure:



A rectangle is attached to the hypotenuse of an isosceles right triangle. What is the area of this figure?

The area of the triangle is easily seen as 8 square centimetres. Since the longer side of the rectangle is the hypotenuse of the triangle, it is $4\sqrt{2}$ centimetres. So, the area of the rectangle is $4\sqrt{2}$ square centimetres.

Total area is $8 + 4\sqrt{2}$ square centimetres.

What if the length of the perpendicular sides of the triangle is some other number? Taking this length as x centimetres, the area of the shape in square centimetres is

$$\frac{1}{2}x^2 + \sqrt{2}\,x$$

This has the square root of 2 in it. But the operations on the changing number x involve only squaring and multiplication by the fixed numbers $\frac{1}{2}$ and $\sqrt{2}$.

So this is also a polynomial.

Let's take another example. If the length of one side of a rectangle of area 25 square centimetres is taken as x centimetres, then its perimeter in centimetres is

$$2x + \frac{50}{x}$$

This involves the operation of taking reciprocal of the changing number and so is not a polynomial.

In a polynomial, we take powers of the changing numbers. The largest power used is called the *degree* of the polynomial. So in the list of polynomials given above, the first is of degree 2, the second is of degree 3 and the third is of degree 1.

Instead of saying polynomial of degree 1, polynomial of degree 2, and so on, we can say first degree polynomial, second degree polynomial and so on.

Based on the degree , we can write the general forms of all polynomials.

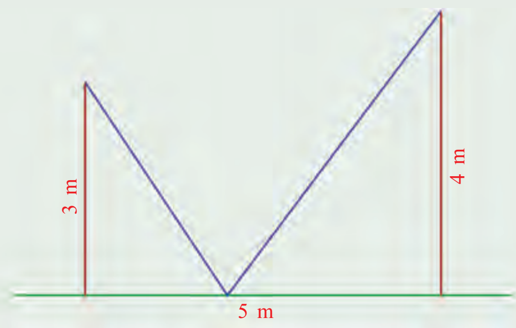
First degree polynomial	:	$ax + b$
Second degree polynomial	:	$ax^2 + bx + c$
Third degree polynomial	:	$ax^3 + bx^2 + cx + d$

Here the letters a, b, c and d denote fixed numbers. That is, in a specific polynomial we take different numbers as x, but a, b, c, d are fixed as specific numbers. They can be any sort of numbers; natural numbers, fractions, numbers which are not fractions or negative numbers. They are called *coefficients* in a polynomial.

(1) Write each of the relations below in algebra and see if it gives a polynomial. Also give reasons for your conclusion.

i) A 1 metre wide path goes around a square ground. The relation between the length of a side of the ground and the area of the path.

ii) A liquid contains 7 litres of water and 3 litres of acid. More acid is added to it. The relation between the amount of acid added and the change in the percentage of acid in the liquid.



iii) Two poles of heights 3 metres and 4 metres are erected upright on the ground, 5 metres apart. A rope is to be stretched from the top of one pole to some point on the ground and from there to the top of the other pole:

The relation between the distance of the point on the ground from the foot of a pole and the total length of the rope.

(2) Write each of the operations below as an algebraic expression, find out which are polynomials and explain why.

i) Sum of a number and its reciprocal.

ii) Sum of a number and its square root.

iii) Product of the sum and difference of a number and its square root.

(3) Find $p(1)$ and $p(10)$ in the following polynomials,

i) $p(x) = 2x + 5$

ii) $p(x) = 3x^2 + 6x + 1$

iii) $p(x) = 4x^3 + 2x^2 + 3x + 7$

(4) Find $p(0)$, $p(1)$ and $p(-1)$ in the following polynomials,

i) $p(x) = 3x + 5$

ii) $p(x) = 3x^2 + 6x + 1$

iii) $p(x) = 2x^2 - 3x + 4$

iv) $p(x) = 4x^3 + 2x^2 + 3x + 7$

v) $p(x) = 5x^3 - x^2 + 2x - 3$

(5) Find polynomials $p(x)$ satisfying each set of conditions below.

i) First degree polynomial with $p(1) = 1$ and $p(2) = 3$.

ii) First degree polynomial with $p(1) = -1$ and $p(-2) = 3$.

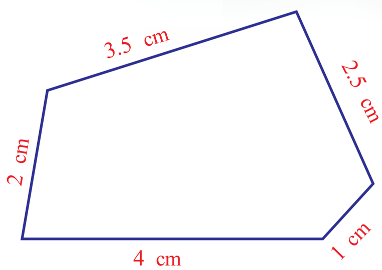
iii) Second degree polynomial with $p(0) = 0$, $p(1) = 2$ and $p(2) = 6$.

iv) Three different second degree polynomials with $p(0) = 0$ and $p(1) = 2$.

9 Circle Measures

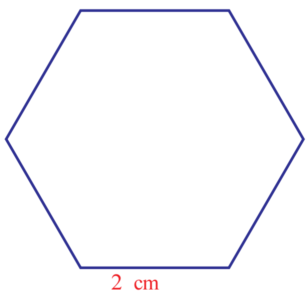
Circle and Polygons

It is easy to calculate the perimeter of a polygon; just add the lengths of sides:



Perimeter $4 + 1 + 2.5 + 3.5 + 2 = 13$ centimetres

For a regular polygon; it's easier still:

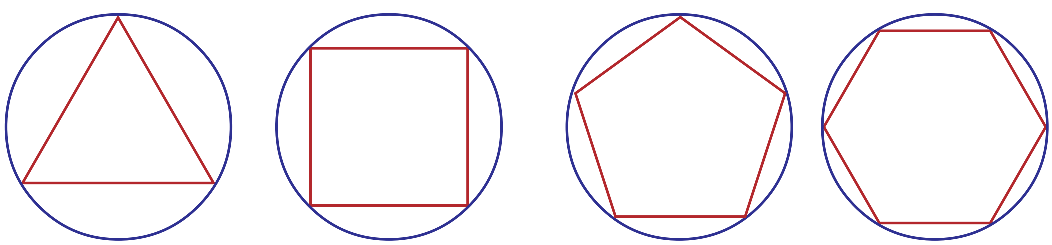


Perimeter $6 \times 2 = 12$ centimetres

What about a circle?

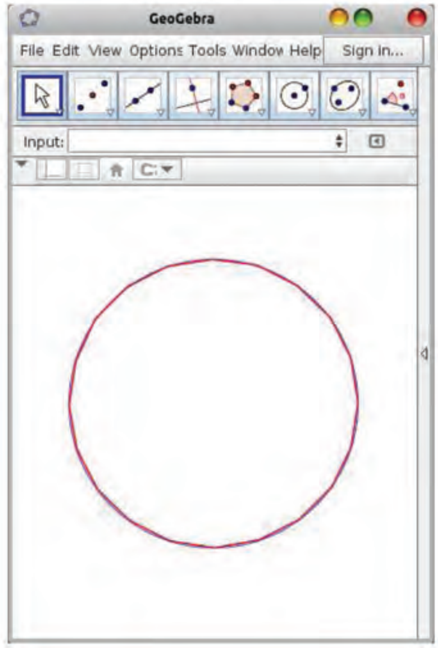
We can place a string around and measure its length. But the real interest of mathematics is calculation instead of actual measurement.

See these pictures:



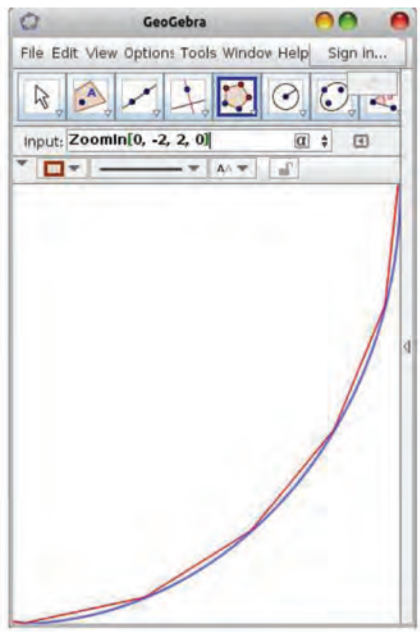
As the number of sides increases, the regular polygon inside the circle gets closer to the circle, right?

This picture shows a regular polygon of 20 sides drawn inside a circle, using GeoGebra. We can hardly distinguish the polygon from the circle.



We can draw regular polygons in a circle easily in GeoGebra. Make an integer slider n with **Min = 3**, **Max = 100**. Draw a circle and mark a point on it. Use **Angle of Given Size** to make an angle of $\left(\frac{360}{n}\right)^{\circ}$ at the centre. We get another point on the circle. Choose **Regular Polygon** and click on the two points on the circle and give the number of sides as n. We get a polygon of n sides. Choose **Distance or Length** and click inside the polygon to get its perimeter. Do this in a circle of radius $\frac{1}{2}$. What happens to the perimeter when n is increased?

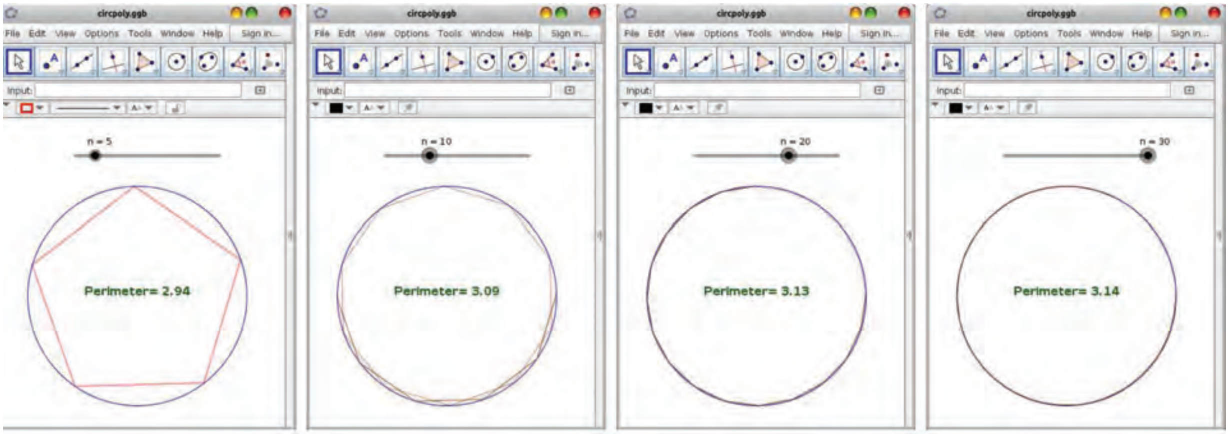
This shows a part of the above picture zoomed in :



Thus however much we increase the number of sides, a polygon never becomes a circle; it gets closer and closer.

Anyway, the perimeters of regular polygons inside the circle with increasing number of sides get closer and closer to the perimeter of the circle. From very early times, mathematicians used this method to measure the perimeter of a circle.

Now we can use a computer to do the calculation, if we give correct mathematical instructions for doing them. See how GeoGebra is used to compute the perimeters of regular polygons of sides 5, 10, 20, 30 drawn inside a circle of diameter 1:



We can see that these numbers get closer and closer to the perimeter of a circle of diameter 1 (centimetre, metre or any other unit). Then some questions arise:

- To which number do the numbers 2.94, 3.09, 3.13, 3.14, . . . get closer and closer?
- Once we get this number, how do we calculate the perimeters of circles of diameter other than 1?

8JURH3

Let's answer the second question first.

But let's do some problems before that.

(1) Prove that the circumcentre of an equilateral triangle is the same as its centroid.

i) Calculate the length of a side of an equilateral triangle with vertices on a circle of diameter 1 centimetre.

ii) Calculate the perimeter of such a triangle.

(2) Calculate the perimeter of a square with vertices on a circle of diameter 1 centimetre.

(3) Calculate the perimeter of a regular hexagon with vertices on a circle of diameter 1 centimetre.

Polygons

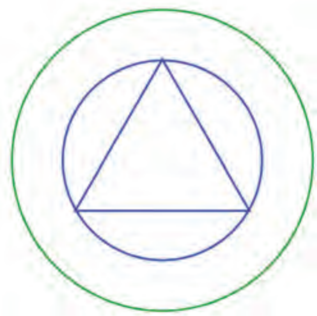
We can see comparisons of the perimeters of squares and hexagons with the perimeters of a circle from very early times. For example, a clay tablet of Babylon, estimated to be made around 1600 BC, says that the perimeter of a regular hexagon with vertices on a circle is $\frac{57}{60} + \frac{36}{60^2}$ that is $\frac{24}{25}$ of the perimeter of the circle. And it is a good estimate. The idea that instead of comparing individual polygons with a circle, the number of sides can be progressively increased to approximate a circle, originated in Greece. It was first proposed by Antiphon in the fifth century BC and made more precise by Eudoxus in the fourth century. A mathematical method based on this to compute the perimeter and area of a circle was developed in the third century by Archimedes, one of the greatest scientists of all time.

Diameter and perimeter

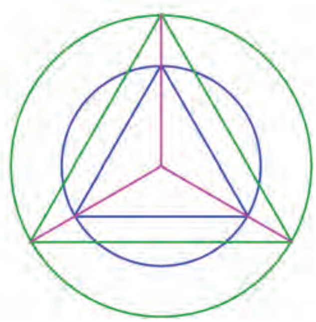
See this picture:

An equilateral triangle is drawn inside a circle.

With the same centre, draw a larger circle:



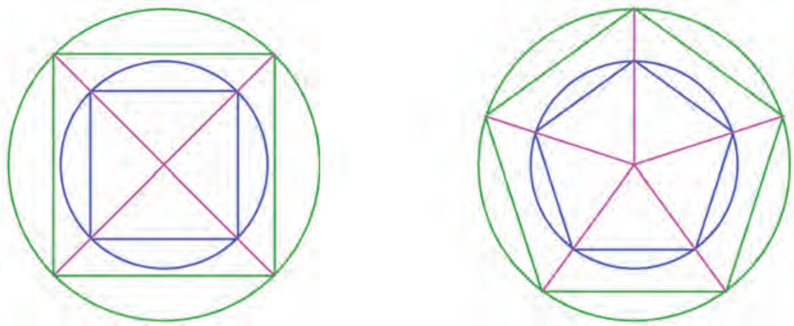
Extend the lines joining the centre of the circle and the vertices of the triangle, to meet the larger circle. Join these points to make a larger triangle:



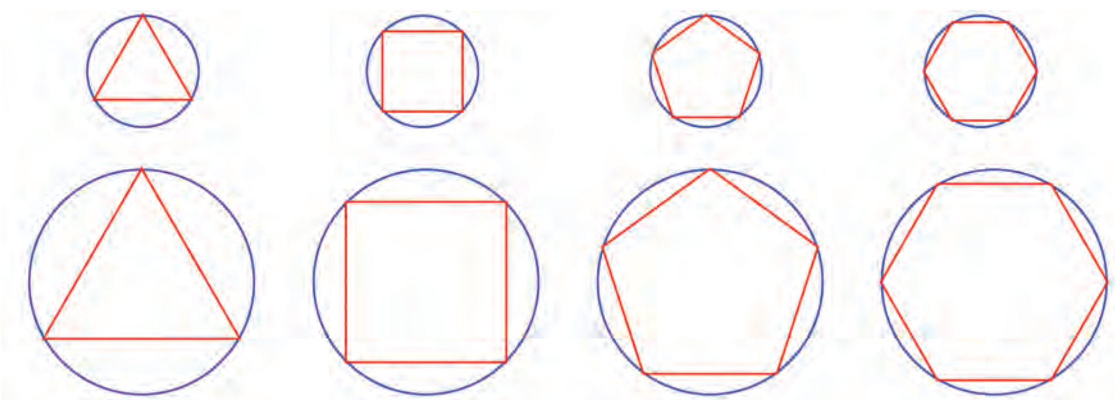
Now the sides of the two triangles are scaled by the same factor as the radii of the circles (the second problem at the end of the section, **The third** way in the chapter, **Similar Triangles**).

So the sides of the two triangles, and hence their perimeters, are in the same ratio as the radii of the circles; and the ratio of the diameters is the same as the ratio of the radii.

Instead of triangles if we take any polygon, we can divide it into triangles as we did to see that the ratio of their perimeters is the same as the ratio of the diameters.



Now suppose we draw such regular polygons inside a circle and inside another circle of double the diameter:



The perimeters of the polygons within each circle get closer and closer to the perimeter of that circle. Since the diameter of the large circle is double that of the small circle, the perimeters of the polygons within the large circle are double the perimeters of the polygons in the small circle.

Let's look at these in terms of numbers. Let's take the perimeter of the triangle inside the small circle as p_1, the perimeter of the square as p_2, that of the pentagon as p_3 and so on, and the perimeter of the small circle as c. Then the numbers $p_1, p_2, p_3, \ldots$ get nearer and nearer to c.

Also, the perimeters of the polygons inside the large circle are $2p_1, 2p_2, 2p_3, \ldots$

In GeoGebra, make a slider a and two integer sliders m and n. Draw two circles with radii a and ma. Draw regular polygons with sides n in these circles and mark their perimeter. When m = 2 (radius of the larger circle twice the radius of smaller circle), what is the relation between the perimeters? Change the number of sides. What if m=3? Do these relations hold, whatever be the radius of the first circle? Change a and see.

Since $p_1, p_2, p_3, \ldots$ get nearer and nearer to c, the numbers $2p_1, 2p_2, 2p_3$, ... get nearer and nearer to $2c$; that is, twice the perimeter of the small circle.

Thus, geometrically we see that the perimeters of the large polygons get closer and closer to the perimeter of the large circle; thinking in terms of numbers, we find that they get nearer and nearer twice the perimeter of the small circle. So, the perimeter of the large circle must be twice the perimeter of the small circle.

If the diameter of the second circle is scaled by some other factor instead of 2, then by the same reasoning we can see that the perimeters are scaled by the same factor.

The perimeters of circles are scaled by the same factor as their diameters.

We can also put it this way:

The perimeters of circles are in the same ratio as their diameters.

So, if we find the perimeter of a circle of diameter 1, then we can calculate the perimeter of any circle, by multiplying the diameter by this number.
Thus we have the answer to the second question asked in the first section.

(1) The perimeter of a regular hexagon with vertices on a circle is 24 centimetres.

i) What is the perimeter of a square with vertices on this circle?

ii) What is the perimeter of a square with vertices on a circle of double the diameter?

iii) What is the perimeter of an equilateral triangle with vertices on a circle of diameter half that of the first circle?

(2) A wire was bent into a circle of diameter 4 centimetres. What would be the diameter of a circle made by bending a wire of half the length?

(3) The perimeter of a circle of diameter 2 metres was measured and found to be about 6.28 metres. How do we compute the perimeter of a circle of diameter 3 metres, without measuring?

A new number

Let's now take up the first question of what the perimeter of a circle of diameter 1 actually is.

As we saw in the first section, if we can compute the perimeters of regular polygons with vertices on such a circle using GeoGebra, we get the decimal forms of numbers approximately equal to the perimeter of the circle. Usually, GeoGebra calculates numbers upto two decimal places, but we can increase it to fifteen places. (Options → Rounding). Here is what we get up to four decimal places.

Sides	Perimeter	Sides	Perimeter
3	2.5981	15	3.1187
4	2.8284	20	3.1287
5	2.9389	25	3.1333
6	3.0000	30	3.1359
7	3.0372	35	3.1374
8	3.0615	40	3.1384
9	3.0782	45	3.1390
10	3.0902	50	3.1395

Thus we see that the perimeter of a circle of diameter 1 is a number approximately equal to 3.14.

We cannot express the perimeter of a circle of diameter 1 as a fraction, as in the case of the diagonal of a square of side 1.

But it is not easy to prove, as in the case of the diagonal of a square. It was proved only in the 18th century.

There is a basic difference between this number and numbers like $\sqrt{2}$ or $2 + \sqrt{3}$. It cannot be computed using roots of rational numbers. A special symbol is used in mathematics to denote it : π

It is the Greek letter read as "pi"

Thus the perimeter of a circle of diameter 1 is π centimetres, the perimeter of a circle of diameter 2 centimetres is 2π centimetres, the perimeter of a circle of diameter $1\frac{1}{2}$ centimetres is $\frac{3}{2}\pi$ centimetres and so on.

8K4HNQ

In short,

The perimeter of a circle is π times its diameter.

Since we usually draw circles of specified radius, this is often stated in terms of the radius.

The perimeter of a circle is 2π times its radius.

Story of the name

Once it was discovered that the perimeters of circles are scaled by the same factor as their diameters, it was realised that the perimeters of all circles are the same multiple of their diameters. The next quest was to find this multiplier. In olden times, fractions approximating this number were used. In different times and in different places, these approximations were made better and better. Though it was proved only much later that this number cannot be expressed as a fraction, it must have been recognised quite early.

This number was named π by the (not very famous) English mathematician, William Jones in 1707AD.

It became popular and finally fixed only after the great mathematician Leonhard Euler, born in Switzerland started using it in his writings.

Since π is not a fraction, we can only compute fractions approximately equal to π. In the third century BC, Archimedes of Greece used a polygon of 96 sides to show that the perimeter of a circle is larger than $3\frac{10}{71}$ of the diameter and less than $3\frac{1}{7}$ of the diameter. In modern terms, this means, up to four decimal places,

$$3.1408 < \pi < 3.1428$$

(The number $3\frac{1}{7} = \frac{22}{7}$ determined by Archimedes was used for a long time to compute the perimeter of a circle.)

In the fourteenth century AD, Madhavan of Kerala found a purely numerical technique to compute π to any degree of accuracy, without geometry.

Using this, we can compute, for example,

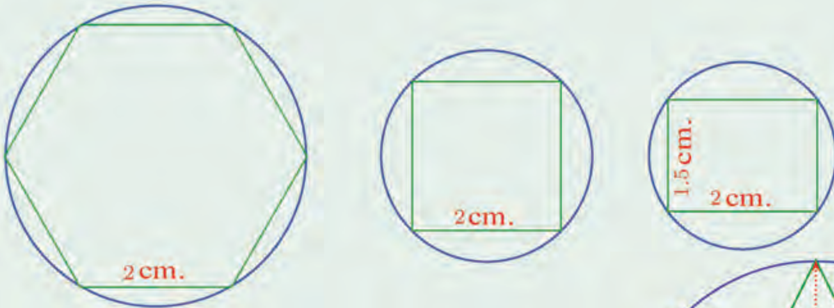
$$\pi = 3.1415926535\ldots$$

Often in practical problems, we need to compute π only upto four decimal places. For example, the perimeter of a circle of radius 5 metres, correct to millimetre is

$$\pi \times 2 \times 5 \approx 31.416 \text{ metres.}$$

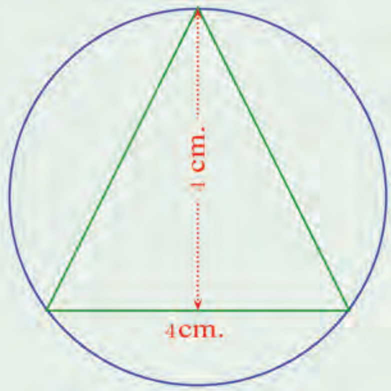
In problems hereafter, you need only write the perimeter as a multiple of π.

(1) In the pictures below, a regular hexagon, square and a rectangle are drawn with their vertices on a circle. Calculate the perimeter of each circle.

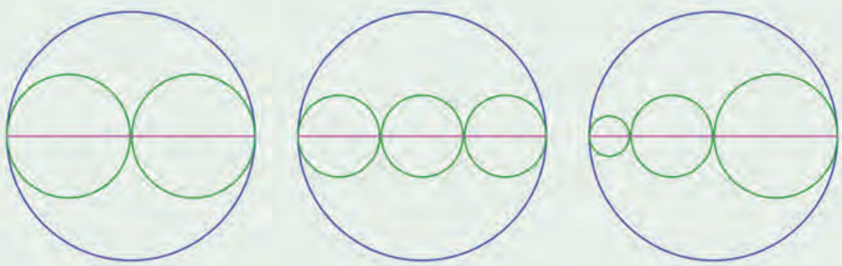


(2) An isosceles triangle with its vertices on a circle is shown in this picture.

What is the perimeter of the circle?

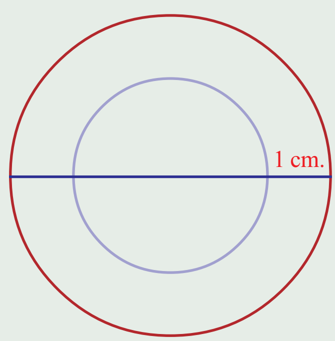


(3) In all the pictures below, the centres of the circles are on the same line. In the first two pictures, the small circles are of the same diameter.



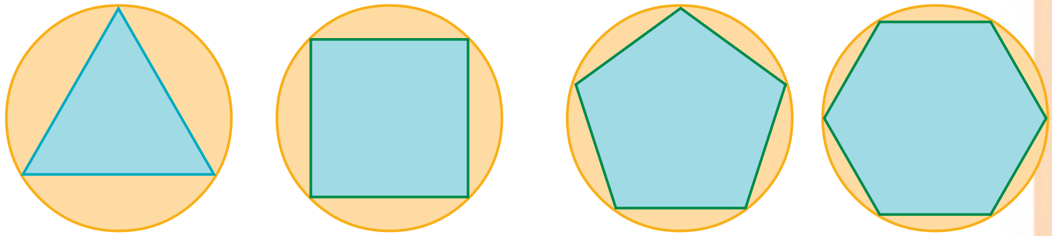
Prove that in all pictures, the perimeter of the large circle is the sum of the perimeters of the small circles.

(4) In this picture, the circles have the same centre and the line drawn is a diameter of the large circle. How much more is the perimeter of the large circle than the perimeter of the small circle?



Area

Just as the perimeters of regular polygons with vertices on the circle get closer and closer to the perimeter of the circle as the number of sides is increased, the areas of these polygons get closer and closer to the area of the circle.



π through the ages

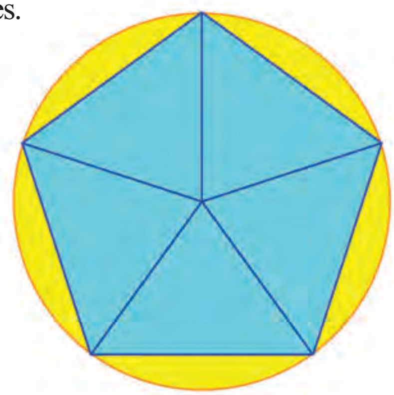
We have seen that thoughts on the methods to find the perimeter and area of circles is about four thousand years old. In the language of the present, these can be interpreted as efforts to compute approximate values of π.

In Class 8, we have mentioned the Ahmose Papyrus from ancient Egypt. The method used to do a problem in that amounts to taking $\frac{256}{81}$ as an approximate value of π. (The decimal form of this is approximately 3.16). From a Babylonian tablet of roughly this time (1500 BC), we find $\frac{25}{8} = 3.125$ as an approximation for π. The Indian text *satapathabrahmana* , gives the approximate value $\frac{339}{108} = 3.138$

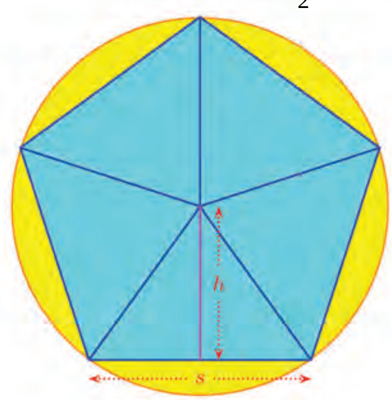
Drawing polygons of 96 sides within and outside a circle, Archimedes found that $\frac{223}{71} < \pi < \frac{22}{7}$; that is $3.1408 < \pi < 3.1428$. In 480 AD, the Chinese mathematician Zu Chong Zhi used polygons of 12288 sides to refine this estimate as $3.1415926 < \pi < 3.1415927$

This is correct up to eight decimal places. Better values could be found only after about a thousand years.

To compute the area of a circle, we must note how the areas of these polygons increase. Joining the vertices of a polygon to the centre of the circle, we can divide the polygon into equal triangles. The area of the polygon is the sum of the areas of these triangles.



In this picture, taking the length of a side of the regular pentagon as s and the length of the perpendicular from the centre of the circle to a side as h, we get the area of a triangle as $\frac{1}{2}sh$



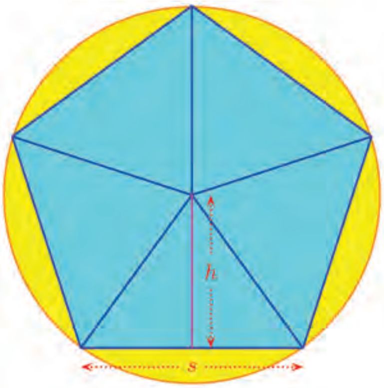
The pentagon is made up of five such triangles. So, its area is

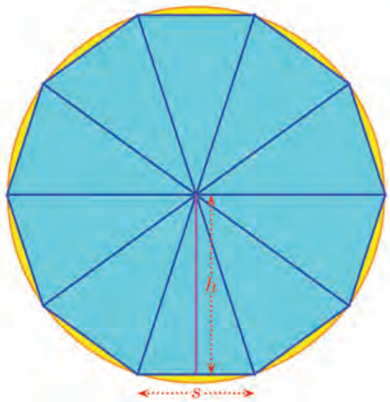
$$5 \times \frac{1}{2} sh = \frac{1}{2} \times 5s \times h$$

Since s is the length of a side of the pentagon, $5s$ is its perimeter.

Writing this as p, the area of the pentagon is $\frac{1}{2} ph$

We can easily see that whatever regular polygon we take instead of the regular pentagon, its area is half the product of its perimeter and the length of the perpendicular from the centre. As we change the polygon in the circle, both the perimeter and the length of the perpendicular change.





If we write the perimeters of the polygons in the circle in order, starting from the equilateral triangle, as $p_1, p_2, p_3, \ldots$ and the lengths of the perpendiculars from the centre to a side as $h_1, h_2, h_3, \ldots$, then the areas would be $\frac{1}{2}p_1h_1, \frac{1}{2}p_2h_2, \frac{1}{2}p_3h_3, \ldots$

In this $p_1, p_2, p_3, \ldots$ get closer and closer to the perimeter of the circle; $h_1, h_2, h_3, \ldots$ get closer and closer to the radius of the circle. So, their products get closer and closer to the product of the perimeter and radius of the circle. And half these products?

To summarise, we see geometrically that the areas of the regular polygons get closer and closer to the area of the circle. Analyzing this in terms of numbers, we find that these areas get closer and closer to half the product of the perimeter and the radius of the circle. What can we say about the area of the circle from this?

π in Kerala

In the fourteenth century AD, Madhavan (Samgamagrama Madhavan) a mathematician and astronomer who lived in Kerala, found a purely numerical method to compute fractions approximating π, which was a turning point in the history of mathematics. He found that the numbers

$$1, 1 - \frac{1}{3}, 1 - \frac{1}{3} + \frac{1}{5}, 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

got by adding and subtracting reciprocals of odd numbers approximate $\frac{\pi}{4}$.

In modern notation, we write this as,

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(This was later rediscovered in the seventeenth century by Gregory of Scotland and Leibniz of Germany in their own ways).

The numbers got by this approach π very slowly.

To get Archimedes approximation, we have to add and subtract about 4000 numbers. But Madhavan himself gave a new method,

$$\frac{1}{\sqrt{12}}\pi = 1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \cdots$$

and used it to compute

$\pi \approx 3.14159265359$

8KDIQD

The area of a circle is half the product of its perimeter and radius.

If we write r for the radius of a circle, then its perimeter is $2\pi r$.

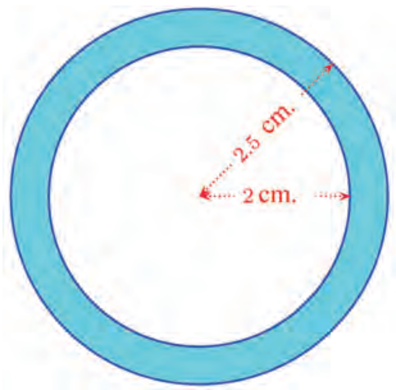
So, the area of the circle is

$$\frac{1}{2} \times 2\pi r \times r = \pi r^2$$

The area of a circle is π times the square of the radius.

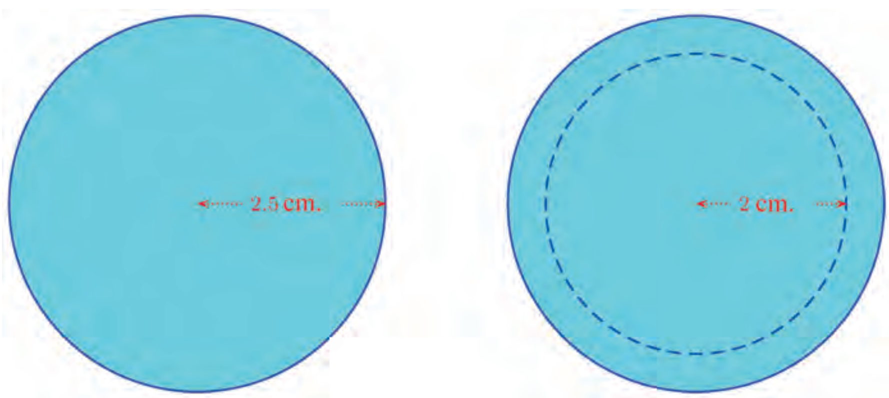
For example, the area of a circle of radius 5 centimetres is 25π square centimetres.

Now see the picture:



What is the area of the ring?

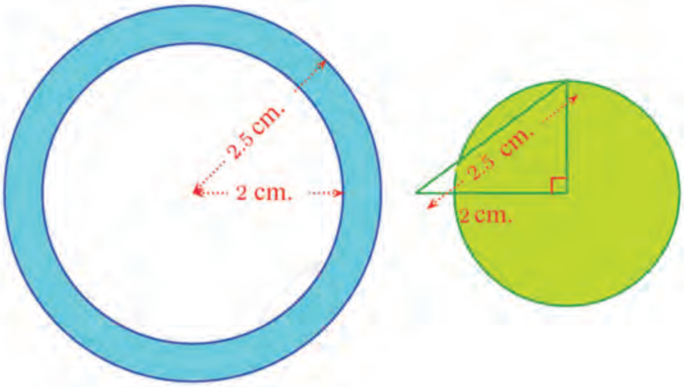
It can be seen as a small circle cut off from a larger one:



So, the area of the ring is

$$6.25\pi - 4\pi = 2.25\pi \text{ square centimetres.}$$

Now suppose we draw a right triangle and a circle as shown below:

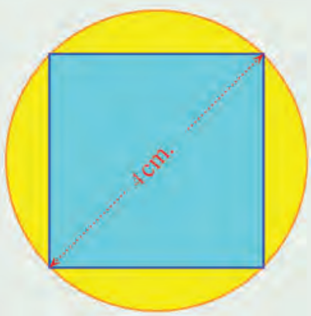


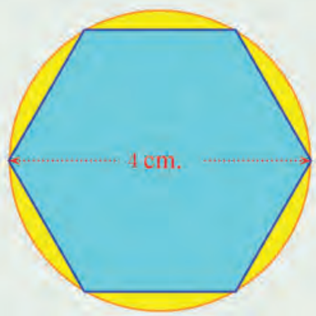
What is the relation between the area of the ring and the area of this new circle?

Maths, Computer and π

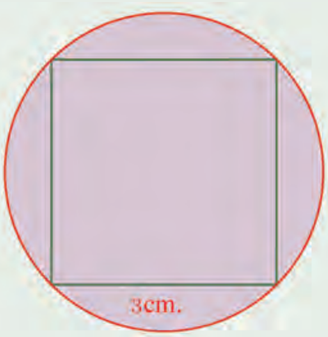
In the twentieth century, the Indian mathematician Srinivasa Ramanujan discovered many methods like that of Madhavan to compute fractions approximating π. Using some of these in computers, π was computed up to a billion decimal places in 1989. Now it is computed up to more than 10^{13} places.

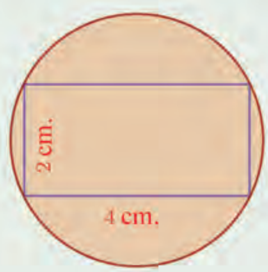
(1) In the pictures below, find the difference between the areas of the circle and the polygon, up to two decimal places:





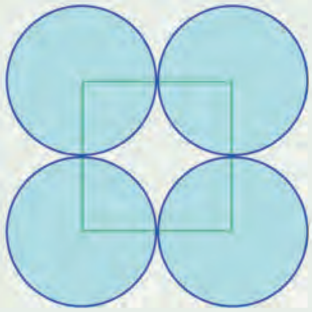
(2) The pictures below show circles through the vertices of a square and a rectangle:

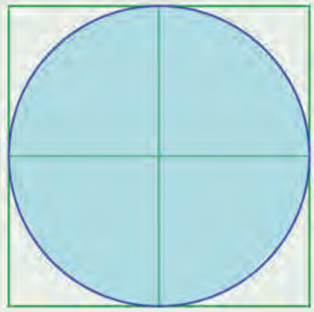




Calculate the areas of the circles.

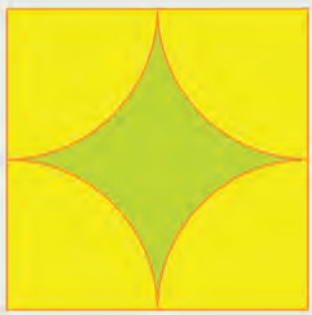
(3) Draw a square and draw circles centred on the corners, of radius half the side of the square. Draw another square formed by four of the first square and a circle just fitting into it:

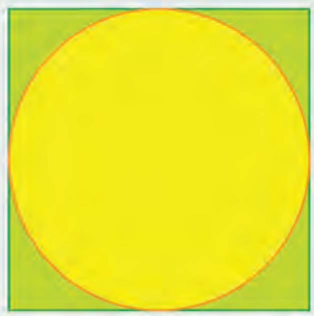




Prove that the area of the large circle is equal to the sum of the areas of the four small circles.

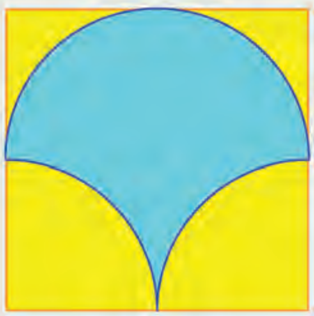
(4) In the two pictures below, the squares are of the same size.





Prove that the green regions are of the same area.

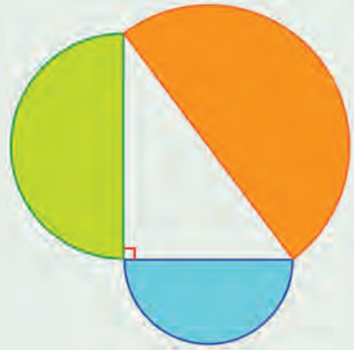
(5) Parts of circles are drawn inside a square as shown in the picture below.



Prove that the area of the blue region is half the area of the square.

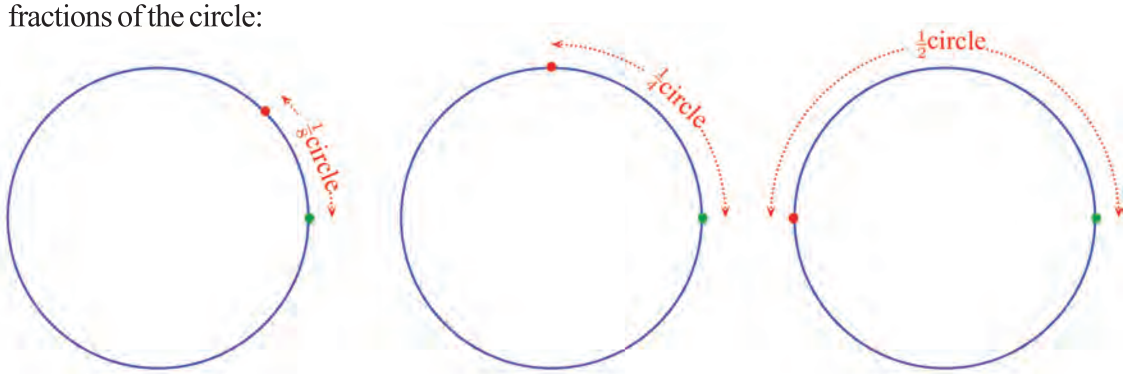
(6) In the figure, semicircles are drawn with the sides of a right triangle as diameters.

Prove that the area of the largest semicircle is the sum of the areas of the smaller ones.



Length and angle

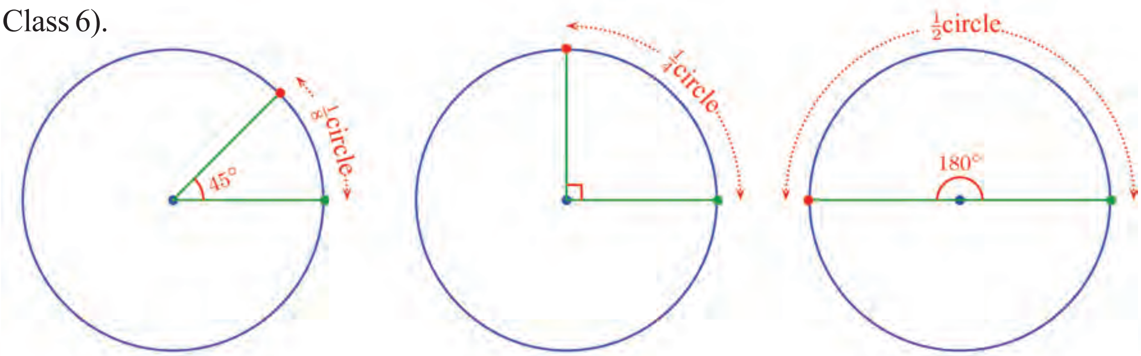
Imagine a point starting at some point on a circle and moving along the circle. The distances travelled by the point at some positions are shown below, as fractions of the circle:



Since this journey is a rotation about the centre of the circle, we can also say how much the point has turned about the centre, instead of stating how much distance it has travelled along the circle.

Remember how we made an angle of 45° at the centre of a circle to get $\frac{1}{8}$ of the circle, and angle of 90° to get $\frac{1}{4}$ of the circle? (The lesson, **Angles** in Class 6).

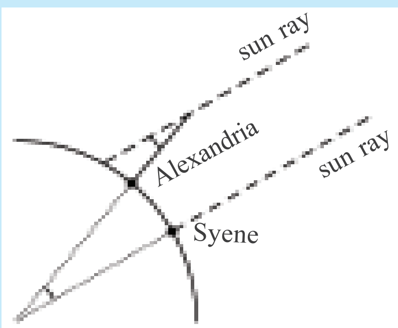
With A as centre, draw a circle of perimeter 24. (Just give the radius as 12/pi). Mark a point B on the cirlce. Make an angle slider α, select. **Angle with Give Size** and click on B and then A give the angle size as α. We get a new point B′. Select **Circular Arc** and click on A, B, B′ in order to draw the arc BB′. Mark its length. See what fraction of the perimeter of the circle is the arc length, for different values of α.



Circumference of the earth

The circumference of the earth was first computed by the Greek scientist and poet Eratosthenes, who lived in the second century BC. (The perimeter of a circle is often called its circumference).

Eratosthenes came to know that on a particular day in the Egyptian town of Syene, the sun would be directly overhead at noon and so, there would be no shadows at that time. Using a pole stuck on the ground, he computed the slant of the sun rays at that particular time in the town of Alexandria, where he worked.



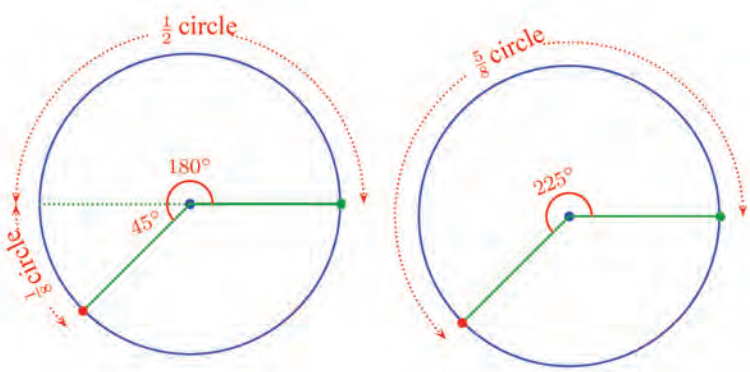
Assuming the sun's rays to be parallel, the central angle of the arc on earth joining Alexandria and Syene is equal to this slant. The length of this arc is the distance between these two towns.

So, if the slant of sun rays at Alexandria is $a°$ and the distance to Syene is d, the circumference of the earth can be computed as $\frac{360}{a} \times d$.

Thus the journey can be described in terms of lengths or in terms of angles. Then a question arises. When the point has moved through half the circle and then $\frac{1}{8}$ of the circle, the total distance covered is $\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$ of the circle. How do we state this as a rotation?

$\frac{1}{2}$ of the circle is $180°$ and $\frac{1}{8}$ of the circle is $45°$. So the point has turned $180°$ and $45°$ more. We can say that it has turned

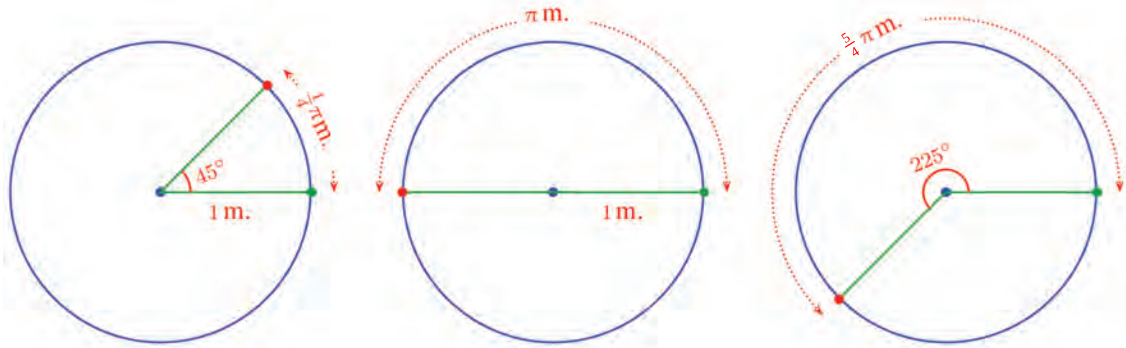
$180° + 45° = 225°$:



Thus at every point of the journey along the circle till the return to the starting point, the distance travelled can be stated either as a fraction of the circle or as an angle between $0°$ and $360°$.

Suppose we also take the radius of the circle as 1 metre in this?

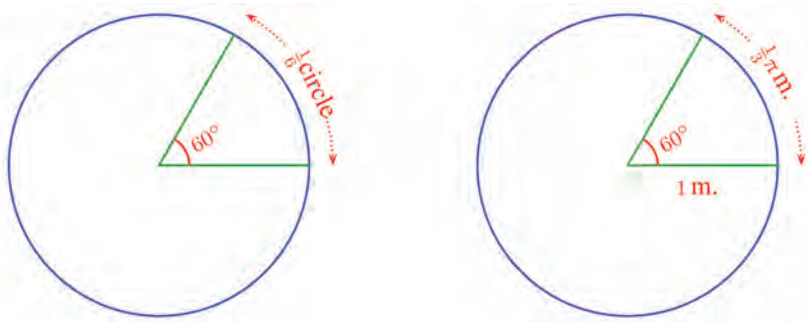
The perimeter of the circle is 2π metres. So the distances can be stated in metres, instead of fractions of the circle:



In other words, at every stage of the journey, we can state either the distance travelled in metres or the amount turned in degrees.

If it has turned 60° about the centre, how many metres has it travelled along the circle?

First let's see what fraction of the circle it has travelled. 1° means $\frac{1}{360}$ of the circle. So 60° means it has turned $60 \times \frac{1}{360} = \frac{1}{6}$ of the circle. Since the perimeter is 2π metres, this is $\frac{1}{3}\pi$ metres.



In general, whatever fraction of 360° it has turned about the centre, the same fraction of 2π metres it has travelled along the circle.

What if we take a circle of radius $1\frac{1}{2}$ metres? The perimeter would be 3π metres. So, to compute the distances corresponding to rotations, we must take fractions of 3π metres. Thus, though the fraction of the circle travelled corresponding to a rotation does not change, the length in metres

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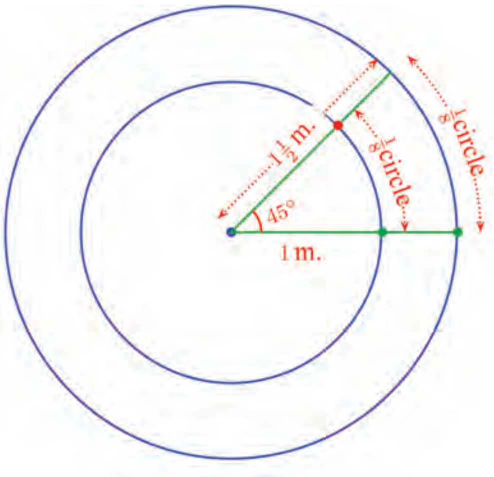
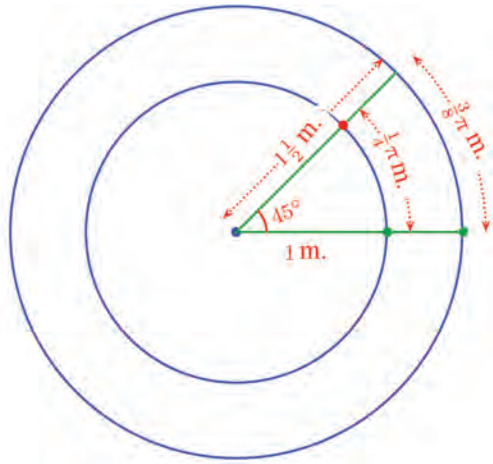
changes. For example, in rotating 45°, the point still travels $\frac{1}{8}$ of this circle. But since the circle is larger, the actual distance travelled is $\frac{3}{8}\pi$ metres.



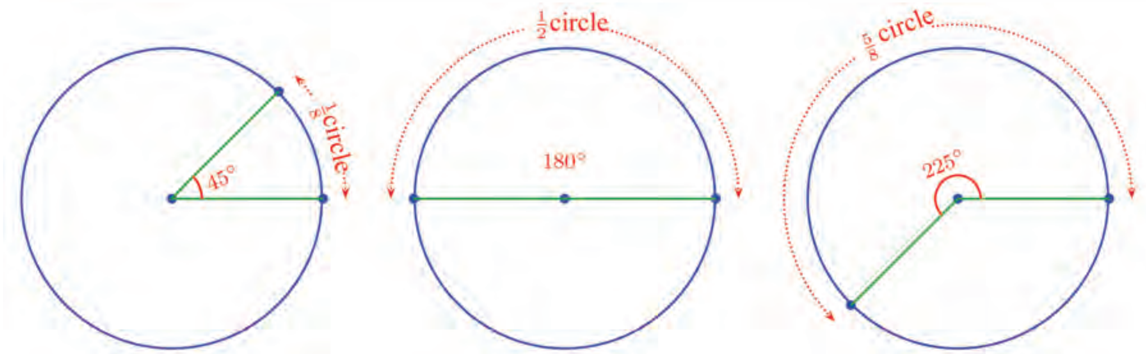


In general, we can say this:

> For a point moving along a circle of radius r metres, the distance travelled along the circle for a rotation of x° about the centre is $2\pi r \times \frac{x}{360}$ metres.

Now let's see how we can state this in purely mathematical terms. Any part of a circle between two points on it, is called an *arc*. The angle between the two radii joining the ends of an arc to the centre of the circle is called the *central angle* of the arc.

So, as we have seen earlier, the central angle of an arc of length $\frac{1}{8}$ of a circle is 45°, that of an arc of length $\frac{1}{2}$ of a circle is 180°, that of an arc of length $\frac{5}{8}$ of a circle is 225° and so on.



Now we can translate our physical principle of motion along a circle into a purely mathematical property of circles:

In a circle of radius r, the length of an arc of central angle x° is

$2\pi r \times \frac{x}{360}$.

In other words,

The length of an arc of a circle is that fraction of the perimeter as the fraction of 360° that its central angle is.

For example, in a circle of radius 3 centimetres, what is the length of an arc of central angle 60°?

We can do this in our mind. The perimeter of the circle is 6π centimetres. Since 60° is $\frac{1}{6}$ of 360°, the length of the arc is $\frac{1}{6}$ of the perimeter. So its length is π centimetres.

What about the length of an arc of central angle 50°, in a circle of radius 2.5 centimetres?

The perimeter of the circle is 5π centimetres and the length of the arc is $\frac{50}{360}$ of this; which means

$$5\pi \times \frac{50}{360} = \frac{25}{36}\pi \approx 2.2 \text{ centimetres.}$$

Let's do another problem : From an iron ring of radius 9 centimetres, a piece of central angle 30° is cut off. This is then bent into a small circle. What is the radius of this circle?

The length of an arc of central angle 30° is $\frac{30}{360} = \frac{1}{12}$ of the perimeter of the circle; so that the length of the piece cut off is $18\pi \times \frac{1}{12} = \frac{3}{2}\pi$ centimetres.

This is the perimeter of the small circle. So its radius is $\frac{3}{2}\pi \div 2\pi = \frac{3}{4}$ centimetres.

This can be calculated a little more easily. The perimeter of the small circle is $\frac{1}{12}$ the perimeter of the large circle. Since perimeter and radius are scaled by the same factor, the radius of the small circle is $\frac{1}{12}$ the radius of the large circle; that is $9 \times \frac{1}{12} = \frac{3}{4}$ centimetres.

(1) In a circle, the length of an arc of central angle $40°$ is 3π centimetres. What is the perimeter of the circle? What is its radius?

(2) In a circle, the length of an arc of central angle $25°$ is 4 centimetres.

i) In the same circle, what is the length of an arc of central angle $75°$?

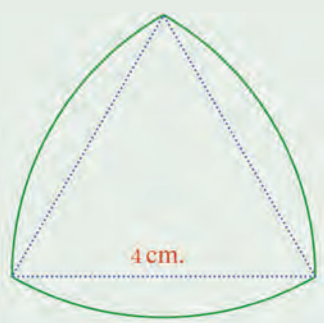
ii) In a circle of radius one and a half times the radius of this circle, what is the length of an arc of central angle $75°$?

(3) From a bangle of radius 3 centimetres, a piece is to be cut out to make a ring of radius $\frac{1}{2}$ centimetres.

i) What should be the central angle of the piece to be cut out?

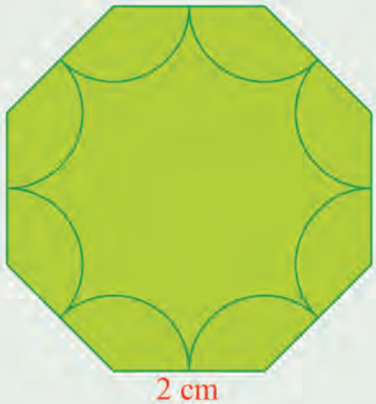
ii) The remaining part of the bangle was bent to make a smaller bangle. What is its radius?

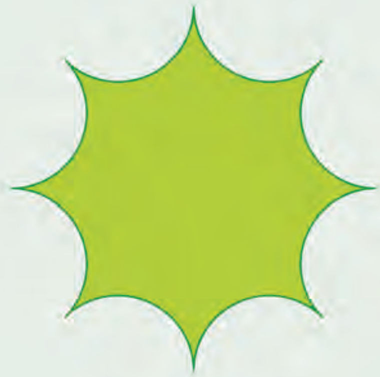
(4) The picture shows the parts of a circle centred at each vertex of an equilateral triangle and passing through the other two vertices.



What is the perimeter of this figure?

(5) Parts of circles are drawn, centred at each vertex of a regular octagon and a figure is cut out as shown below:





Calculate the perimeter of the figure.

Angle and area

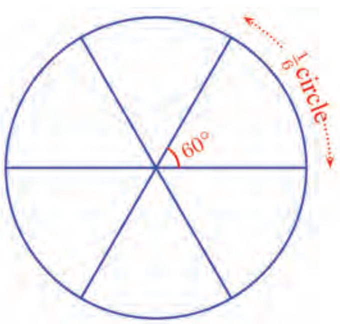
An arc is a part of the perimeter of a circle. An arc and the radii through its ends enclose a part of the area of the circle.

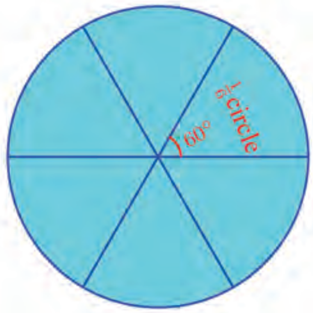
Such a piece of a circle is called a *sector*. The central angle of its arc is called the central angle of the sector.

Draw a circle of area 24 centred at A by giving the radius as sqrt(24/pi). Mark a point B on it. Make a slider α and mark by B′ with $\angle BAB' = \alpha$. Select **Circular Sector** and click on A, B, B′ to draw the sector. Mark its area. Find the relation between the angle, area of the sector and the area of the circle.

Just as the length of an arc changes according to its central angle, so does the area of the sector. And the scale factor is the same in both.

For example, an arc of central angle 60° is $\frac{1}{6}$ the perimeter of the circle, a sector of central angle 60° is $\frac{1}{6}$ the area of the circle.





Similarly an arc of central angle 1° is $\frac{1}{360}$ of the perimeter of the circle and a sector of central angle 1° is $\frac{1}{360}$ of the area of the circle.

So, we can state the relation between the central angle of a sector and its area in much the same as the relation between the central angle of an arc and its length:

The area of a sector is that fraction of the area of the circle as the fraction of 360° that its central angle is.

Using algebra, we can put it like this

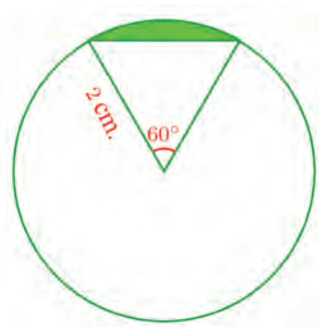
In a circle of radius r, the area of a sector of central angle x° is

$$\pi r^2 \times \frac{x}{360}$$

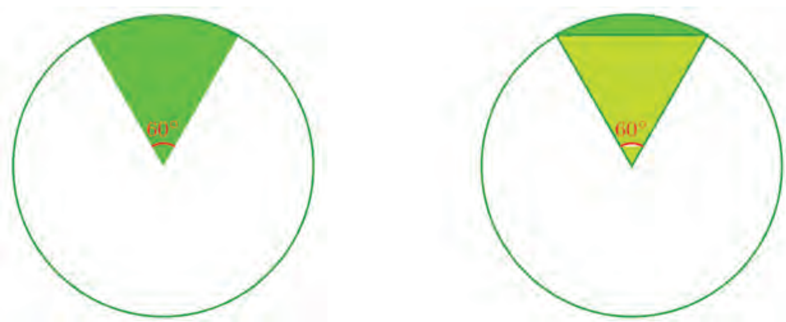
For example, in a circle of radius 3 centimetres, the area of a sector of central angle 40° is $\frac{40}{360} = \frac{1}{9}$ of the area of the circle; and the area of the circle is 9π square centimetres. So the area of the sector is π square centimetres.

Now look at this picture.

What is the area of the coloured part?



We can see this as a triangle removed from a sector:



The area of the sector is $\frac{1}{6}$ the area of the circle.

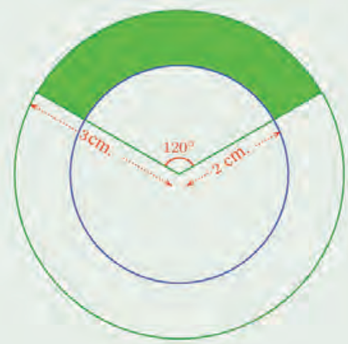
That is, $4\pi \times \frac{1}{6} = \frac{2}{3}\pi$ square centimetres.

The triangle is equilateral (why?). Its area is $\frac{\sqrt{3}}{4} \times 4 = \sqrt{3}$ square centimetres.

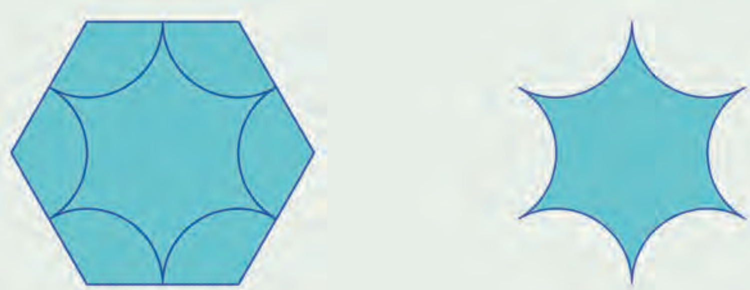
So, the area of the coloured part of the circle in the first picture is $\frac{2}{3}\pi - \sqrt{3}$ square centimetres.

(1) What is the area of a sector of central angle 120° in a circle of radius 3 centimetres? What is the area of a sector of the same central angle in a circle of radius 6 centimetres?

(2) Calculate the area of the green coloured part of this picture.

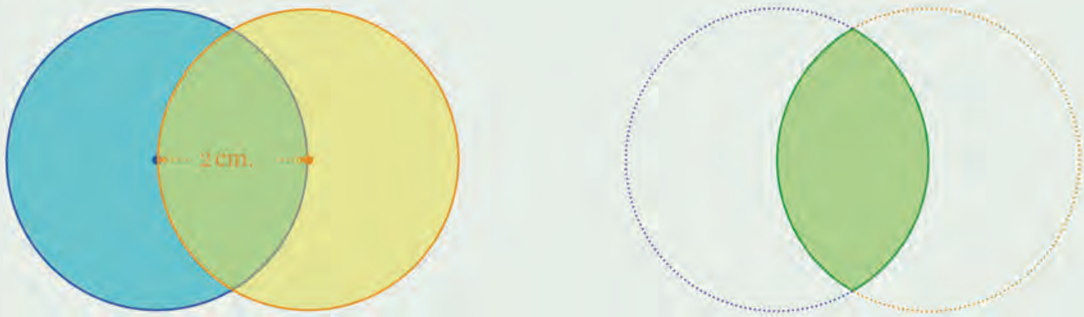


(3) Centred at each corner of a regular hexagon, a part of a circle is drawn and a figure is cut out as shown below:



What is the area of this figure?

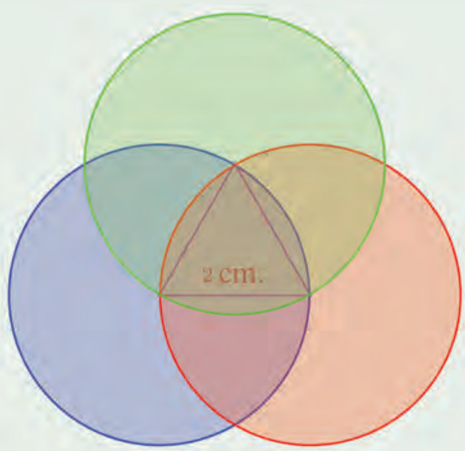
(4) The picture below shows two circles, each passing through the centre of the other:



Calculate the area of the region common to both.

(5) The figure shows three circles drawn with their centres on each vertex of an equilateral triangle and passing through the other two vertices;

Find the area of the region common to all three.



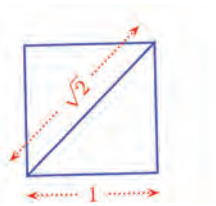
10 Real Numbers

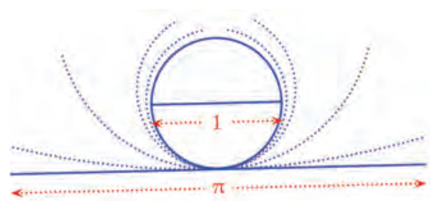
Points and Numbers

How do we express the length of lines as numbers? We take some fixed length as 1 and then take double that length as 2, half that length as $\frac{1}{2}$, one and a half times that length as $1\frac{1}{2}$ and so on.

The fixed length we take as 1 is called the *unit* of length. Once we have fixed a unit, then many other lengths can be expressed as natural numbers or fractions, as we have just seen.

But there are some lengths which we cannot express as natural numbers or fractions of a chosen unit. For example, the diagonal of a square with this unit as the length of a side or the perimeter of a circle with this unit as diameter:





When we express the relations between measures and relations between operations on pure numbers as algebraic equations, we sometimes use negative numbers for convenience (The section **Uses**, of the lesson **Negative Numbers**, in Class 8). Thus we need numbers like $-\sqrt{2}$ and $-\pi$ as the negatives of $\sqrt{2}$ and π.

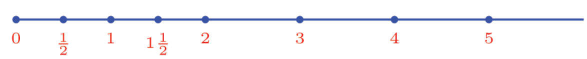
The natural numbers, fractions and their negatives with zero as well are collectively called *rational numbers*. All other numbers are called *irrational numbers*.

We can write natural numbers also as fractions. For example, we can write 5 in various ways such as $\frac{5}{1}$ or $\frac{10}{2}$.

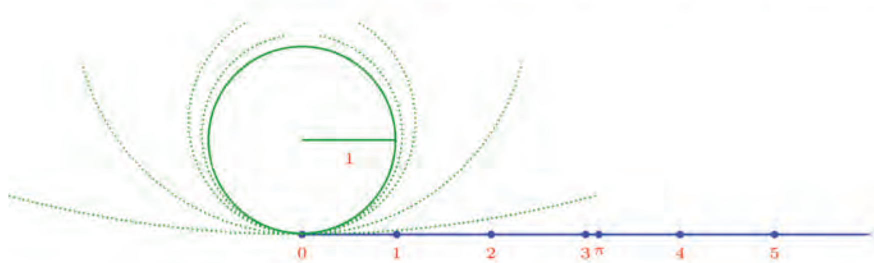
Negatives of natural numbers can also be written like this, by taking numerator or denominator as negative. Zero, we can write as $\frac{0}{1}$. So, all rational numbers have a common form $\frac{x}{y}$ where x and y are natural numbers or their negatives, with x possibly zero also. But among irrational numbers, there are numbers like $\sqrt{2}$ and $\sqrt{3}$ and also numbers like π which cannot be expressed by any operations on rational numbers; they are not bound to any common form.

The rational and irrational numbers together are called *real numbers*.

We can visualise real numbers geometrically. For this, imagine a line with a point marked at its left end and another point to its right. Taking the distance from the first point to the second as 1 (the unit of length), we can write the distances to all other points on the right as numbers:

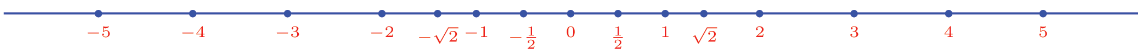


To mark the distances to all points, we need irrational numbers also.



We can extend this line from the point 0 to the left also. How do we mark the distance to this side as numbers?

For that we use the negatives of the numbers to the right.



Thus, all points on this line can be marked using real numbers. On the other hand, all real numbers can be seen as the points on this line.

Such a line is called *number line* or *real line*.

In the number line, as we move right from 0, the numbers get larger and larger. And as we move to the left?

Which is larger, -1 or -2?

-1 is 1 less than 0; what about -2? 2 less than 0, that is, 1 less than -1, so, -2 is less than -1. In symbols, $-2 < -1$.

Thus we see larger numbers on moving rightward along the number line and smaller numbers on moving leftward.

This is true even when we start from any point other than 0. So for any two real numbers, the position of the larger number on the number line is to the right of the smaller number.

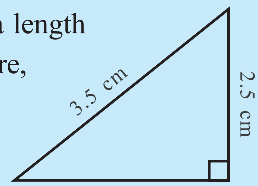
Thus the numerical relation of small and large is transformed to the geometrical relation of left and right on the number line.

Now let's see how the geometric notion of the distance between two points on the number line can be expressed in terms of the numbers denoting these points. First let's take the distance from zero.

Irrational lengths

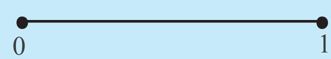
Not only lengths, but areas and volumes also may be irrational numbers. The area of a rectangle of sides $\sqrt{3}$ and $\sqrt{2}$ is $\sqrt{3} \times \sqrt{2} = \sqrt{6}$, isn't it?

$\sqrt{6}$ may arise as a length also. See this picture, what is the length of the third side? It is convenient to see all positive irrational numbers as lengths.

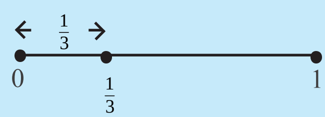


Number density

How many numbers are there between 0 and 1? No natural numbers; but there are innumerable other numbers such as the rational numbers $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$ and so on, and irrational numbers such as $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$, $\frac{3}{\pi}$ and so on. We can see this geometrically also. Draw a line and mark its ends as 0 and 1.

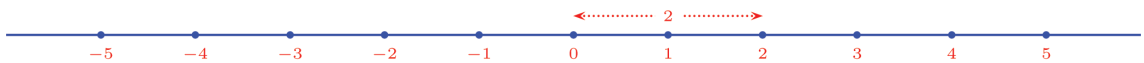


Now if we take any point on this line, its distance from 0 can be marked by that number.

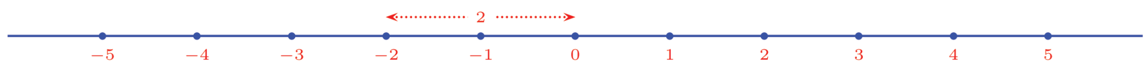


Thus every point on the line gives a number. How many points are there on the line?

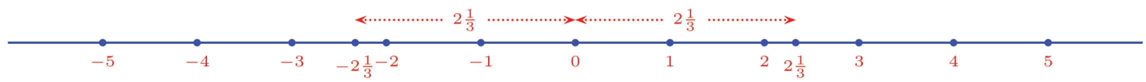
The very scheme of denoting points by numbers is based on distance, right? For example, the point labelled 2 is at a distance 2 to the right of the point labelled 0.



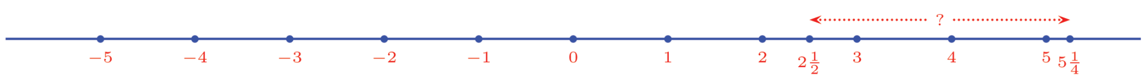
And the point at the same distance to the left of the point marked 0 is labelled −2.



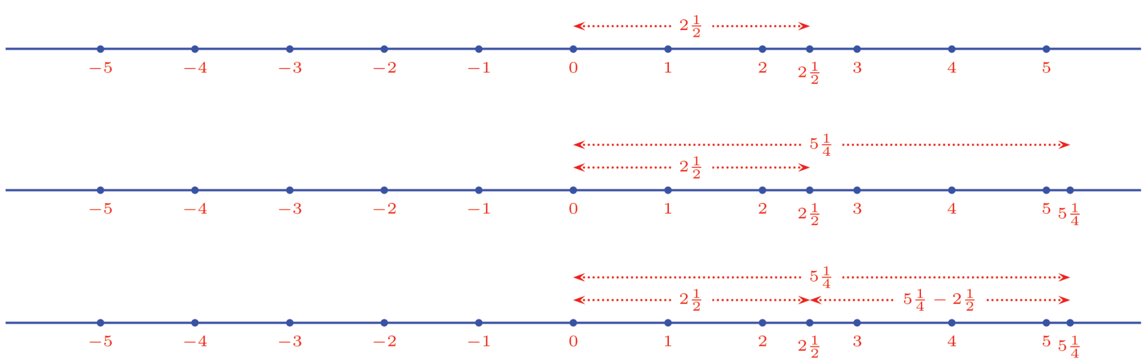
Similarly, the distance of the point marked $2\frac{1}{3}$ from the point 0, and the distance of the point marked $-2\frac{1}{3}$ from the point 0 are both $2\frac{1}{3}$.



Now let's see in general the distance between two points. For example, what is the distance between the points $2\frac{1}{2}$ and $5\frac{1}{4}$?

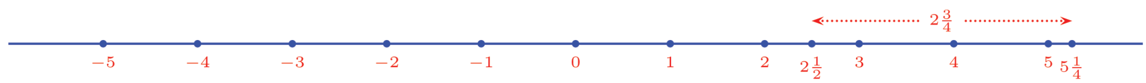


The distance between them is the difference of the distances from 0 to each, isn't it?



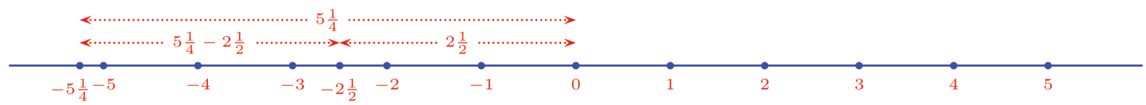
Thus the distance between the points $2\frac{1}{2}$ and $5\frac{1}{4}$ is

$$5\frac{1}{4} - 2\frac{1}{2} = 2\frac{3}{4}$$

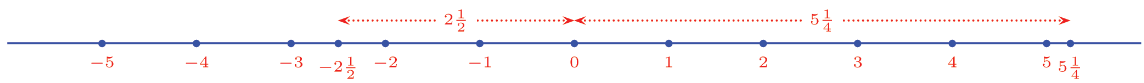


What about the distance between $-2\frac{1}{2}$ and $-5\frac{1}{4}$?

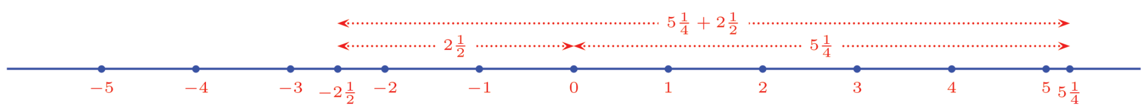
Again, we can get it by finding the distances from 0 to these points and subtracting the smaller distance from the larger:



How about $-2\frac{1}{2}$ and $5\frac{1}{4}$?



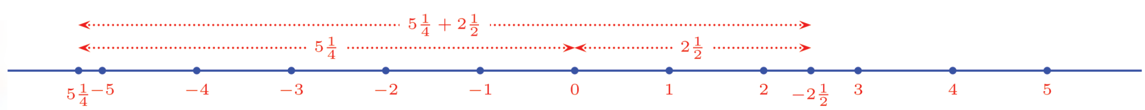
Since they are on different sides of 0, to find the distance between them, we have to add the distances from 0:



Thus the distance between $-2\frac{1}{2}$ and $5\frac{1}{4}$ is

$$2\frac{1}{2} + 5\frac{1}{4} = 7\frac{3}{4}$$

And the distance between $2\frac{1}{2}$ and $-5\frac{1}{4}$ is also the same.



Let's write all the distances we have calculated:

Points	Distance
$2\frac{1}{2}$, $5\frac{1}{4}$	$5\frac{1}{4} - 2\frac{1}{2} = 2\frac{3}{4}$
$-2\frac{1}{2}$, $-5\frac{1}{4}$	$5\frac{1}{4} - 2\frac{1}{2} = 2\frac{3}{4}$
$-2\frac{1}{2}$, $5\frac{1}{4}$	$5\frac{1}{4} + 2\frac{1}{2} = 7\frac{3}{4}$
$2\frac{1}{2}$, $-5\frac{1}{4}$	$5\frac{1}{4} + 2\frac{1}{2} = 7\frac{3}{4}$

In this, the distance between the first pair is found by subtracting from the larger number $5\frac{1}{4}$, the smaller number $2\frac{1}{2}$.

What about the second pair? The larger number is $-2\frac{1}{2}$; subtracting the smaller number $-5\frac{1}{4}$ from it gives,

$$-2\frac{1}{2} - \left(-5\frac{1}{4}\right) = -2\frac{1}{2} + 5\frac{1}{4} = 5\frac{1}{4} - 2\frac{1}{2} = 2\frac{3}{4}$$

And this is what we get as the distance, right? So in this pair also, the distance is the difference of the smaller number from the larger number. What about the third pair? $5\frac{1}{4}$ is the larger number and $-2\frac{1}{2}$ is the smaller. Subtracting the smaller from the larger gives

$$5\frac{1}{4} - \left(-2\frac{1}{2}\right) = 5\frac{1}{4} + 2\frac{1}{2} = 7\frac{3}{4}$$

So for this pair also, distance is the difference of the smaller number from the larger. Let's look at the last pair also. The larger number is $2\frac{1}{2}$ and the smaller number, $-5\frac{1}{4}$.

$$2\frac{1}{2} - \left(-5\frac{1}{4}\right) = 2\frac{1}{2} + 5\frac{1}{4} = 7\frac{3}{4}$$

So whether both points are on the right side or on the left of zero or one on the right and the other on the left, the

Some history

The Pythagorean belief that all measures could be expressed through natural numbers and their ratios was proved wrong by Hippasus, as we have mentioned earlier. But there is no concept of irrational numbers in ancient Greek mathematics. So all number principles are stated in geometric language in Greek texts of that period.

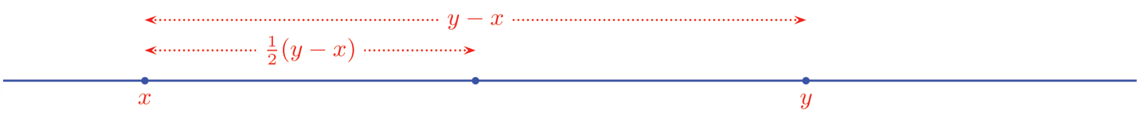
distance between them is the difference of the smaller of the numbers denoting them from the larger.

Is this true even when one of the numbers is zero? For example, the distance between 0 and 2 is 2. Now for 0 and -2 also, the distance is 2 itself; that is $0 - (-2) = 2$.

The distance between any two points on the number line is the smaller of the numbers denoting them subtracted from the larger.

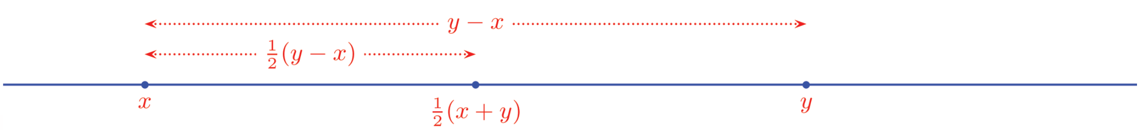
Using this, we can find the mid point of two points on the number line. Let's write the smaller of the numbers denoting the points as x and the larger as y. So, x is to left of y and the distance between them is $y - x$.

The point y is at a distance $y - x$ from x; the mid point is half this distance away from x:



So, the mid point is

$$x + \frac{1}{2}(y - x) = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}(x + y)$$



The midpoint of two points on the number line is that point denoted by half the sum of the numbers denoting these points.

For example, the midpoint of $-2\frac{1}{2}$ and $4\frac{3}{4}$ is

$$\frac{1}{2}\left(-2\frac{1}{2} + 4\frac{3}{4}\right) = \frac{1}{2} \times 2\frac{1}{4} = 1\frac{1}{8}$$

(1) Find the distance between the two points on the number line, denoted by each pair of numbers given below:

i) $1, -5$ ii) $\frac{1}{2}, \frac{2}{3}$ iii) $-\frac{1}{2}, -\frac{1}{3}$

iv) $-\frac{1}{2}, \frac{3}{4}$ v) $-\sqrt{2}, -\sqrt{3}$

(2) Find the mid point of each pair of points in the first problem.

(3) The part of the number line between the points denoted by the numbers $\frac{1}{3}$ and $\frac{1}{2}$ is divided into four equal parts. Find the numbers denoting the ends of each such part.

Algebra

The distance between the points 3 and 0 on the number line is 3 itself. The distance between -2 and 0 is 2.

In general, the distance between a positive number and zero is the number itself. The distance between a negative number and zero is the number with the negative sign discarded.

How do we say this in algebra?

If x is a positive number, the distance between x and 0 is x itself. What if x is a negative number?

(We have seen in the lesson **Negative Numbers**, in Class 8, that when we denote numbers by letters, we don't distinguish between positive and negative numbers and write both in the same way as x or y).

So, we have to describe the operation of discarding the negative sign in a different way. Do you remember seeing in Class 8 that the negative of the negative of a number is the number itself?

For example,

$$-(-2) = 2$$

Thus the removal of the negative sign of a number is the same as taking the negative of the number. So, if x is a negative number, to get the positive number discarding its negative sign, we should take $-x$. For example, if $x = -3$ then

$$-x = -(-3) = 3$$

Now we can say that the distance between a negative number x and zero on the number line is $-x$.

This operation of taking x itself if $x > 0$ (that is, if x is positive) and $-x$ if $x < 0$ is shortened as $|x|$. It is called the *absolute value* of x.

For example,

$$|5| = 5 \qquad |-5| = 5$$

$$\left|\frac{2}{3}\right| = \frac{2}{3} \qquad \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$|\pi| = \pi \qquad |-\pi| = \pi$$

The absolute value of 0 is taken as 0 itself. All this can be combined like this:

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

What we have said so far can be summarised like this:

On the number line, the distance between the point denoted by 0 and point denoted by another number is the absolute value of that number.

We put it in algebra like this:

The distance between the points denoted by 0 and x on the number line is $|x|$.

Now let's see how we can write the distance between any two points x and y on the number line. We have seen that the distance is the smaller number subtracted from the larger. So, the distance depends on the larger among x and y.

If $x > y$, then the distance is $x - y$

If $x < y$, then the distance is $y - x$

Instead of saying x is larger than y, we can say that $x - y$ is positive; or we can say $x - y > 0$. Similarly instead of saying x is smaller than y, we can say that $x - y$ is negative or $x - y < 0$.

If $x - y > 0$, then the distance is $x - y$

If $x - y < 0$, then the distance is $y - x$

Now can you recall any relation between the numbers $x - y$ and $y - x$?

We have seen in Class 8 that the number got by subtracting one number from another and the number got by subtracting in reverse are negatives of each other (The section **Uses**, of the lesson **Negative Numbers**)

That is, for any two numbers x and y,

$$y - x = -(x - y)$$

So we can rewrite our distance computation again:

If $x - y > 0$, then the distance is $x - y$

If $x - y < 0$, then the distance is $-(x - y)$

Look at this again. We take $x - y$ itself, if it is positive and if $x - y$ is negative, we take its negative, right? But it is just the absolute value of the number $x - y$.

So, what we have seen in two pieces about the distances can now be combined together:

> The distance between the points denoted by the numbers x and y on the number line is $|x - y|$.

We can put it in ordinary language like this:

The distance between two points on the number line is the absolute value of the difference of the numbers denoting these points.

For example, the distance between the points 2 and 5 on the number line is,

$$|2 - 5| = |-3| = 3$$

What about the distance between 2 and -5?

$$|2 - (-5)| = |2 + 5| = |7| = 7$$

Now let's look at some problems.

What are the numbers x for which $|x - 1| = 3$?

This can be done in several ways. Looking at it geometrically, $|x-1|$ is the distance between the numbers x and 1 on the number line; and it is to be 3. The number on the right of 1, at a distance 3 from it is $1+3=4$, and the number on the left of 1, at a distance 3 from it is $1-3=-2$. So $x=4$ or $x=-2$.

Now let's think about this algebraically. If $x>1$, then $|x-1|=x-1$; and if $x-1=3$ then $x=4$.

What if $x<1$? Then $|x-1|=1-x$ and if $1-x=3$, then

$$x=1-3=-2$$

Let's change the question slightly and ask:

What are the numbers x for which $|x+1|=3$.

To do this geometrically, $|x+1|$ must be seen as a distance. And distance is expressed as the absolute value of a difference. So we must first write the sum $x+1$ as a difference;

$$x+1=x-(-1)$$

Square root and absolute value

The number $|x|$ is positive, whether x is positive or negative. Similarly, x^2 is positive, whether x is positive or negative.

Every positive number has two square roots, one positive and one negative. It is the positive square root that we denote by the symbol $\sqrt{\ }$.

So, what is $\sqrt{x^2}$?

For example, $x=4$ then $x^2=16$ and

$$\sqrt{x^2}=\sqrt{16}=4=x$$

What if $x=-4$? Then $x^2=16$

$$\sqrt{x^2}=\sqrt{16}=4=-x$$

In general, for any number x,

$$\sqrt{x^2}=|x|$$

Thus $\left(\sqrt{x}\right)^2=x$ for all numbers, but $\sqrt{x^2}$ may not be x itself.

So, $|x+1|$ can be seen as the distance between x and -1 on the number line. Then as in the first problem, we can find the number on the right of -1, at a distance 3 as $-1+3=2$ and the number on the left of -1, at a distance 3 as $-1-3=-4$.

Do this problem also using algebra.

One more problem:

Prove that $|x|^2=x^2$ for any number x.

If x is a positive number, then $|x|=x$ and so

$$|x|^2=x^2$$

What if x is negative? Then $|x|=-x$. and so

$$|x|^2=(-x)^2=(-x)\times(-x)=x\times x=x^2$$

Finally, if $x = 0$, then $|x| = 0$ and

$$|x|^2 = 0^2 = 0$$

Again, since $x = 0$, we have

$$x^2 = 0^2 = 0$$

Thus in this case,

$$|x|^2 = 0 = x^2$$

(1) Find those x satisfying each of the equations below:

i) $|x - 1| = |x - 3|$

ii) $|x - 3| = |x - 4|$

iii) $|x + 2| = |x - 5|$

iv) $|x| = |x + 1|$

(2) Prove that if $1 < x < 4$ and $1 < y < 4$, then $|x - y| < 3$

(3) Prove that if $x < 3$ and $y > 7$, then $|x - y| > 4$

(4) Find two numbers x, y such that $|x + y| = |x| + |y|$

(5) Are there numbers x, y such that $|x + y| < |x| + |y|$?

(6) Are there numbers x, y such that $|x + y| > |x| + |y|$?

(7) What are the numbers x, for which $|x - 2| + |x - 8| = 6$?

(8) What are the numbers x, for which $|x - 2| + |x - 8| = 10$?

Taking x as different numbers, what all numbers do we get as $|x - 2| + |x - 8|$?

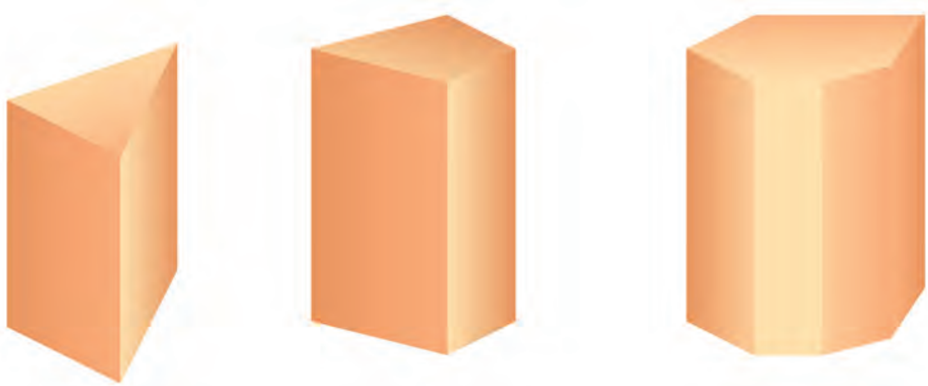
11 Prisms

Different bases

We have learnt about rectangular blocks and their volumes in Class 6.

Its outer surface is made up of many rectangles: two identical rectangles at the top and bottom, two on the left and right and a third pair at the front and back, making six rectangles in all.

Look at these pictures:



They have horizontal spreads and vertical heights.

They are *solids*, also called *three dimensional* objects.

And, they share some other common features.

Solids in GeoGebra

We can draw solids in GeoGebra. We must make some preparations first:

- Open the **Algebra**, **Graphic** and **3D Graphic** windows.
- Right click on the **3D Graphic** window and in preferences, deselect **Show Axes**, **Use Clipping**, **Show Clipping**.
- Choose **Options** → **Labelling** → **No New Objects** to avoid labelling of objects.

Draw a polygon such as a triangle or rectangle in the **Graphics** window (using **Grid**, if needed). This will be seen in the **3D Graphics** window also. Choose **Extrude to Prism or Cylinder** from **3D Graphics** and click on the polygon. In the window opening up, give the height of the prism. We get a prism, which can be rotated. The shape of the prism changes according to the polygon drawn in **Graphics**. A slider may be used to control the height of the prism.

The surface of the first is made up of two identical triangles and three rectangles. The second has quadrilaterals instead of triangles and the third, hexagons.

In general, the surface of any of them is made up of two identical polygons and rectangles of the same height, with the polygons as opposite sides. Such solids are collectively called *prisms*.

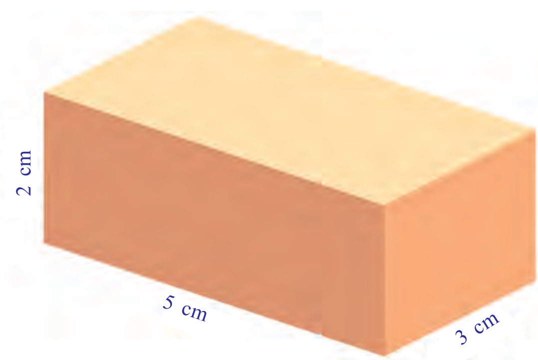
The polygons and rectangles in a prism are called its *faces*. The polygons on the top and bottom are called *bases* and the rectangles are called *lateral faces*. Depending on the shape of the bases, prisms can be classified as triangular prisms, quadrangular prisms and so on.

The three pictures above are of a triangular prism, a quadrangular prism and a hexagonal prism. What we have called a rectangular block may now be called (a little more formally) a rectangular prism.

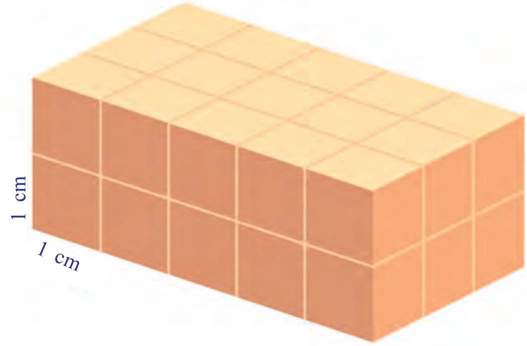
Try to make hollow prisms, by cutting out polygons and rectangles from cardboard.

Volume

Remember how we calculated the volumes of rectangular prisms (blocks) in Class 6? For example, look at this rectangular prism.



We can split this into cubes of side one centimetre, as shown below:

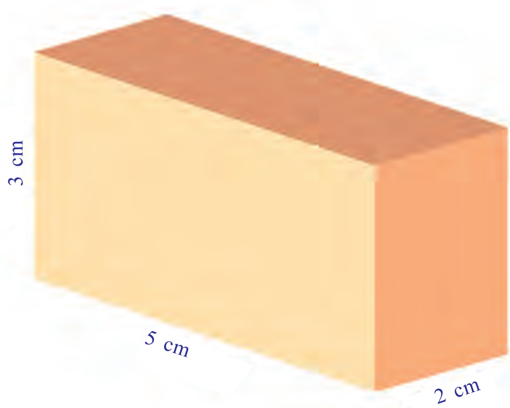


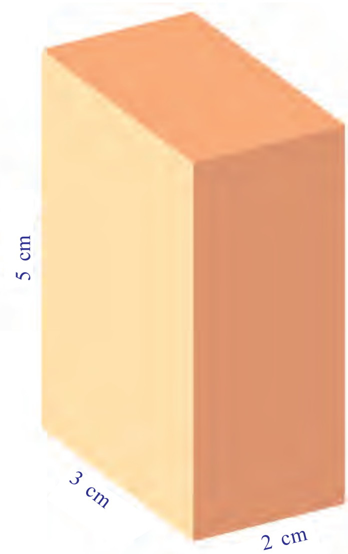
There are $5 \times 3 \times 2 = 30$ small cubes in it and so its volume is 30 cubic centimetres.

We can show that the volume of a prism is the product of these lengths, even if they are fractions, as we did for areas in Class 6 (the section **Fractional areas**, of the lesson **Part of Parts**). Also, we can show that this is true even if the lengths are irrational numbers, as we explained for areas of rectangles in the section **Multiplication**, of the lesson **New Numbers**.

We can describe the volume of a rectangular prism in another way. In the picture above, the base of the prism is a rectangle of sides 5 and 3 centimetres, so that its area is 5×3 square centimetres. The volume is the product of this area and the height 2 centimetres of the prism.

Since all faces of a rectangular prism are rectangles, any face can be taken as the base.

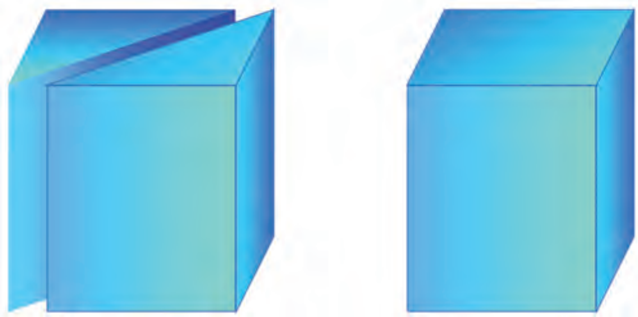




And whichever way we take, the volume is the product of the base area and height, isn't it?

Let's see whether the volume of any prism can be computed like this. First we take a right triangular prism:

Joining with an identical prism turns it into a rectangular prism.

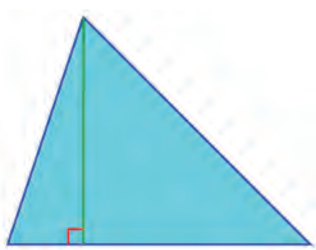


To find the volume of a prism drawn in GeoGebra, select Volume and click on the prism. In the algebra window we see a letter and a number below prism. This number give the volume of the prism.

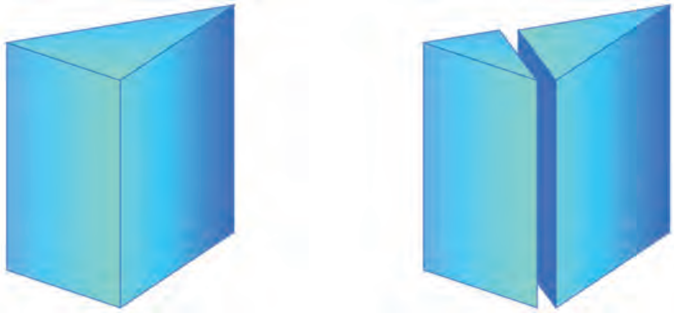
8L66VB

If the area of the right triangle is taken as a, the base area of the rectangular prism is $2a$. Its height is the same as that of the original triangular prism. Taking it as h, the volume of the rectangular prism is $2ah$. This is the volume of two identical triangular prisms put together. So the volume of the triangular prism is ah. That is, the product of base area and height.

Now, what if the base is a non-right triangle? We can split it into two right triangles by drawing a perpendicular from a vertex.



So, the prism itself can be vertically split through this perpendicular and a parallel line in the upper triangle to make two right triangular prisms:



The sum of the volumes of these two prisms is the volume of the original prism. Taking the base area before division as a and the base area of the two prisms got by splitting as b and c, we have $a = b + c$. All these prisms have the same height. Taking it as h, the sum of the volumes of the right triangular prisms is $bh + ch = (b + c)h = ah$. And this sum is the volume of the original prism, right?

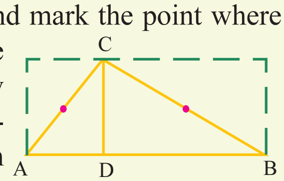
Thus we see that the volume of any triangular prism is the product of the base area and height.

We can divide any polygon into triangles, by joining one vertex to all other vertices; and the area of the polygon is the sum of the areas of these triangles:

So, any prism can be split into triangular prisms:

Draw a right triangle and a prism based on it. Mark the midpoint of the hypotenuse. **Select Reflect about Point** and click on the triangle and this point. We get a copy of the first triangle, making a rectangle. Based on this new triangle draw another prism of the same height. These two right triangular prisms together make a rectangular prism.

Draw a triangle and draw a prism based on it. Draw the perpendicular from a vertex of a triangle to the opposite side and mark the point where it meets the side. Draw two right triangles with this point as the right angled corner. Draw the reflection of each on the midpoint of its hypotenuse. (use **Reflect about Point**). Hide the first prism and draw four prisms of the same height, based on the four right triangles we now have. These four together make a rectangular prism. What is the relation between the volume of this and that of the first triangular prism?



Let's write the base area of a prism as a and its height as h. If the base is split into n triangles as above, the prism can be split into n triangular prisms. Taking their base areas as $b_1, b_2, \ldots, b_n$, their volumes are $b_1h, b_2h, \ldots, b_nh$. So the volume of the original prism is

$$b_1h + b_2h + \ldots + b_nh = (b_1 + b_2 + \ldots + b_n)h = ah$$

Thus we have the following:

The volume of any prism is the product of its base area and height.

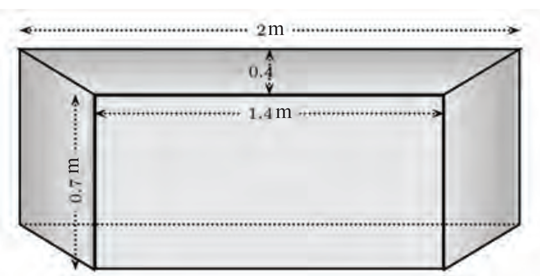
8LF2HY

For example, if the base of a prism is an equilateral triangle of side 4 centimetres and its height is 10 centimetres, then its volume is

$$\frac{\sqrt{3}}{4} \times 16 \times 10 = 40\sqrt{3} \text{ cubic centimetres.}$$

Here's another problem: the picture below is that of a water trough:

This is a prism with identical isosceles trapeziums as the front and back faces. To see that it is actually a prism, think about the trough on its side like this. How many litres of water can it hold?



The area of a trapezium is

$$\frac{1}{2} \times (2 + 1.4) \times 0.4 = 0.68 \text{ square metres}$$

The volume of the trough is

$$0.68 \times 0.7 = 0.476 \text{ cubic metres}$$

One cubic metre is thousand litres. So the trough can hold 476 litres.

(A prism need not always be placed with the base at the bottom)

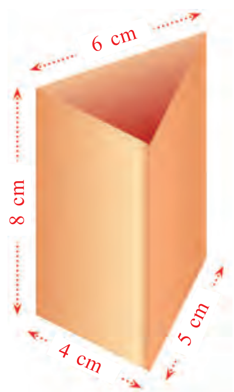
(1) The base of a prism is an equilateral triangle of perimeter 15 centimetres and its height is 5 centimetres. Calculate its volume.

(2) A hexagonal hole of each side 2 metres is dug in the school ground to collect rain water. It is 3 metres deep. It now has water one metre deep. How much litres of water is there in it?

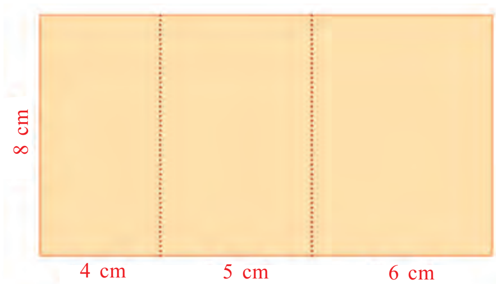
(3) A hollow prism of base a square of side 16 centimetres contains water 10 centimetres high. If a solid cube of side 8 centimetres is immersed in it, by how much would the water level rise?

Area

A hollow tube is to be made of cardboard with dimensions as shown in this picture.



We can make it by pasting three separate rectangles. Or by folding and pasting a single rectangle like this:



How much paper is needed for this?

The area of the rectangle is,

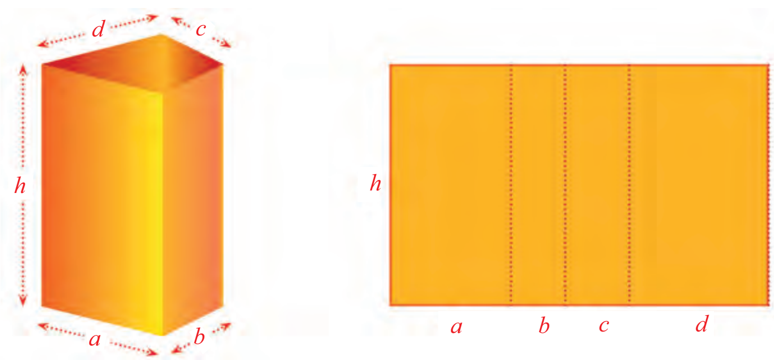
$(4 + 5 + 6) \times 8 = 15 \times 8 = 120$ square centimetres

We can see how a hollow prism would look on cutting and spreading it out, using GeoGebra. Draw a prism as explained in **Solids in GeoGebra**. Select **Net** from **3D Graphics** and click on the prism. We get the shape got by cutting the prism and spreading it out flat. We also get a slider in **Graphics**. We can see the prism being formed by moving the slider. In the **Algebra** view, if we deselect the **Prism**, then it would be hidden. To hide all vertices, click **Points** in the **Algebra** view to select all points and then right click to deselect **Show Object**.

This is the sum of the areas of all lateral faces of the triangular prism. In general, the sum of the areas of all the lateral faces of a prism is called its *lateral surface area*.

To get the lateral surface area of the triangular prism in our problem, we multiplied 15 by 8. In this, $4 + 5 + 6 = 15$ is the perimeter of the triangular base and 8 is the height. A little further thought convinces that the lateral surface area of any triangular prism can be computed like this.

What if the base is a quadrilateral?



The lateral surface area of this quadrangular prism is $(a + b + c + d)\, h$; that is, the product of the perimeter of the base quadrilateral and the height. We can calculate the lateral surface area of any prism like this:

The lateral surface area of any prism is the product of the base perimeter and height.

For a closed prism, the total surface area can be calculated by adding the base areas to the lateral surface area.

Look at this problem:

The lateral surface area of a wooden prism of base an equilateral triangle is 48 square centimetres, and its height is 4 centimetres. Six of these are put together to make a hexagonal prism:

When we use **Net** to get the cut and spread shape from a prism, we get a letter and a number (for example h = 22). The number gives the surface area.

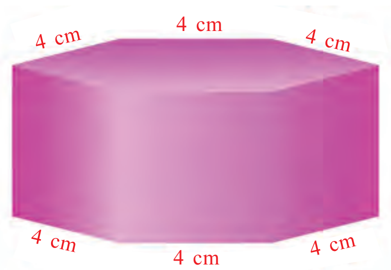
To cover it completely with coloured paper, how much paper do we need?

We want the total surface area of the hexagonal prism. For that, we must add the lateral surface area and the base areas.

To calculate the lateral surface area, we need the perimeter of the hexagon; and for that we have to calculate the sides of a triangle.

For any prism, the lateral surface area divided by the height gives the base perimeter. So the base perimeter of the base of a triangular prism in the problem is $48 \div 4 = 12$ centimetres.

Since the base is an equilateral triangle, the base perimeter is three times the length of a side. So, the length of a side is $12 \div 3 = 4$ centimetres.



The perimeter of a regular hexagon of side 4 centimetres is $6 \times 4 = 24$ centimetres. Since the height is also 4 centimetres, the lateral surface area is $24 \times 4 = 96$ square centimetres.

Next, we must add the areas of both bases. The area of one triangular base is

$$\frac{\sqrt{3}}{4} \times 4^2 = 4\sqrt{3} \text{ square centimetres.}$$

The area of the hexagon formed by six of these is $6 \times 4\sqrt{3} = 24\sqrt{3}$ square centimetres.

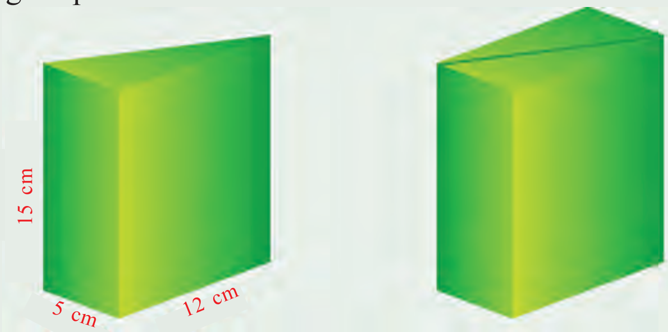
So, the total surface area of the hexagonal prism is

$$96 + \left(2 \times 24\sqrt{3}\right) = 96 + 48\sqrt{3} = 48\left(2 + \sqrt{3}\right) \text{ square centimetres.}$$

If we take 1.73 as an approximation for $\sqrt{3}$, we can see this to be a little more than 179 square centimetres. Anyway, 180 square centimetres of paper would be enough.

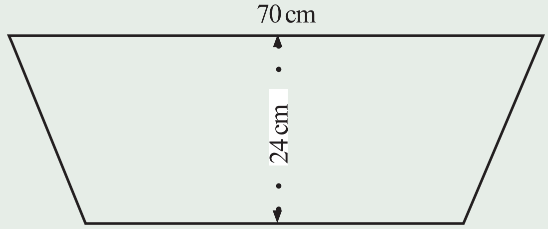
(1) The base of a prism is an equilateral triangle of perimeter 12 centimetres and its height is 5 centimetres. What is its total surface area?

(2) Two identical prisms with right triangles as base are joined to form a rectangular prism as shown below:



What is its total surface area?

(3) A water trough in the shape of a prism has trapezoidal faces. The dimensions of a base are shown in this picture:



The length of the trough is 80 centimetres. It is to be painted inside and outside. How much would be the cost at 100 rupees per square metre?

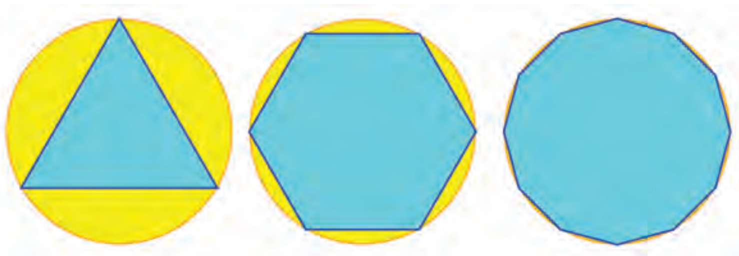
Cylinder

Prisms are solids with equal polygons at two ends and rectangles all around. There are solids with circles at both ends and a smoothly curving surface all around. You have seen several instances of such objects, solid and hollow, haven't you?

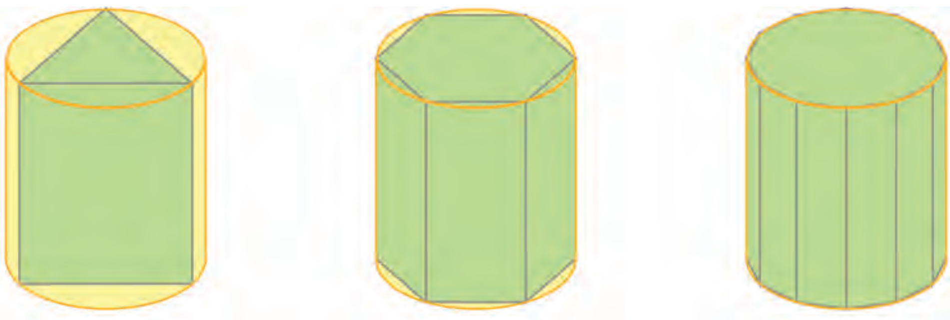
Such a solid is called a *cylinder*. Is the volume of a cylinder also, the product of base area and height?

We calculated the area of a circle by increasing the number of sides of the regular polygons drawn inside the circle, remember?

Make an Integer Slider n. Draw a circle and draw α regular polygon of n sides with vertices on the circle Open the **3D Graphics** window and draw a prism based on the polygon and cylinder based on the circle. Increase n and see the polygonal prism getting closer and closer to the cylinder.



So if we increase the number of sides of the bases of prisms, they get closer and closer to a cylinder:



8LNXYL

Taking the base areas of such prisms as p_1, p_2, p_3,.... and the base area of the cylinder as c, we see that the numbers p_1, p_2, p_3,... get closer and closer to the number c. All the prisms have the same height. Taking it as h, the volumes of the prisms are p_1h, p_2h, p_3h, ... and these numbers get closer and closer to ch. From the picture, we see that the volumes of the prisms get closer and closer to the volume of the cylinder. Thus we see that the volume of the cylinder is ch.

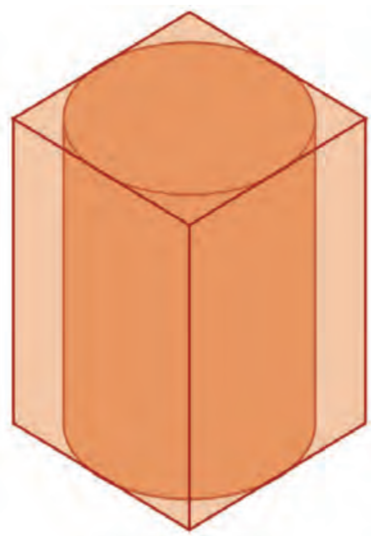
The volume of a cylinder is the product of its base area and height.

The area of a circle is the product of the square of its radius and π. So, if the base radius of a cylinder is 3 centimetres and its height if 5 centimetres, then its volume is $\pi \times 3^2 \times 5 = 45\pi$ cubic centimetres.

Another problem:

> The base of a rectangular block of wood is a square of side 10 centimetres and its height is 20 centimetres. What is the volume of the largest cylinder that can be carved out of this?

The base of the cylinder is the largest circle that can be drawn within the square base and its height is the same as that of the square prism:



That is, the diameter of the circle must be equal to a side of the square.

So, the base radius of the cylinder is 5 centimetres and hence its area is 25π square centimetres. Since the height is 20 centimetres, its volume is $25\pi \times 20 = 500\pi$ cubic centimetres.

(1) The base radius of an iron cylinder is 15 centimetres and its height is 32 centimetres. It is melted and recast into a cylinder of base radius 20 centimetres. What is the height of this cylinder?

(2) The base radii of two cylinders of the same height are in the ratio 3 : 4. What is the ratio of their volumes?

(3) The base radii of two cylinders are in the ratio 2 : 3 and their heights are in the ratio 5 : 4.

i) What is the ratio of their volumes?

ii) The volume of the first cylinder is 720 cubic centimetres. What is the volume of the second?

Curved Surface

We can make a cylindrical tube by rolling a rectangular sheet of paper or metal. On the other hand, if a hollow cylinder with both ends open is cut and spread flat, we get a rectangle:

The area of this rectangle is the *curved surface area* of the cylinder.

One side of this rectangle is the height of the cylinder: The other side is the base circle stretched straight; that is, its length is the perimeter of the circle. The curved surface area of the cylinder is the product of these lengths.

The curved surface area of a cylinder is the product of the base perimeter and height.

The perimeter of a circle is π times its diameter. So if the base radius of a cylinder is 3 centimetres and its height is 5 centimetres, then its curved surface area is $\pi \times 6 \times 5 = 30\pi$ square centimetres.

If this is a closed cylinder, then to get the total surface area, we must add the areas of the circles at the ends also.

That is, $30\pi + (2 \times 3^2 \times \pi) = 48\pi$ square centimetres.

When we draw a cylinder in GeoGebra, in the **Algebra** window we see a number below **Cylinder** which gives the volume, and a number below **Surface**, which gives the surface area.

(1) The inner diameter of a well is 2.5 metres and it is 8 metres deep. What would be the cost of cementing its inside at 350 rupees per square metre?

(2) The diameter of a road roller is 80 centimetres and it is 1.20 metres long:

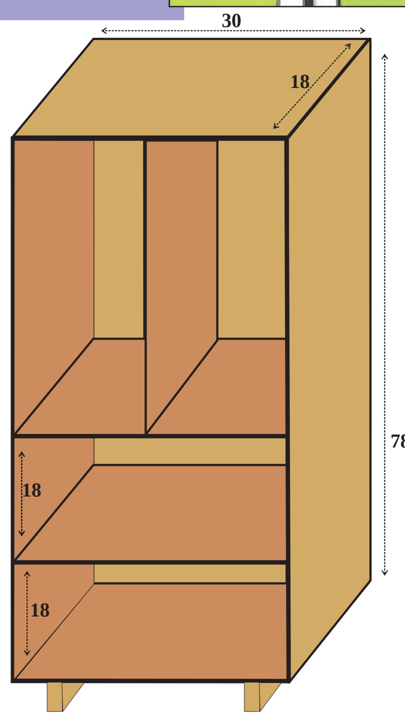
What is the area of levelled surface, when it rolls once?

(3) The base area and the curved surface area of a cyclinder are equal. What is the ratio of the base radius and height?

The picture shows a cupboard to be made. All lengths are in inches. Two doors are also to be made. Draw the various pieces of plywood needed to make this separately and mark the lengths. (Identical pieces needed are to be drawn only once, indicating the number of pieces needed)

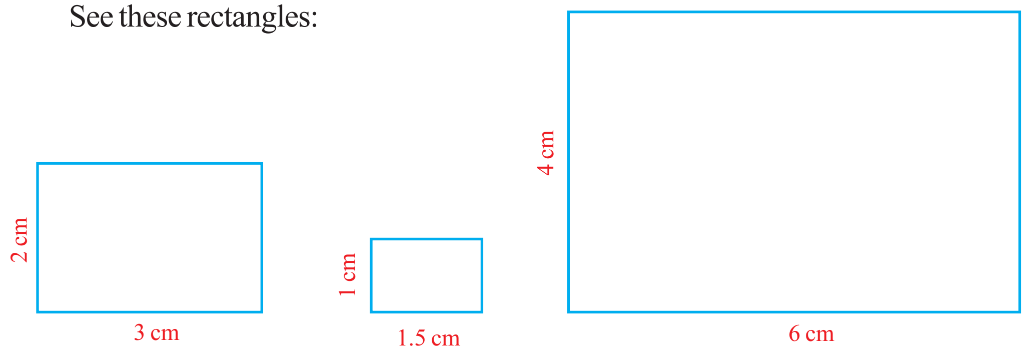
The thickness of plywood is to be 18 millimetres (about $\frac{3}{4}$ inch): Two types of sheets, 96×48 and 72×48 (in inches) are available in the market

To make the cupboard, how many of each sheet are to be bought? (Try to use the maximum possible part of each sheet): Try to make a model of this using thick cardboard.



12 Proportion

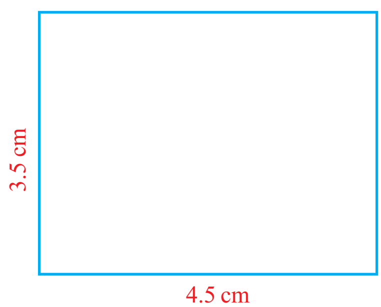
See these rectangles:



The widths and heights are different, but isn't there a connection? The width of the second rectangle is half that of the first; and the width of the third is twice that of the first. And the same goes for height also, right?

That is, the width and height are scaled by the same factor.

Now look at this rectangle:



Can we include it in the group?

Its width is one and a half times that of the first and the height is one and three quarters times. Since the width and height are not scaled by the same factor, this rectangle does not belong to the first group.

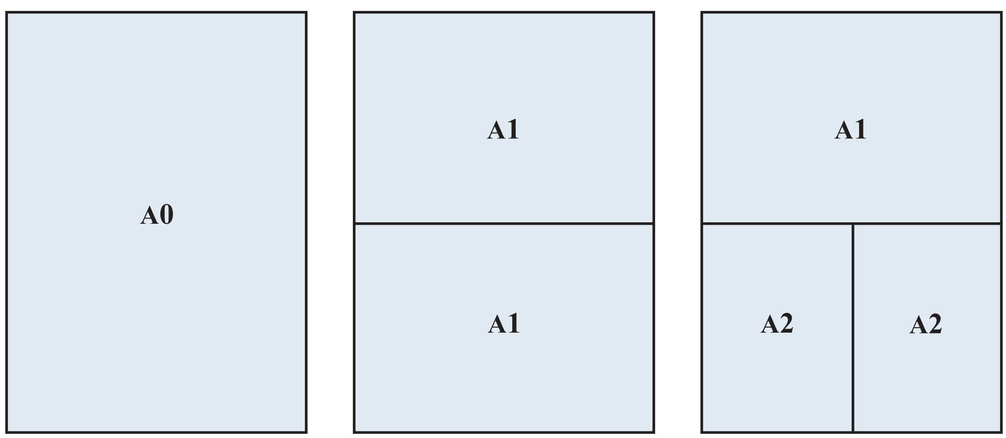
We came to this conclusion by comparing this rectangle with the first one in our group. We can see it in another way also. In all the rectangles of the group, the width is one and a half times the height, right? But it is not so in the new rectangle. In other words, the ratio of width to height in all three rectangles of the first group is 3 : 2. In the fourth rectangle, it is 9 : 7. And these ratios are not equal.

Equality of ratios is called *proportion*. Thus we can say that in the first three rectangles, widths and heights are *proportional*.

Rectangles of proportional width and height are often needed in various contexts. For example, we have seen that though television sets are made in different sizes, the width to height ratio is 16 : 9 in all; also that flags of different countries have definite width to height ratio (The lesson, **Ratio** in Class 7)

Let's look at another such instance. We usually use A4 paper for writing and printing. There are other paper sizes such as A0, A1, A2... What do these symbols stand for?

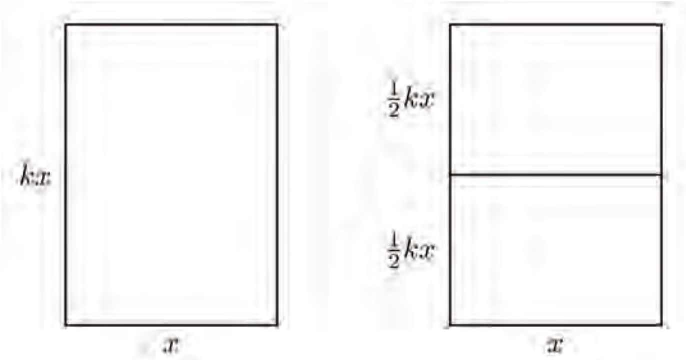
A1 is half the size of A0, half the size of A1 is A2 and so on:



There is another thing: The lengths of the sides of these rectangles are proportional.

Let's see how it is done. For that, we take any one size from these, say A1. Let's take the length of the shorter side as x and the length of the longer side as k times this.

So, what are the lengths of the sides of this cut in half?



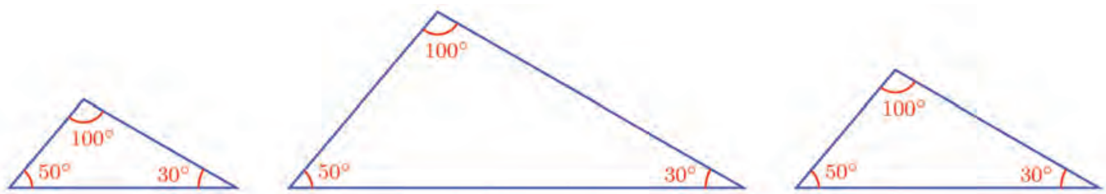
According to the specifications said earlier, in the half-rectangle also, the longer side should be k times the shorter side. So we get

$$x = k \times \frac{1}{2}\ kx = \frac{1}{2}k^2x$$

From this, we can see that , $\frac{1}{2}k^2 = 1$ and hence $k = \sqrt{2}$.

Thus in all paper sizes A0, A1, A2, ... the length of the longer side is $\sqrt{2}$ times the length of the shorter side.

We can talk about proportionality of more than two quantities also. See these triangles.



Since the angles are the same, the sides are scaled by the same factor. That is, taking any one triangle among these, the lengths of its sides are multiplied by the same number to get the lengths of the sides of any other. (The lesson **Similar Triangles**). In other words, the ratio of the lengths of the sides is the same for all these triangles.

In our new terminology, the sides of these triangles are proportional. Such proportional relations occur in other sciences also. The law of definite proportion in chemistry states that a chemical compound always contains the same proportion of elements by mass. For example, the mass of oxygen and hydrogen in any sample of water is approximately in the ratio 8 : 1. More precisely, 100 grams of water contains about 88.8 grams of oxygen and 11.2 grams of hydrogen. (What about one kilogram of water?)

(1) A person invests 10000 rupees and 15000 rupees in two different schemes. After one year, he got 900 rupees as interest for the first amount and 1200 rupees as interest for the second amount.

i) Are the interests proportional to the investments?

ii) What is the ratio of the interest to the amount invested in the first scheme? What about the second?

iii) What is the annual rate of interest in the first scheme? And in the second?

(2) The area of A0 paper is one square metre. Calculate the lengths of the sides of A4 paper correct to a millimetre, using a calculator.

(3) In calcium carbonate, the masses of calcium, carbon and oxygen are in the ratio 10 : 3 : 12. When 150 grams of a compound was analysed, it was found to contain 60 grams of calcium, 20 grams of carbon and 70 grams of oxygen. Is it calcium carbonate?

Constancy in proportionality

If all sides of a square are doubled to make a larger square, what happens to the perimeter?

If the lengths of the sides were originally 1 centimetre, it would become 2 centimetres. And the perimeter, originally 4 centimetres would become 8 centimetres. Thus the perimeter also is doubled.

Is it true for all squares?

To check whether a relation between numbers hold in general, a good method is to use algebra. If we take the lengths of the sides of the original rectangle as x centimetres, the perimeter is $4x$ centimetres upon doubling, it becomes $4 \times 2x = 8x$ centimetres. That is, the perimeter is also doubled.

What if the sides are halved ? Or increased to one and a half times the original length?

In general, the side and perimeter of a square are scaled by the same factor. In other words, however much we alter a square, the ratio of side to perimeter does not change.

We can also say that the perimeter of a square varies in proportion to the length of a side. So, the relation between the length of a side and perimeter can be stated in various forms:

- In any square, the perimeter is 4 times the length of a side.
- In any square, the ratio of the length of a side to the perimeter is 1 : 4
- The length of a side and perimeter of a square are scaled by the same factor.
- The perimeter of a square varies proportionally as the length of a side.

We have seen in the lesson, **New Numbers**, that the length of a diagonal of a square is $\sqrt{2}$ times the length of a side. In what all ways can we state this, as stated above?

Now let's take areas of squares instead of perimeters. The area of a square of side 1 centimetre is 1 square centimetre; if the sides are doubled, the area would be 4 square centimetres. So, the length of side and the area are not scaled by the same factor; that is, they are not proportional.

We have seen in the lesson, **Similarity of Triangles**, how similar figures can be drawn in GeoGebra, using **Dilate from Point**. Draw a regular polygon and a similar polygon. Mark their lengths of sides, perimeters and areas. Move the slider to see which of the following pairs are proportional:

i) Length of a side and perimeter
ii) Length of a side and area
iii) Perimeter and area

Let's take an example from physics. An object moving at a steady speed of 10 metres/second travels 10 metres in 1 second, 20 metres in 2 seconds and 5 metres in $\frac{1}{2}$ second.

In general, in x seconds, it travels $10x$ metres. Thus the distance travelled varies as 10 times the time. In other words, the ratio of time to distance is always 1 : 10. That is, distance is proportional to time.

Now suppose the speed is always changing. For example, the speed of an object falling from a height changes every instant. In x seconds, it travels $4.9x^2$ metres. Thus it travels 4.9 metres in one second and 19.6 metres in two seconds. So in this motion, the change in time and distance are not by the same factor. Then the ratio changes every instant; they are not proportional.

In this example, if we take the speed at x second as y metres/second, then the time-speed equation is $y = 9.8x$. Is the speed proportional to time?

Let's look at these examples together;

Context	Quantities		Equation	Proportional
	x	y		
Square	Side	Perimeter	$y = 4x$	Yes
	Side	Diagonal	$y = \sqrt{2}x$	Yes
	Side	Area	$y = x^2$	No
Motion Steady speed	Time	Distance	$y = 10x$	Yes
Motion Varying speed	Time Time	Distance Speed	$y = 4.9x^2$ $y = 9.8x$	No Yes

In GeoGebra, make an angle slider α. Draw a line AB and another line AB′ making angle α with AB. Mark a point C on this line and draw the perpendicular from C to AB. Mark the point of intersection of this perpendicular with AB as D. Now hide the perpendicular and mark the lengths of CA and CD. Change the position of C on AB′ . Do the lengths of CA and CD change proportionally? Compute the constant of proportionality for α equal to 30°, 45^0, 60°.

What do we see in all these? When one quantity changes, all related quantities change. If the dependent quantity changes as a fixed multiple of the independent quantity, then the ratio of these quantities does not change; the change is proportional.

Let's put this in algebra. Let's take the independent quantity as x and the dependent quantity as y. If y is always x multiplied by a fixed number k (which does not change with x), then we can write the relation between them as

$$y = kx$$

We can also write it as

$$\frac{y}{x} = k$$

We see that the ratio of x to y remains unchanged as $1 : k$.

That is, y changes proportionally to x.

The fixed number occurring in the equation of proportional change is called the *constant of proportionality*.

For example, in the case of an object falling towards the earth, the fixed number in time-speed equation is 9.8. It is the constant of proportionality in the time-speed relation. It is interpreted in physics as the acceleration due to gravity of the earth.

Again, for objects made of the same material, the mass is proportional to the volume. The constant of proportionality in this case is called density of the material. For example, the density of iron is 7.87 and that of copper is 8.96. Thus the mass of an object made of iron is 7.87 times the volume and the mass of an object made of copper is 8.96 times the volume.

(1) For each pair of quantities given below, check whether the first is proportional to the second. For proportional quantities, calculate the constant of proportionality.

i) Perimeter and radius of circles

ii) Area and radius of circles.

iii) The distance travelled and the number of rotations of a circular ring moving along a line.

iv) The interest got in a year and the amount deposited in a scheme in which interest is compounded annually.

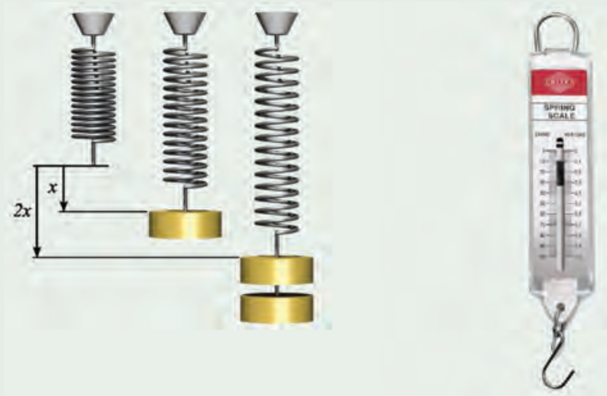
v) The volume of water poured into a hollow prism and the height of the water level.

(2) During rainfall, the volume of water falling in each square metre may be considered equal.

i) Prove that the volume of water falling in a region is proportional to the area of the region.

ii) Explain why the heights of rainwater collected in different sized hollow prisms kept near one another are equal.

(3) When a weight is suspended by a spring, the extension is proportional to the weight. Explain how this can be used to mark weights on a spring balance.



(4) In the angle shown below, for different points on the slanted line, as the distance from the vertex of the angle changes, the height from the horizontal line also changes.

i) Prove that height is proportional to the distance.

ii) Calculate the constant of proportionality for 30°, 45° and 60° angles.

Different proportions

We have seen the relation between the number of sides of a polygon and the sum of its inner angles, in Class 8. Is this change proportional?

The sum of the angles of a triangle is 180° and the sum of the angles of a hexagon is 720°. Thus when the number of sides is doubled, the sum of the inner angles becomes greater than the double. So this relation is not proportional.

We know that the sum of the inner angles is got by multiplying two less than the number of sides by 180°. That is, if we write the number of sides as n and the sum of the inner angles as s, then.

$$s = 180\,(n - 2)$$

Let's write $n - 2$ as m.

Then the equation becomes,

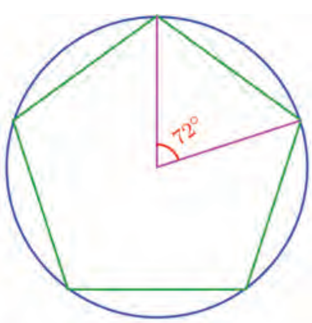
$$s = 180\,m$$

So the quantity s is proportional to the quantity m. In ordinary language, the sum of the inner angles is proportional to two less than the number of sides.

There are several such instances, where a quantity is not proportional to another quantity, but is proportional to the second quantity changed a little bit. For example, the area of a circle is the product of the square of the radius by the fixed number π and so the area is not proportional to the radius, but area is proportional to the square of the radius.

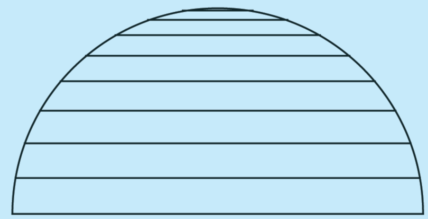
Similarly, the distance travelled by a falling body is not proportional to time; but proportional to the square of the time.

Let's look at another problem: For any regular polygon, we can draw a circle through all vertices. What can we say about the angle made by two adjacent vertices at the centre of the circle?

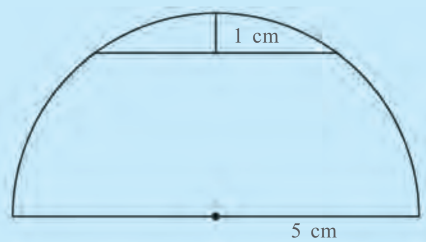


Proportionality problem

The picture shows several chords of a semicircle.

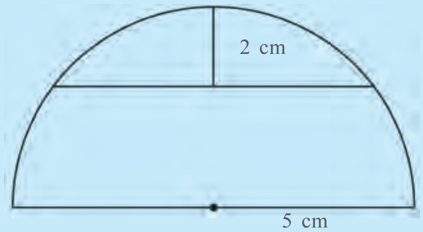


As the distance from the top increases, so does the length of the chord. Is the change proportional? Let's look at an example.



In the picture above, the length of the chord can be computed as 6 centimetres (Try it!).

Now look at this picture:

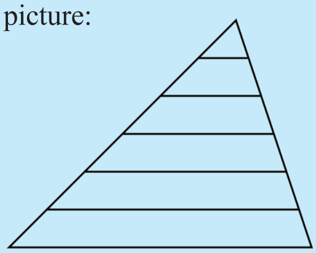


Now the length of the chord is 8 centimetres.

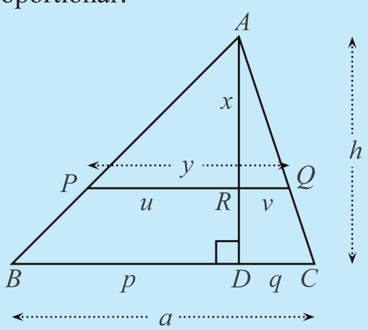
When the distance from the top is doubled, the length of the chord is not doubled. So this change is not proportional.

Height and width

See this picture:



Several lines are drawn parallel to the base of a triangle. As the distance from the top vertex increases, so does the length of the line. Is this proportional?



Since ΔAPR and ΔABD are similar,

$$\frac{u}{p} = \frac{x}{h}$$

Since ΔAQR and ΔACD are similar

$$\frac{v}{q} = \frac{x}{h}$$

From these , we get

$$\frac{u}{x} = \frac{p}{h}, \quad \frac{v}{x} = \frac{q}{h}$$

So

$$\frac{y}{x} = \frac{u+v}{x} = \frac{p+q}{h} = \frac{a}{h}$$

For different lines, x and y change, but not a or h. That is, the ratio of y to x does not change.

8LXU19

If this central angle for a regular polygon of x sides is taken as y^{o}, then

$$y = \frac{360}{x} \qquad y = 360 \times \frac{1}{x}$$

Thus in this case, y is proportional to the reciprocal of x.

Such changes in which a quantity is proportional to the reciprocal of another quantity is seen often in physics. Such a change is called *inverse proportion*. That is, if the equation of the change in a quantity y depending on a quantity x, is of the form $y = \frac{k}{x}$, where k is a fixed number, we say that y is inversely proportional to x.

To contrast with this, a change given by the equation $y = kx$ is also called *direct proportion*.

Here's another example of inverse proportion.

Imagine an object travelling from one point to a point 100 metres away at a steady speed along a straight line. If the speed is 10 metres/second, the time to reach the second point is 10 seconds. If the speed is increased to 25 metres/second, then the time needed is only 4 seconds.

In general, if we write the speed as x metres/second and the time taken to reach the destination as y seconds, then

$$y = \frac{100}{x}$$

So, y is inversely proportional to x.

Many of the laws of physics are stated in terms of proportions. One of the most important of such laws is Newton's Law of Universal Gravitation.

> Any two bodies in the Universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Taking the masses of two objects as m_1 and m_2, and the distance between them as r, the algebraic form of this law can be written like this;

$$F = G\frac{m_1 m_2}{r^2}$$

(1) i) Prove that for equilateral triangles, area is proportional to the square of the length of a side. What is the constant of proportionality?

ii) For squares, is area proportional to square of the length of a side? If so, what is the constant of proportionality?

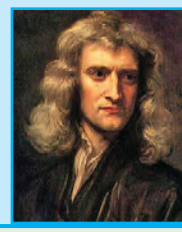
(2) In rectangles of area one square metre, as the length of one side changes, so does the length of the other side. Write the relation between the lengths as an algebraic equation. How do we say this in the language of proportions?

(3) In triangles of the same area, how do we say the relation between the length of the longest side and the length of the perpendicular from the opposite vertex? What if we take the length of the shortest side instead?

(4) In regular polygons, what is the relation between the number of sides and the degree measure of an outer angle? Can it be stated in terms of proportion?

Newton

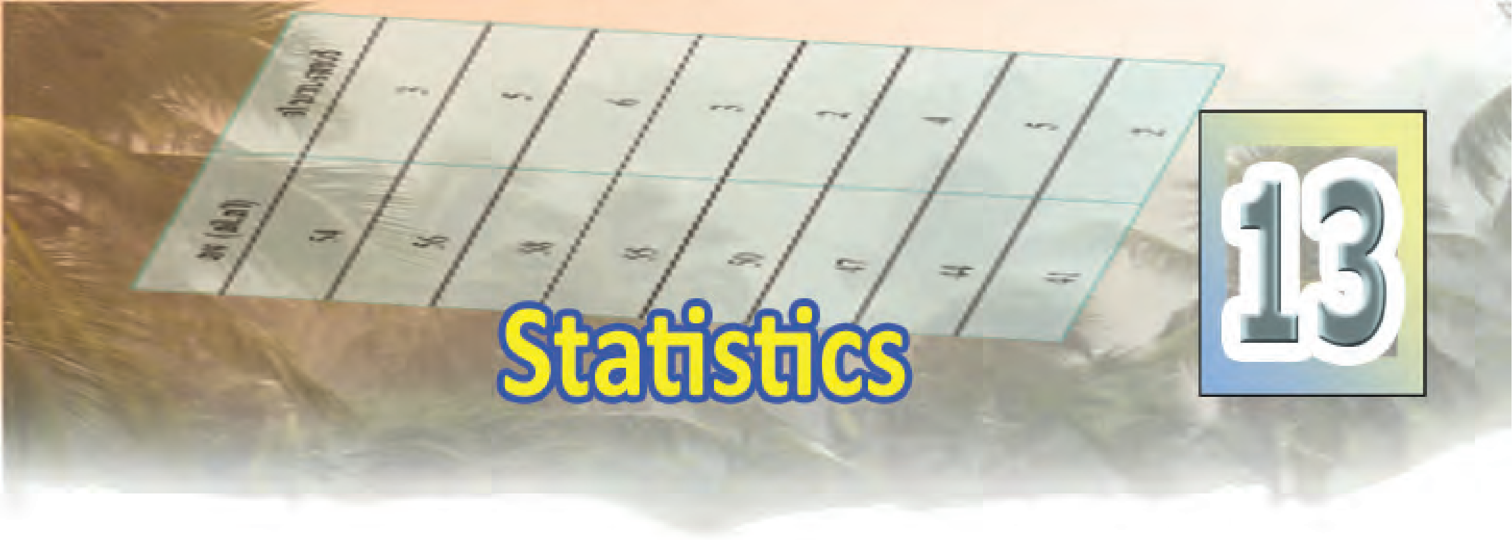
It was Galileo in the sixteenth century who first presented the idea that the laws of nature are to be studied using mathematics. The greatest example of this is the book, *Philosophia Naturalis Principia Mathematica*. (The mathematical principles of natural philosophy) published by Newton in the seventeenth century. He presented the mathematical laws of motion and the law of universal gravitation in this. For this, he invented a new mathematical technique, which later developed into the branch of mathematics called calculus.



(5) A fixed volume of water is to flow into a rectangular water tank. The rate of flow can be changed by using different pipes. Write the relations between the following quantities as an algebraic equation and in terms of proportions.

i) The rate of water flow and the height of the water level.

ii) The rate of water flow and the time taken to fill the tank.



13

Statistics

Average

Remember studying averages in Class 6? Let's do a problem on average.

The daily earnings of five friends who work in a factory are these:

350 rupees, 400 rupees, 350 rupees, 450 rupees, 450 rupees

What is the average daily earning of a worker in this group?

The total daily earning of all five is to be divided by five. Here if we note that the numbers 350 and 450 occur twice, the summing up can be speeded up a little:

$$(2 \times 350) + (2 \times 450) + 400 = 2000$$

8M7Q2H

And the average is 400 rupees.

Instead of mentioning the daily earning of each separately if only the average daily earning is given as 400 rupees, we would still get a rough idea of the economic condition of these five people.

Now look at this problem:

The table below shows the number of people earning different daily wages in a factory.

Daily wages (Rs)	Number of workers
300	2
350	4
400	6
450	4
500	4

What is the average daily wage in this factory?

There are 20 workers altogether. We have to first find the total daily wages. As before, repeated addition can be written as multiplication.

Daily wages (Rs)	Number of workers	Total wages (Rs)
300	2	600
350	4	1400
400	6	2400
450	4	1800
500	4	2000
Total	20	8200

And then, we can compute the average as $8200 \div 20 = 410$ rupees.

Note that the daily wage of each is between 300 and 500 rupees, and so is the average. Is it true always?

For example, consider any 8 numbers between 100 and 200. Since all numbers are greater than or equal to 100, the total must be greater than or equal to 800. The average got by dividing the total by 8 must also be greater than or equal to 100.

In the same way, we can see the average is less than or equal to 200.

For any other number instead of 100, 200 and 8, we can reason like this. In general, we can say this.

The average of any set of numbers between two fixed numbers is also between these two numbers.

(1) The weight of 6 players in a volley ball team are all different and the average weight is 60 kilograms.

i) Prove that the team has at least one player weighing more than 60 kilograms.

ii) Prove that the team has at least one player weighing less than 60 kilograms.

(2) Find two sets of 6 numbers with average 60, satisfying each of the conditions below:

i) 4 of the numbers are less than 60 and 2 of them greater than 60.

ii) 4 of the numbers are greater than 60 and 2 of them less than 60.

(3) The table shows the children in a class, sorted according to the marks they got for a math test.

Marks	No. of Children
2	1
3	2
4	5
5	4
6	6
7	11
8	10
9	4
10	2

Calculate the average mark of the class.

(4) The table below shows the days in a month sorted according to the amount of rainfall in a locality.

Rainfall (mm)	No. of Days
54	3
56	5
58	6
55	3
50	2
47	4
44	5
41	2

What is the average rainfall per day during this month?

(5) The details of rubber sheets a farmer got during a month are given below.

Rubber (Kg)	No. of Days
9	3
10	4
11	3
12	3
13	5
14	6
16	6

i) How many kilograms of rubber did he get a day on average in this month?

ii. The price of rubber is 120 rupees per kilogram. What is his average income per day this month from selling rubber?

Frequency tables

We have seen in Class 8, how data is tabulated in classes and frequencies, when the amount of data is large. Let's look at such a problem.

The table shows the labourers in a factory sorted according to their daily wages:

Daily Wages (Rs)	Number of workers
250 - 300	8
300 - 350	4
350 - 400	16
400 - 450	7
450 - 500	5

What is the average daily wage in this factory?

How do we calculate the total daily wages paid? From the first row of the table, we see that 8 workers are paid between 250 and 300 rupees, but we do not know how much exactly each is paid. So with only this information, we cannot compute their total daily wages.

In such cases, we have to make some assumptions about the missing information. Though we do not know the individual incomes of the 8 workers in the first row of the table, we know that all of them are between 250 rupees and 300 rupees; so the average wage of these 8 workers must also be between 250 rupees and 300 rupees. Moreover, this average is usually somewhere around the middle of 250 and 300 rupees.

So, to compute the average from such tables, we make the assumption that the average of each class is exactly the mid-value of that class.

Under this assumption, we can expand the table in this problem as shown below:

Daily wages (Rs.)	Number of workers	Mid value of class	Total wages
250 - 300	8	275	2200
300 - 350	4	325	1300
350 - 400	16	375	6000
400 - 450	7	425	2975
450 - 500	5	475	2375
Total	**40**		**14850**

Now we can calculate the average.

Very often we have to deal with large sets of data, such as the heights and weights of all school children in Kerala, the monthly income of all people in Kerala and so on. And we need to compute a few numbers which give some estimate of their general nature. There are several methods to do this. Dividing the total by the number is only one of these methods. Some numbers calculated thus are generally called *averages* or *measures of central tendency* in statistics. The number got on dividing the sum by the number, which we usually call average is technically called *arithmetic mean* or simply, *mean*.

For example, the mean daily wage in the problem we have been discussing now is 371.25 rupees.

(1) Find different sets of 6 different numbers between 10 and 30 with each number given below as mean:

i) 20 ii) 15 iii) 25

(2) The table below shows the children in a class, sorted according to their heights.

Height (cm)	Number of children
148 - 152	8
152 - 156	10
156 - 160	15
160 - 164	10
164 - 168	7

What is the mean height of a child in this class?

(3) The teachers in a university are sorted according to their ages, as shown below.

Age	Number of Persons
25 - 30	6
30 - 35	14
35 - 40	16
40 - 45	22
45 - 50	5
50 - 55	4
55 - 60	3

What is the mean age of a teacher in this university?

(4) The table below shows children in a class sorted according to their weights.

Weight (kg)	21 - 23	23 - 25	25 - 27	27 - 29	29 - 31	31 - 33
Number of children	4		7	6	3	1

The mean weight is calculated as 26 kilograms. How many children have weights between 23 and 25 kilograms?

NOTES

Let's know about cyber safety

There is absolutely no need to mention the advantages of Internet and Social Networking sites. We have embraced their potential for communication, entertainment and information seeking.

But over the period, it is seen that a lot of teenagers are being harassed and fall prey to the abuse of Social Media. You can easily prevent yourself from being a victim, if you take a few precautionary measures while being online.

How Social Networking sites can be dangerous

- Sharing and posting too much of personal information such as phone number, address, location, photos, etc., can be misused.
- Trusting strangers believing their profile to be true can be dangerous, as they may not be the same as stated.
- Snapshots of chats, photos, videos, etc., are saved and will be used for blackmailing and threatening.
- Being cyber bullied by posting negative, derogatory comments, posts, photos, etc. to tarnish one's image.
- Lots of predators and adult criminals are lurking online to trap children.

Tips for safe Social Networking

- Always keep your personal information strictly personal.
- Customize your privacy settings so that others can see only the basic information.
- Just know about and manage your friends. Don't trust all the online friends.
- Let your friends know that you are uncomfortable if they post something inappropriate about you.
- Do not publish any information that reveals your identity.
- Always use strong passwords. Don't share them with others.
- Never share your pictures, photographs, email accounts, etc., with anyone.
- Keep your personal messages strictly personal. Once posted they are published for ever.
- If ever threatened or bullied seek the help of parents/teachers.

Helpline Phone Numbers

Crime Stopper: 1090
Cyber Cell (Tvm): 9497975998
Control Room: 100
Child Helpline: 1098 / 1517