EXAM NO: | | | | | |

STD: XII	MATHEMATICS	MARKS: 90
DATE:	GOVT. PUBLIC EXAM (SEP 2020)	TIME: 3.00hr

PART – A

Choose the correct answer **20 x 1 = 20**

1. If $A^T A^{-1}$ is symmetric, then $A^2 =$

(1) A^{-1} **(2) $(A^T)^2$** (3) A^T (4) $(A^{-1})^2$

2. If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then the values of x and y are respectively,

(1) $e^{(\Delta_2/\Delta_1)}, e^{(\Delta_3/\Delta_1)}$ (2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$ **(4) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$**

3. If z is a non zero complex number, such that $2iz^2 = \bar{z}$ then $|z|$ is

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3

4. If $a = 3 + i$ and $z = 2 - 3i$ then the points on the Argand diagram representing $az, 3az$ and $-az$ are

(1) Vertices of a right angled triangle (2) Vertices of an equilateral triangle

(3) Vertices of an isosceles triangle **(4) Collinear**

5. The polynomial $x^3 - kx^2 + 9x$ has three real roots if and only if, k satisfies

(1) $|k| \le 6$ (2) $k = 0$ (3) $|k| > 6$ **(4) $|k| \ge 6$**

6. The $\cot^{-1} x = \frac{2\pi}{5}$ for some $x \in \mathbb{R}$, the value of $\tan^{-1} x$ is

(1) $-\frac{\pi}{10}$ (2) $\frac{\pi}{5}$ **(3) $\frac{\pi}{10}$** (4) $-\frac{\pi}{5}$

7. $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid for

(1) $-\pi \le x \le 0$ **(2) $0 \le x \le \pi$**

(3) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (4) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

8. The length of the latus rectum of the parabola $y^2 - 4x + 4y + 8 = 0$ is

(1) 8 (2) 6 **(3) 4** (4) 2

9. If the coordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are $(11, 2)$, the coordinates of the other end are

(1) $(-5, 2)$ (2) $(2, -5)$ (3) $(5, -2)$ (4) $(-2, 5)$

10. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j}, \vec{c} = \hat{i}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = \lambda\vec{a} + \mu\vec{b}$, then the value of $\lambda + \mu$ is

(1) 0 (2) 1 (3) 6 (4) 3

11. The coordinates of the point where the line $\vec{r} = (6\hat{i} - \hat{j} - 3\hat{k}) + t(-\hat{i} + 4\hat{k})$ meets the plane $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 3$ are

(1) $(2, 1, 0)$ (2) $(7, -1, -7)$ (3) $(1, 2, -6)$ **(4) $(5, -1, 1)$**

12. Angle between $y^2 = x$ and $x^2 = y$ at the origin is

(1) $\tan^{-1}\frac{3}{4}$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$ **(3) $\frac{\pi}{2}$** (4) $\frac{\pi}{4}$

13. The number given by the Rolle's theorem for the function $x^3 - 3x^2, x \in [0,3]$ is

(1) 1 (2) $\sqrt{2}$ (3) $\frac{3}{2}$ **(4) 2**

14. If $w(x,y) = x^y, x > 0$, then $\frac{\partial w}{\partial x}$ is equal to

(1) $x^y \log x$ (2) $y \log x$ **(3) yx^{y-1}** (4) $x \log y$

15. The value of $\int_{-\pi/2}^{\pi/2} \left(\frac{\sin x}{1+\cos x}\right) dx$ is

(1) 0 (2) 2 (3) $\log 2$ (4) $\log 4$

16. If $f(x) = \int_0^x t \cos t \, dt$, then $\frac{df}{dx} =$

(1) $\cos x - x \sin x$ (2) $\sin x + x \cos x$

(3) $x \cos x$ (4) $x \sin x$

17. If $\cos x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$ then P

(1) $-\cot x$ (2) $\cot x$ (3) $\tan x$ **(4) $-\tan x$**

18. The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(1) $x\,\phi\left(\frac{y}{x}\right) = k$ **(2) $\phi\left(\frac{y}{x}\right) = kx$** (3) $y\,\phi\left(\frac{y}{x}\right) = k$ (4) $\phi\left(\frac{y}{x}\right) = ky$

19. In 16 throws of a die getting an even number is considered a success. Then the variance of the successes is

(1) 4 (2) 6 (3) 2 (4) 256

20. The operation $*$ defined by $a * b = \frac{ab}{7}$ is not a binary operation on

(1) $\mathbb{Q}^+$ **(2) $\mathbb{Z}$** (3) $\mathbb{R}$

PART - B

Answer any SEVEN questions

Question number 30 is compulsory **7 x 2 = 14**

21. Find the least positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1$

Solution:

$$\text{Consider } \frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1-1)+i(1+1)}{1^2+(-1)^2} = \frac{0+2i}{1+1} = \frac{2i}{2} = i$$

$$\text{G.T. } \left(\frac{1+i}{1-i}\right)^n = 1$$

$$\therefore i^n = 1$$

$$\text{Since } i^4 = i^8 = i^{12} = \cdots = 1$$

Hence the least positive integer $n = 4$

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22. Obtain the Cartesian form of the locus of $z = x + iy$ in $|z + i| = |z - 1|$.

Solution:

$$\text{G.T. } z = x + iy$$

$$|z + i| = |z - 1|$$

$$\Rightarrow |x + iy + i| = |x + iy - 1|$$

$$\Rightarrow |x + i(y+1)| = |(x-1) + iy|$$

$$\Rightarrow \sqrt{x^2 + (y+1)^2} = \sqrt{(x-1)^2 + y^2}$$

Square on both sides,

$\Rightarrow x^2 + (y+1)^2 = (x-1)^2 + y^2$

$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$

$\Rightarrow 2x + 2y = 0$

$\Rightarrow x + y = 0,$ the locus of z in Cartesian form

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23. If α, β, γ and δ are the roots of the polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a quadratic equation with integer coefficients whose roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$.

Solution:

G.T. $2x^4 + 5x^3 - 7x^2 + 8 = 0$

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{5}{2}$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{8}{2} = 4$$

$$S.R. = -\frac{5}{2} + 4 = \frac{-5+8}{2} = \frac{3}{2}$$

$$P.R. = \left(-\frac{5}{2}\right)(4) = -10$$

The quadratic equation is

$$x \;\; - (S.R.)x + (P.R.) = 0$$

$$x \;\; - \frac{3}{2}x - 10 = 0$$

$2x \;\; - 3x - 20 = 0$ is the req. quadratic equation with integer coefficients

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24. $\tan \;\; (\sqrt{3})$

Solution:

Let $y = \tan \;\; (\sqrt{3})$

$$\tan y = \sqrt{3}$$

$$\tan y = \sqrt{3} = \tan\frac{\pi}{3}$$

$$y = \frac{\pi}{3} \qquad \because \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

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25. If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$.

Solution:

G.T. $\hat{a}, \hat{b}, \hat{c}$ are the unit vectors

i.e., $|\hat{a}| = |\hat{b}| = |\hat{c}| = 1$

$$\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$$

$$\Rightarrow (\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b}$$

Comaring $\hat{b}$ on both sides

$$\hat{a}.\hat{c} = \frac{1}{2}$$

$$\Rightarrow |\hat{a}||\hat{c}|\cos\theta = \frac{1}{2}$$

$$\Rightarrow (1)(1)\cos\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{3}}$$

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26. Evaluate the limit $\lim_{x\to 0}\left(\frac{\sin mx}{x}\right)$.

Solution:

$$\text{G. T.} \quad \lim_{x\to 0}\left(\frac{\sin mx}{x}\right) \qquad \left(\because \frac{0}{0} \text{ form}\right)$$

$$\lim_{x\to 0}\left(\frac{\sin mx}{x}\right) = \lim_{x\to 0}\left(\frac{m\cos mx}{1}\right)$$

$$= \frac{m\cos 0}{1} = \frac{m(1)}{1} = m$$

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27. Evaluate $\int_3^4 \frac{dx}{x^2-4}$.

Solution:

$$^{4}_{3}\ \frac{}{x^2 - 4} \qquad ^{4}_{3}\ \frac{}{x^2 - 2^2} \qquad \frac{}{x^2 - a^2} \qquad \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$$

$$= \left[\frac{1}{2\times 2}\log\left|\frac{x-2}{x+2}\right|\right]_3^4$$

$$= \frac{1}{4}\left[\log\left(\frac{2}{6}\right) - \log\left(\frac{1}{5}\right)\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{2}{6}\times\frac{5}{1}\right)\right] = \frac{1}{4}\log\left(\frac{5}{3}\right)$$

$$\boxed{\therefore \int_3^4 \frac{dx}{x^2 - 4} = \frac{1}{4}\log\left(\frac{5}{3}\right)}$$

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28. Find the differential equation of the family of $y = ax^2 + bx + c$, where a, b are parameters and c is a constant.

Solution:

$$\text{G. T.} \quad y = ax^2 + bx + c \quad \rightarrow \quad (1)$$

$$y' = 2ax + b \quad \rightarrow \quad (2)$$

$$y'' = 2a \Rightarrow \frac{y''}{2} = a \quad \rightarrow \quad (3)$$

form (2), $y' = y''x + b \Rightarrow b = y' - y''x$

Sub a, b values in (1)

$$y = \frac{y''}{2}x^2 + (y' - y''x)x + c$$
$$2y = y'\,x^2 + 2y'x - 2y'\,x^2 + 2c$$
$$x^2y'' - 2xy' + 2y - 2c = 0$$

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29. Examine the binary operation of the operation $a * b = \left(\frac{a-1}{b-1}\right), \forall\, a, b \in \mathbb{Q}$.

Solution:

$$\text{G. T.} \quad a * b = \left(\frac{a\ -\ 1}{b\ -\ 1}\right)$$

Since denominator $b - 1$ must be non zero

Let $b - 1 = 0 \Rightarrow b = 1 \in \mathbb{Q}$,

$*$ is not a binary operation on $\mathbb{Q}$

Omitting 1 from $\mathbb{Q}$, the output $a * b$ exists in $\mathbb{Q} \setminus \{1\}$

Hence, $*$ is a binary operation on $\mathbb{Q} \setminus \{1\}$

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30. Show that, if $x = r\cos\theta, y = r\sin\theta$, then $\frac{\partial r}{\partial x}$ is equal to $\cos\theta$.

Solution:

$$x = r\cos\theta, y = r\sin\theta$$
$$x^2 = r^2\cos^2\theta, y^2 = r^2\sin^2\theta$$
$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2(1)$$
$$\therefore x^2 + y^2 = r^2$$
$$\frac{dr}{}$$
$$\frac{dr}{dx} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

PART – C

Answer any SEVEN questions

Question number 40 is compulsory **7 x 3 = 21**

31. Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$.

Solution:

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|}\ \text{adj}\ (AB)$$

$$(AB)^{-1} = \frac{1}{0+6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \rightarrow \text{①}$$

$$A^{-1} = \frac{1}{0+3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{2+0}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \rightarrow ②$$

From ① and ②, we get

$$\therefore \boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

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32. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj}A)A = |A|\, I$.

Solution:

$$|A| = 24 - 20 = 4$$

$$\text{adj } A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A(\text{adj } A) = |A|I_2 \rightarrow ①$$

$$(\text{adj } A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore (\text{adj } A)A = |A|I \rightarrow ②$$

from ① and ②

$$A(\text{adj } A) = (\text{adj}A)A = |A|I_2$$

33. Obtain the condition that the roots of $x + px + qx + r = 0$ are in A.P.

Solution:

$$\text{G.T} \quad x^3 + px^2 + qx + r = 0$$

Let us assume the roots $\alpha - d, \alpha, \alpha + d$

$$\sum\nolimits_1 = \alpha - d + \alpha + \alpha + d = -\frac{p}{1}$$

$$3\alpha = -p$$

$$\alpha = -\frac{p}{3}$$

Since α is a root of $x^3 + px^2 + qx + r = 0$

$$\alpha^3 + p\alpha^2 + q\alpha + r = 0$$

$$\left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$$

$$-\frac{p^3}{27} + \frac{p^3}{9} - \frac{pq}{3} + r = 0$$

$$-p^3 + 3p^3 - 9pq + 27r = 0$$

$$2p^3 - 9pq + 27r = 0$$

$\therefore 2p^3 + 27r = 9pq$ is a required condition

34. A circle of area 9π square units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$. Find the equation of the circle.

Solution:

$$\text{G.T. circle of area } 9\pi$$
$$\text{W.K.T. Area of circle} = \pi r^2$$
$$\pi r^2 = 9\pi \implies r^2 = 9$$

$$\begin{array}{l|l} x + y = 5 \quad \rightarrow \ (1) & \text{Sub } x = 3 \text{ in } (1) \\ x - y = 1 \quad \rightarrow \ (2) & 3 + y = 5 \\ \hline 2x \quad = 6 & y = 2 \\ \implies x = 3 & \end{array}$$

Equation of circle is

$$(x - h)^2 + (y - k)^2 = r^2$$
$$\text{Here } h = 3,\ k = 2 \text{ and } r^2 = 9$$
$$(x - 3)^2 + (y - 2)^2 = 9$$
$$x^2 - 6x + 9 + y^2 - 4y + 4 = 9$$
$$x^2 + y^2 - 6x - 4y + 4 = 0$$

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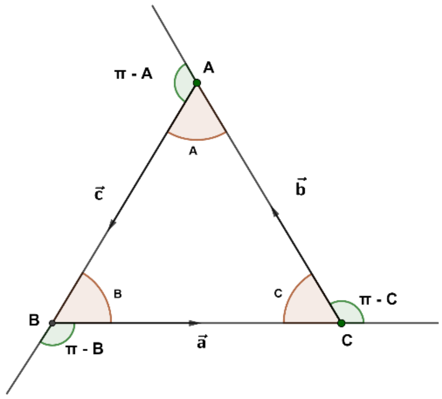
35. Prove that with usual notations $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ by using area of the triangle property.

Solution:

Let $\overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}, \overrightarrow{AB} = \vec{c}$



$$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{2}|\vec{c} \times \vec{a}|$$

$$\implies |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$|\vec{a}||\vec{b}|\sin(\pi - C) = |\vec{b}||\vec{c}|\sin(\pi - A) = |\vec{c}||\vec{a}|\sin(\pi - B)$$
$$\implies ab\sin C = bc\sin A = ca\sin B$$
$$\implies \frac{ab\sin C}{abc} = \frac{bc\sin A}{abc} = \frac{ca\sin B}{abc}$$
$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b}$$

Taking reciprocal and arranging,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

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36. Find the absolute extrema of the function $f(x) = x^2 - 12x + 10$; $[1, 2]$

Solution:

$$\text{G.T. } f(x) = x^2 - 12x + 10$$
$$f'(x) = 2x - 12$$
$$f'(x) = 0$$
$$2x - 12 = 0 \implies 2x = 12$$
$$\implies x = 6 \notin (1, 2)$$

$$f(1) = 1^2 - 12(1) + 10 = -1$$
$$f(2) = 2^2 - 12(2) + 10 = -10$$
Absolute maximum $= -1$
Absolute minimum $= -10$

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37. Suppose that $z = ye^{x^2}$, where $x = 2t$ and $y = 1 - t$ then find $\frac{dz}{dt}$.

Solution:

$z = ye^{x^2}$ $\frac{\partial z}{\partial x} = ye^{x^2}(2x)$ $\frac{\partial z}{\partial y} = e^{x^2}$	$x = 2t$ $\frac{dx}{dt} = 2$	$y = 1 - t$ $\frac{dy}{dt} = -1$

W.K.T. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$\frac{dz}{dt} = 2xye^{x^2}(2) + e^{x^2}(-1)$$
$$= e^{x^2}(4xy - 1)$$
$$= e^{4t^2}(4(2t)(1-t) - 1)$$
$$\frac{dz}{dt} = e^{4t^2}(8t - 8t^2 - 1)$$

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38. Two fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

Solution:

Let $S = \{T, H\} \times \{T, H\}$
$S = \{TT, TH, HT, HH\}$
Let X be the number of heads
$X(TT) = 0, X(TH) = X(HT) = 1, X(HH) = 2$
$\therefore$ The random varibles takes the values 0 1 and 2

The values of random variable of X	0	1	2	Total
The no. of element in inverse image	1	2	1	4

The possible probability are

$$f(0) = P(x = 0) = \frac{1}{4}$$
$$f(1) = P(x = 1) = \frac{2}{4} = \frac{1}{2}$$
$$f(2) = P(x = 2) = \frac{1}{4}$$

The probability mass function is given by

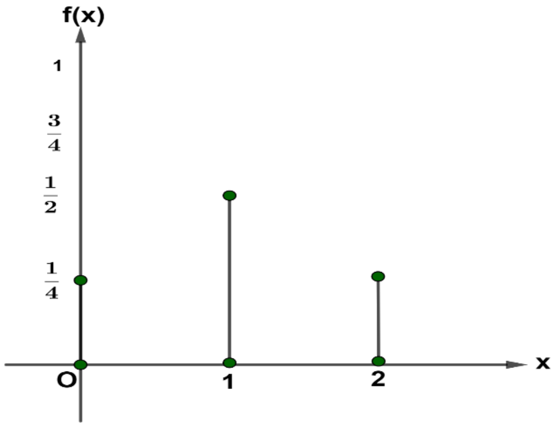
Tabulor form:

X	0	1	2
f(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Exression form:

$$f(x) = \begin{cases} \frac{1}{4}, & x = 0,2 \\ \frac{1}{2}, & x = 1 \end{cases}$$

Graping form:



39. The mean and variance of a binomial variate X are respectively 2 and 1.5. Find $P(X = 0)$.

Solution:

$$\text{mean} = 2 \text{ and variance} = 1.5$$

$$np = 2 \quad \text{and} \quad npq = 1.5 = \frac{3}{2}$$

$$\frac{3}{2} \qquad \frac{3}{4}$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore n\left(\frac{1}{4}\right) = 2 \quad \Longrightarrow n = 8$$

$$X \sim B\left(8, \ \frac{1}{4}\right)$$

$$P(X = x) = \ {}^{8}C_{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{8-x} \quad , \quad x = 0,1,2, \dots \dots 8$$

$$P(X = 0) = \ {}^{8}C_{0} \left(\frac{1}{4}\right)^{0} \left(\frac{3}{4}\right)^{8-0} = (1)(1)\left(\frac{3}{4}\right)^{8} = \left(\frac{3}{4}\right)^{8}$$

40. Show that $((\neg q) \wedge p) \wedge q$ is a contradiction.

Solution:

Truth table for $((\neg q) \wedge p) \wedge q$

p	q	$\neg q$	$(\neg q) \wedge p$	$((\neg q) \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

Since the last column contains only F

Hence, $\left((\neg q) \wedge p\right) \wedge q$ is a contradiction

PART – D

Answer ALL questions **7 x 5 = 35**

41. (a) Test for consistency and if possible, solve the following systems of equations by rank method. $2x+2y+z=5, x-y+z=1, 3x+y+2z=4$

Solution:

The matrix form of the system is $AX = B$

$$\text{Where } A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \;;\; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;;\; B = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

it is a row echelon form

$$\rho(A) = 2 \;;\; \rho([A|B]) = 3$$

$$\therefore \rho(A) \neq \rho([A|B])$$

Then the system of equation is inconsistent and it has no solution

OR

(b) Prove that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Solution:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\arg(z_1) = \theta_1 \;;\; \arg(z_2) = \theta_2$$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

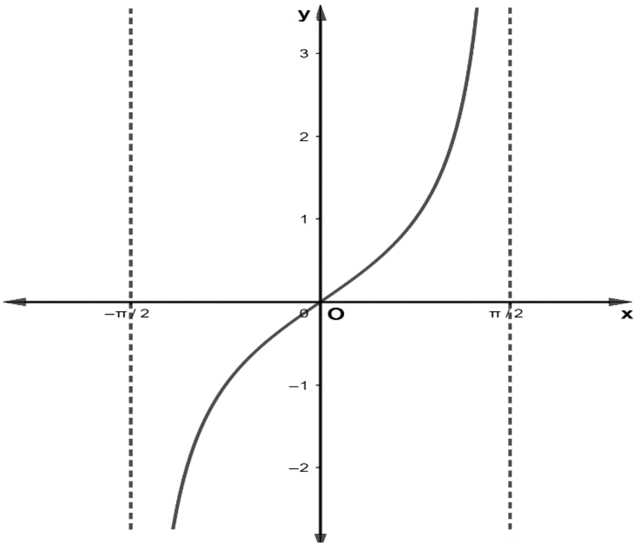
$$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$\therefore \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

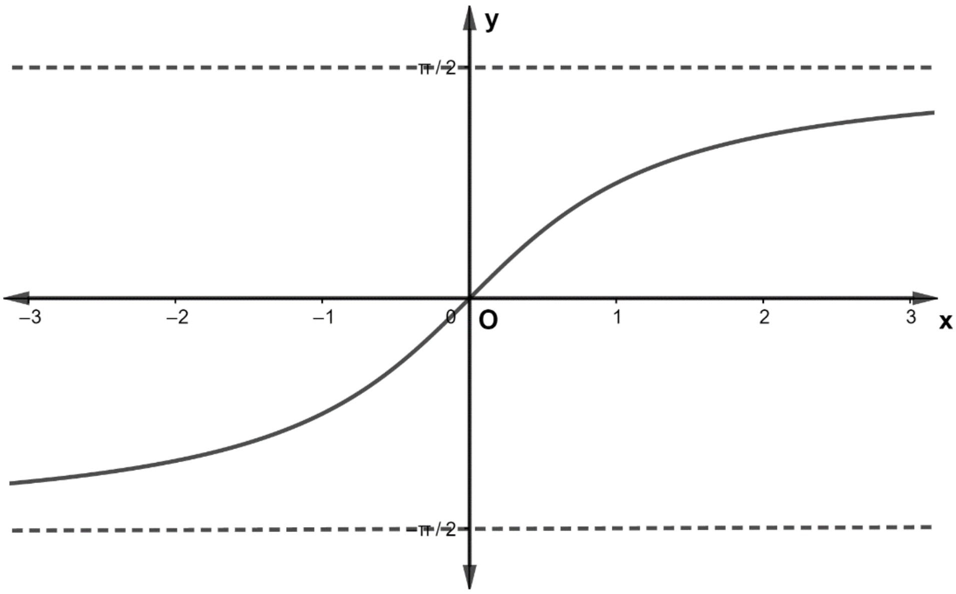
42. (a) Draw the graph of $\tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan^{-1} x$ in $(-\infty, \infty)$.

Solution:

Graph of $y = \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Graph of $y = \tan^{-1} x$ in $(-\infty, \infty)$



OR

(b) Find the centre, foci, and eccentricity of the hyperbola : $11x^2 - 25y^2 - 44x + 50y - 256 = 0$.

Solution:

$$11x^2 - 25y^2 - 44x + 50y - 256 = 0$$
$$11x^2 - 44x - 25y^2 + 50y = 256$$
$$11(x^2 - 4x) - 25(y^2 - 2y) = 256$$
$$11(x^2 - 4x + 4) - 25(y^2 - 2y + 1) = 256 + 11(4) - 25(1)$$
$$11(x-2)^2 - 25(y-1)^2 = 256 + 44 - 25$$
$$11(x-2)^2 - 25(y-1)^2 = 275$$
$$\frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1$$

Transverse axis is parallel to $x-$axis

Here $h = 2, k = 1$ and $a^2 = 25, b^2 = 11$

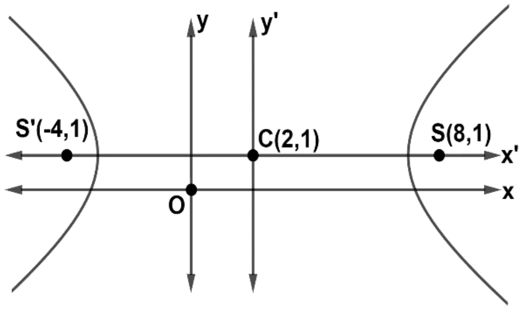
$$c^2 = a^2 + b^2 = 25 + 11 = 36$$

$$c = 6$$

centre $= C(h, k) = C(2,1)$

foci $= S(h \pm c, k) = S(2 \pm 6, 1)$

$\Longrightarrow S(8,1)$ and $S'(-4, 1)$



eccentricity $= e = \dfrac{c}{a} = \dfrac{6}{5}$

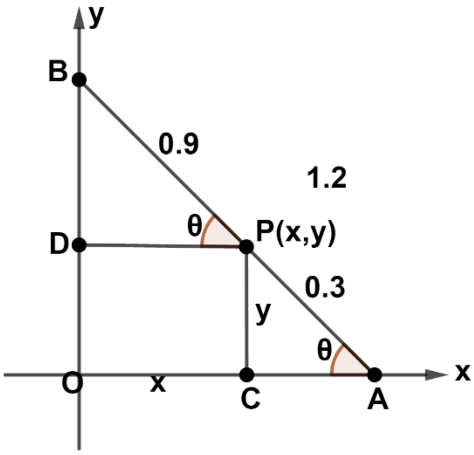
43. (a) A rod of length 1.2m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3m from the end in contact with x – axis is an ellipse. Find the eccentricity.

Solution:

Let AB be the rod with $AB = 1.2$

Let $P(x, y)$ be the point on the rod such that $AP = 0.3$

Let θ be the angle of the rod with positive $x -$ axis



From the diagram

$$\Delta CAP \; ||| \; \Delta DPB$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{x}{0.9} \quad ; \quad \sin\theta = \frac{}{\text{hyp.}} = \frac{}{0.3}$$

W K.T. $\cos^2\theta + \sin^2\theta = 1$

$$\left(\frac{x}{0.9}\right)^2 + \left(\frac{y}{0.3}\right)^2 = 1$$

$$\frac{x^2}{(0.9)^2} + \frac{y^2}{(0.3)^2} = 1$$

$$a^2 = (0.9)^2 \quad ; \quad b^2 = (0.3)^2$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{(0.3)^2}{(0.9)^2}} = \sqrt{1 - \left(\frac{0.3}{0.9}\right)^2}$$

$$= \sqrt{1 - \left(\frac{3}{9}\right)^2} = \sqrt{1 - \frac{1}{9}}$$

$$= \sqrt{\frac{9-1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\boxed{\therefore e = \frac{2\sqrt{2}}{3}}$$

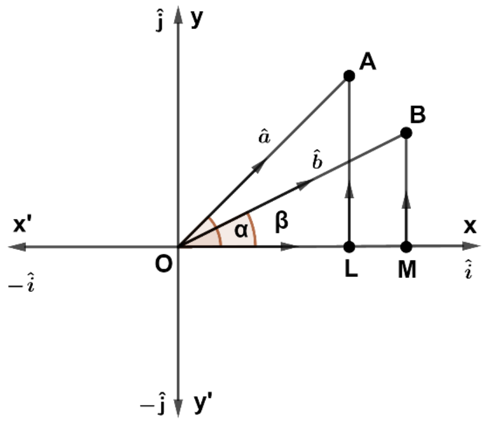
OR

(b) By vector method, prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.

Solution:

Let $\hat{a}=\overrightarrow{OA}$ and $\hat{b}=\overrightarrow{OB}$ be the unit vectors and which makes angle α and β with positive $x-$axis respectively

$\therefore$ The angle between $\hat{a}$ and $\hat{b}$ is $\alpha+\beta$



From the diagram

$$\cos\alpha=\frac{OL}{OA}=\frac{OL}{1}=OL \qquad \cos\beta=\frac{OM}{OB}=\frac{OM}{1}=OM$$

$$\sin\alpha=\frac{LA}{OA}=\frac{LA}{1}=LA \qquad \sin\beta=\frac{MB}{OB}=\frac{MB}{1}=MB$$

$$\hat{a}=\overrightarrow{OA}=\overrightarrow{OL}+\overrightarrow{LA} \qquad \hat{b}=\overrightarrow{OB}=\overrightarrow{OM}+\overrightarrow{MB}$$

$$=OL\,\hat{\imath}+LA\,\hat{\jmath} \qquad =OM\,\hat{\imath}+MB\,\hat{\jmath}$$

$$\hat{a}=\cos\alpha\,\hat{\imath}+\sin\alpha\,\hat{\jmath} \qquad \hat{b}=\cos\beta\,\hat{\imath}+\sin\beta\,\hat{\jmath}$$

$$\hat{a}\cdot\hat{b}=|\hat{a}||\hat{b}|\cos(\alpha-\beta)=(1)(1)\cos(\alpha-\beta)=\cos(\alpha-\beta)\rightarrow (1)$$

$$\hat{a}\cdot\hat{b}=(\cos\alpha\,\hat{\imath}+\sin\alpha\,\hat{\jmath})\cdot(\cos\beta\,\hat{\imath}+\sin\beta\,\hat{\jmath})$$

$$\hat{a}\cdot\hat{b}=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

From (1) and (2), we get

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

44. (a) Find the vector and Cartesian equations of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.

Solution:

The requied equation of plane passing through a point $(1,-2,4)$ and parallel to vectors $\hat{\imath}+2\hat{\jmath}-3\hat{k}$ and $3\hat{\imath}-\hat{\jmath}+\hat{k}$

$$\text{Let } \vec{a}=\hat{\imath}-2\hat{\jmath}+4\hat{k}$$

$$\vec{b}=\hat{\imath}+2\hat{\jmath}-3\hat{k}$$

$$\vec{c}=3\hat{\imath}-\hat{\jmath}+\hat{k}$$

Vector equation:

$$\vec{r}=\vec{a}+s\vec{b}+t\vec{c} \quad \text{where } s,t\in\mathbb{R}$$

$$\vec{r}=(\hat{\imath}-2\hat{\jmath}+4\hat{k})+s(\hat{\imath}+2\hat{\jmath}-3\hat{k})+t(3\hat{\imath}-\hat{\jmath}+\hat{k}) \quad \text{where } s,t\in\mathbb{R}$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$(x_1, y_1, z_1) = (1, -2, 4)$$
$$(b_1, b_2, b_3) = (1, 2, -3)$$
$$(c_1, c_2, c_3) = (3, -1, 1)$$
$$\begin{vmatrix} x-1 & y+2 & z-4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$$
$$(x-1)(2-3) - (y+2)(1+9) + (z-4)(-1-6) = 0$$
$$(x-1)(-1) - (y+2)(10) + (z-4)(-7) = 0$$
$$-x + 1 - 10y - 20 - 7z + 28 = 0$$
$$-x - 10y - 7z + 9 = 0$$
$$x + 10y + 7z - 9 = 0$$

OR

(b) Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let x, y be the sides of the rectangle

Perimeter of the rectangle is $2x + 2y = k$ (Given)

$$2y = k - 2x$$
$$\Longrightarrow y = \frac{k-2x}{2} = \frac{k}{2} - x \quad \rightarrow \quad ①$$
$$\text{Area} = lb = xy$$
$$A(x) = x\left(\frac{k}{2} - x\right) = \frac{k}{2}x - x$$
$$A'(x) = \frac{\ }{2} - 2x$$
$$A\ \ (x) = -2$$
$$\text{Since } A\ (x) = 0$$
$$\frac{k}{2} - 2x = 0$$
$$\frac{k}{2} = 2x \Longrightarrow x = \frac{k}{4}$$
$$A''\left(\frac{k}{4}\right) = -2 < 0$$
$$\therefore A(x) \text{ is local maximum at } x = \frac{k}{4}$$
$$\text{i.e., Area is local maximum at } x = \frac{k}{4}$$
$$\text{sub } x = \frac{k}{4} \text{ in } ①$$
$$y = \frac{k}{2} - \frac{k}{4} = \frac{k}{4}$$
$$\therefore x = y = \frac{k}{4}$$

Hence, the maximum area rectangle of given perimeter is a square.

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45. (a) Show that $\int_0^1 (\tan^{-1} x + \tan^{-1}(1-x))\,dx = \frac{\pi}{2} - \log_e 2$.

Solution:

$$\text{Let } I = \int_0^1 (\tan^{-1} x + \tan^{-1}(1-x))\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}\big(1-(1-x)\big)\,dx$$

Use $f(a-x) = f(x)$ in the second integral only for our convenience

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1} x\,dx$$

$$= 2\int_0^1 \tan^{-1} x\,dx$$

$$= 2\int_0^1 u\,dv \qquad \text{here } u = \tan^{-1} x, \qquad dv = dx$$

$$= 2\left[uv - \int v\,du\right]_0^1$$

$$= 2\left[\tan^{-1} x\,(x) - \int x\frac{1}{1+x^2}\,dx\right]_0^1$$

$$= 2\left[x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx\right]_0^1 = 2\left[x\tan^{-1} x - \frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= 2\left[\left(\tan^{-1} 1 - \frac{1}{2}\log 2\right) - \left(0 - \frac{1}{2}\log 1\right)\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$$

Hence $\boxed{\int_0^1 (\tan^{-1} x + \tan^{-1}(1-x))\,dx = \frac{\pi}{2} - \log_e 2}$

OR

(b) A pot of boiling water at $100°C$ is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to $80°C$, and another 5 minutes later it has dropped to $65°C$. Determine the temperature of the kitchen.

Solution:

Let T be the temperature of boiling water at time t and S be the room (kitchen) temperature.

$$\frac{dT}{dt} \propto (T - S) \implies \frac{dT}{dt} = k(T - S)$$

$$\frac{dT}{(T - S)} = k\,dt$$

$$\int \frac{dT}{(T - S)} = k \int dt$$

$$\log|T - S| = kt + \log|C|$$

$$\log\left|\frac{T - S}{C}\right| = kt$$

$$T - S = Ce^{kt} \rightarrow \quad ①$$

When $t = 0,\ \ T = 100$

$$100 - S = Ce^{0} \implies C = 100 - S$$

$$\text{Sub } C = 100 - S \text{ in } ①$$

$$T - S = (100 - S)e^{kt} \rightarrow \quad ②$$

When $t = 5,\ \ T = 80$

$$80 - S = (100 - S)e^{k(5)}$$

$$\frac{80 - S}{100 - S} = \left(e^{k}\right)^{5}$$

$$e^{k} = \left(\frac{80 - S}{100 - S}\right)^{\frac{1}{5}}$$

$$\frac{80 - S}{\quad}\ ^{\frac{t}{5}}$$

When $t = 10,\ \ T = 65$

$$65 - S = (10 - S)\left(\frac{80 - S}{100 - S}\right)^{\frac{10}{5}}$$

$$65 - S = (10 - S)\frac{(80 - S)^2}{(100 - S)^2}$$

$$(65 - S)(100 - S) = (80 - S)^2$$

$$6500 - 65S - 100S + S^2 = 6400 - 160S + S^2$$

$$6500 - 6400 = 165S - 160S$$

$$5S = 100$$

$$S = 20°C$$

$\therefore$ The temperature of the kitchen is 20°C

---

46. (a) Solve $\frac{dy}{dx} = e^{x+y} + x^3e^y$.

Solution:

$$\frac{dy}{dx} = e^{x+y} + x^3e^y$$

$$\frac{dy}{dx} = e^xe^y + x^3e^y = e^y(e^x + x^3)$$

$$\frac{dy}{e^y} = (e^x + x^3)dx$$

$$\int e^{-y}\,dy = \int (e^x + x^3)\,dx$$

$$-e^{-y} = e^x + \frac{x^4}{4} + c$$

$$e^x + e^{-y} + \frac{x^4}{4} = -c$$

$$\therefore e^x + e^{-y} + \frac{x^4}{4} = C$$

OR

(b) If $X \sim B(n,p)$ such that $4P(X=4) = P(X=2)$ and $n = 6$. Find the distribution, mean and standard deviation of X.

Solution:

$$n = 6$$

$$\text{W.K.T.}\quad P(X=x) = {}^nC_x\, p^x\, q^{n-x} \quad ,x = 0,1,2,\ldots..n$$

$$P(X=x) = {}^6C_x\, p^x\, q^{6-x} \quad ,x = 0,1,2,\ldots..6$$

$$4P(X=4) = P(x=2)$$

$$4\left({}^6C_4\, p^4\, q^{6-4}\right) = \left({}^6C_2\, p^2\, q^{6-2}\right)$$

$$4\left({}^6C_2\, p^4\, q^2\right) = \left({}^6C_2\, p^2\, q^4\right) \qquad \because\ {}^nC_r = {}^nC_{n-r}$$

$$4p^2 = q^2$$

$$4p = (1-p)$$

$$4p = 1 - 2p + p$$

$$3p + 2p - 1 = 0$$

$$(p+1)\left(p - \frac{1}{3}\right) = 0$$

$$p + 1 = 0 \quad ; \quad p - \frac{1}{3} = 0$$

$$p = -1(\text{not possible}) \quad ; \quad p = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

The probability mass function is

$$P(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x} \quad ,x = 0,1,2,\ldots..6$$

$$\text{Mean} = np = 6\left(\frac{1}{3}\right) = 2$$

$$\text{Standard deviation} = \sqrt{npq} = \sqrt{6\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

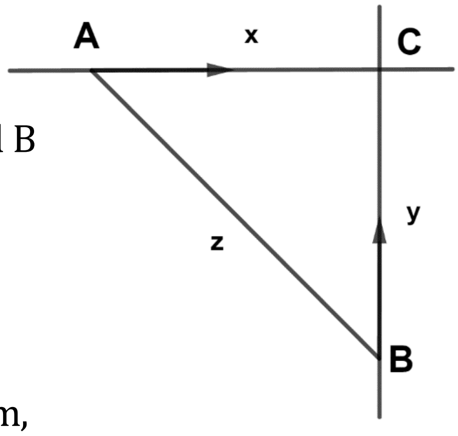
---

47. (a) A car A is travelling from west at 50 km/hr and car B is traveling towards north at 60 km/hr. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 kilometers and car B is 0.4 kilometers from the intersection?

Solution:

Let C is the intersection of the two roads. At a given time t,

Let x be the distance from car A to C,
Let y be the distance from car B to C and
Let z be the distance between the cars A and B



G. T. $\frac{dx}{dt} = -50$ and $\frac{dy}{dt} = -60$

$x = 0.3$ and $y = 0.4$

find $\frac{dz}{dt} = ?$

By Pythagoras theorem,

$$x^2 + y^2 = z^2 \rightarrow (1)$$

$$z^2 = (0.3)^2 + (0.4)^2 = 0.09 + 0.16 = 0.25$$

$$z = 0.5$$

Equ (1), diff. w. r. to "t"

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

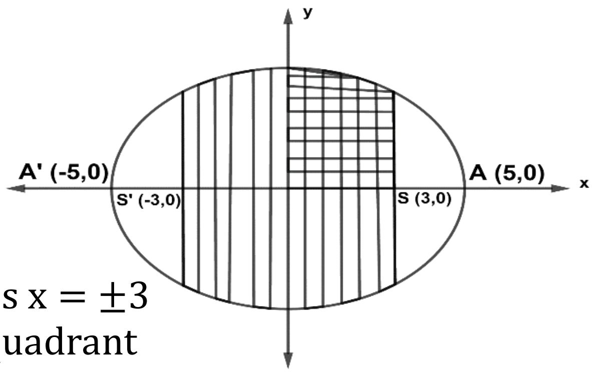
$$(0.5)\frac{dz}{dt} = (0.3)(-50) + (0.4)(-60)$$

$$\frac{dz}{dt} = \frac{-15-24}{0.5} = -\frac{34}{1/2} = -78$$

i.e., The cars are approaching each other at a rate of 78 km/hr.

(b) Find the area of the region bounded by the ellipse ${}_{25} + {}_{16} = 1$ and its latus rectums.

Solution:



$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$a^2 = 25, \quad b^2 = 16$$

$$c^2 = a^2 - b^2 = 25 - 16 = 9$$

$$\Rightarrow c = 3$$

The equation of latus rectum is $x = \pm 3$

The req. area $= 4 \times$ area of I quadrant

$$= 4\int_0^3 y \, dx$$

$$= 4\int_0^3 \frac{4}{5}\sqrt{25 - x^2} \, dx$$

$$= \frac{16}{5}\int_0^3 \sqrt{5^2 - x^2} \, dx$$

$$= \frac{16}{5}\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right)\right]_0^3$$

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$$= \frac{16}{5}\left[\left(\frac{3}{2}\sqrt{25-3^2}+\frac{25}{2}\sin^{-1}\left(\frac{3}{5}\right)\right)-0\right]$$

$$= \frac{8}{5}\left[12+25\sin^{-1}\left(\frac{3}{5}\right)\right]$$

$$\text{The req. area } = \frac{96}{5}+40\sin^{-1}\left(\frac{3}{5}\right)$$

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