# MATRICES

## Focus area class - 4

### Transpose of a matrix

Let A be any matrix then transpose of the matrix A is denoted by $A'$ or $A^T$. The transpose of a matrix is obtained by interchanging its rows and columns.

eg: ① $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

② $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 5 & 4 \end{bmatrix}$ then $B^T = \begin{bmatrix} 1 & -1 & 5 \\ -2 & 0 & 4 \end{bmatrix}$

③ $C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ then $C^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

### Properties

1, $(A^T)^T = A$

2, $(A \pm B)^T = A^T \pm B^T$

3, $(AB)^T = B^T A^T$

4, $(KA)^T = K \cdot A^T$

① If $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ then show that $(AB)^T = B^T A^T$

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & 1-2 \\ 6+0 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -1 & 6 \\ -1 & 2 \end{bmatrix} \text{ --- } (1)$$

$$B^T A^T = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & 6+0 \\ 1-2 & 2+0 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ -1 & 2 \end{bmatrix} \text{ --- } (2)$$

From (1) and (2) $\underline{(AB)^T = B^T A^T}$

(2) If $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 6 \\ 0 & 4 \end{bmatrix}$ then find

(i) $(A+2B)^T$ (ii) S.T $\left(\frac{1}{2} B\right)^T = \frac{1}{2} B^T$

(i) $A+2B = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 12 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 4 & 10 \end{bmatrix}$

$$(A+2B)^T = \begin{bmatrix} 5 & 4 \\ 15 & 10 \end{bmatrix}$$

(ii) $\frac{1}{2} B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

$$\left(\frac{1}{2}B\right)^T = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \quad \text{—— ①}$$

$$\frac{1}{2}B^T = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \quad \text{—— ②}$$

From ① and ②

$$\underline{\underline{\left(\frac{1}{2}B\right)^T = \frac{1}{2}B^T}}$$

## Symmetric matrix

A square matrix A is said to be symmetric if $A^T = A$.

eg: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, $B = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

## Skew-symmetric matrix

A square matrix A is said to be skew-symmetric if $A^T = -A$

eg: $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

## Note

Every square matrix can be expressed as the sum of a symmetric and skew-symmetric matrices.

(1) Express $A=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrices

$$A^T=\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A^T)=\frac{1}{2}\begin{bmatrix}6 & 6\\ 6 & -2\end{bmatrix}$

clearly $P^T=P$, hence $P$ is symmetric

Let $Q=\frac{1}{2}(A-A^T)=\frac{1}{2}\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}$

clearly $Q^T=-Q$, hence $Q$ is skew-symmetric.

Now $P+Q=\frac{1}{2}\begin{bmatrix}6 & 10\\ 2 & -2\end{bmatrix}$

$$=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}=A$$

(2) Express $A=\begin{bmatrix}2 & 3 & -1\\ 0 & 4 & 2\\ 1 & 5 & 1\end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrices.

$$A^T=\begin{bmatrix}2 & 0 & 1\\ 3 & 4 & 5\\ -1 & 2 & 1\end{bmatrix}$$

Let $P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 4 & 3 & 0 \\ 3 & 8 & 7 \\ 0 & 7 & 2 \end{bmatrix}$

Clearly $P^T = P$, hence $P$ is symmetric

Let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$

Clearly $Q^T = -Q$, hence $Q$ is skew-symmetric.

Now $P + Q = \frac{1}{2}\begin{bmatrix} 4 & 6 & -2 \\ 0 & 8 & 4 \\ 2 & 10 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 1 & 5 & 1 \end{bmatrix} = A$

③ If $A$ and $B$ are symmetric matrices then, show that $AB - BA$ is skew-symmetric.

Since $A$ and $B$ are symmetric matrices

$A^T = A$ and $B^T = B$.

Let $Q = AB - BA$

$Q^T = (AB - BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T = BA - AB$

$= -[AB - BA] = -Q$

$\therefore AB - BA$ is skew-symmetric