Class-5

Application of Derivatives

Tangents and normals

Let $y = f(x)$ be the equation of a curve then

Slope of the tangent at (x_1, y_1) is $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

$$\therefore m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

Slope of the normal at (x_1, y_1) is $\frac{-1}{m}$

ie slope of the normal $= \frac{-1}{\left(dy/dx\right)_{(x_1,y_1)}}$

Eq: of the tangent at (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Eq: of the normal is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

If tangent line is $\parallel^{el}$ to x-axis then slope of the tangent is zero. If tangent line is $\parallel^{el}$ to y-axis then slope of the normal is zero.

① Find slope of the tangent to the curve $y = 3x^2 - 4$ at $x = 4$

$$\frac{dy}{dx} = 6x$$

slope of the tangent $= \left(\frac{dy}{dx}\right)_{x=4}$

$$= 6 \times 4 = \underline{\underline{24}}$$

② Find slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$

$$\frac{dy}{dx} = \frac{(x-2)\cdot 1 - (x-1)\cdot 1}{(x-2)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(x-2)^2}$$

slope of the tangent $= \left(\frac{dy}{dx}\right)_{x=10}$

$$= \underline{\underline{\frac{-1}{64}}}$$

③ Find slope of the tangent and normal to the curve $y = x^3 - x + 1$ at $x = 2$

$$\frac{dy}{dx} = 3x^2 - 1$$

$m =$ slope of the tangent $= \left(\frac{dy}{dx}\right)_{x=2}$

$= 12 - 1 = 11$

slope of the normal $= \frac{-1}{m}$

$= \frac{-1}{11}$

(4) Find slope of the tangent and normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \pi/4$

$\frac{dx}{d\theta} = a \cdot 3\cos^2\theta \times -\sin\theta$

$\frac{dy}{d\theta} = a \times 3\sin^2\theta \times \cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta \cdot \sin\theta}$

$\frac{dy}{dx} = -\tan\theta$

slope of the tangent $= \left(\frac{dy}{dx}\right)_{\theta=\pi/4}$

$= -\tan\pi/4 = -1$

slope of the normal $= \frac{-1}{m} = \frac{-1}{-1} = 1$

5, Find slope of the tangent and normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$, at $\theta = \pi/2$

$$\frac{dx}{d\theta} = -a\cos\theta$$

$$\frac{dy}{d\theta} = b \times 2\cos\theta \times -\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta}$$

$$= \frac{2b}{a}\sin\theta$$

Slope of the tangent $= \left(\frac{dy}{dx}\right)_{\theta = \pi/2}$

$$\therefore m = \frac{2b}{a}$$

Slope of the normal $= \frac{-1}{m} = \frac{-a}{2b}$

⑥ Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to x-axis

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Since tangent is $\|^{el}$ to X-axis

Slope of the tangent $= 0$

$$\therefore \frac{dy}{dx} = 0$$

$$3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$x = 3, -1$$

When $x = 3$, $y = 27 - 27 - 27 + 7 = -20$

When $x = -1$, $y = -1 - 3 + 9 + 7 = 12$

$\therefore$ Points are $(3, -20)$ and $(-1, 12)$

(7) Find Points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

(i) $\parallel^{el}$ to X-axis, (ii) $\parallel^{el}$ to Y-axis

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

diff: w.r.t x

$$\frac{2x}{9} + \frac{2y}{16} \cdot \frac{dy}{dx} = 0$$

$$\frac{y}{16} \frac{dy}{dx} = -\frac{x}{9}$$

$$\therefore \frac{dy}{dx} = \frac{-16x}{9y}$$

(i) Since the tangent is $\parallel^{el}$ to x-axis

$$\frac{dy}{dx} = 0$$

$$-\frac{16x}{9y} = 0$$

$$\therefore x = 0$$

when $x = 0$, $y = \pm 4$

$\therefore$ Points are $(0,4)$ and $(0,-4)$

(ii) Since the tangent is $\parallel^{el}$ to y-axis

$$\frac{-1}{dy/dx} = 0$$

$$\frac{9y}{16x} = 0$$

$$\therefore y = 0$$

when $y = 0$, $x = \pm 3$

$\therefore$ Points are $(3,0)$ and $(-3,0)$