Dear students and teachers,

This material is prepared by a team of higher secondary Mathematics teachers in our District for Malappuram District Panchayath Vijayabheri Programme, based on the focus point in Mathematics formulated by SCERT for the academic year 2020-21. This is useful for students of all levels.

This book includes short notes, important results, solved problems and self-practice problems of every chapter in plus two Mathematics.

We hope that this humble attempt will build confidence among students and with the help of teachers they are able to bring off atleast pass mark in Mathematics.

With regards,

Team Mathematics

Malappuram

Prepared by:

- **MOHAMMED ARIF P** **[GHSS Anamangad 9745866628]**
- **ANANDA KUMAR MK** **[SHM Govt VHSS Edavanna 9446543161]**
- **ASHFAQ SHANAVAS T.P** **[ASM HSS Velliyanchery 9846635762]**
- **ABOOBACKER A** **[GBHSS Malappuram 9745894348]**
- **ANEESH C** **[PKMM HSS Edarikode 9946007201]**
- **SAJITH K** **[MSM HSS Kallingaparamba 9567689927]**
- **MUHAMMED ILIYAS P** **[DUHSS Panakkad 9447927713]**
- **SUMA P** **[MSP HSS Malappuram [9446022444]**
- **Dr. SREEJA M** **[GHSS PANDIKKAD 9061318212]**

CONTENTS

1

RELATIONS AND FUNCTIONS

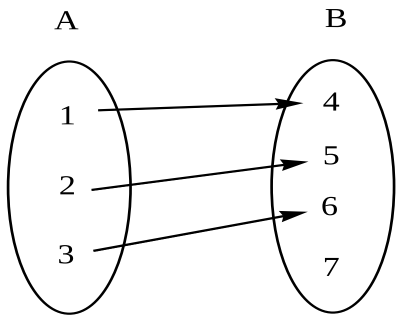
KEY NOTES

- Functions (Mappings) : A relation R from a set A to another set B is called a function or mapping if it satisfies the following conditions

  (i) Every element in A should have an image in B

  (ii) For any element in A, there should not be more than one image in B

  Eg : Consider the sets A={1,2,3} and B={4,5,6,7} . Let R be a relation from A to B such that R={(1,4), (2,5), (3,6)}. The arrow diagram of the above relation is given by



The relation R is a function since every element in A has image and no element has more than one image.

Here, domain = {1,2,3} and Range = {4,5,6} , co-domain is {4,5,6,7}

Note :

If the elements of a set A of a given function are $x_1, x_2, \cdots\cdots x_n$, the images are represented by $f(x_1), f(x_2), \cdots\cdots f(x_n)$

Types of functions

One-one function

A function $f : A \to B$ is said to be one-one if different elements have different images

i.e., if x_1 and x_2 are different then $f(x_1)$ and $f(x_2)$ are different.

In other words, if $f(x_1)$ and $f(x_2)$ are same, then x_1 and x_2 must be same.

or

$$\boldsymbol{f(x_1) = f(x_2) \Longrightarrow x_1 = x_2}$$

Many-one function

A function which is not one-one is called many one function.

- **On-to function**

  If every elements in B have a preimage in A, then the mapping is said to be on-to.
  In an on-to mapping; **Range of f = co-domain**

- **In-to function**

  A function which is not on-to is called into function.
  In an in-to mapping; Range of $f \subset$ co-domain

- **Bijective function**

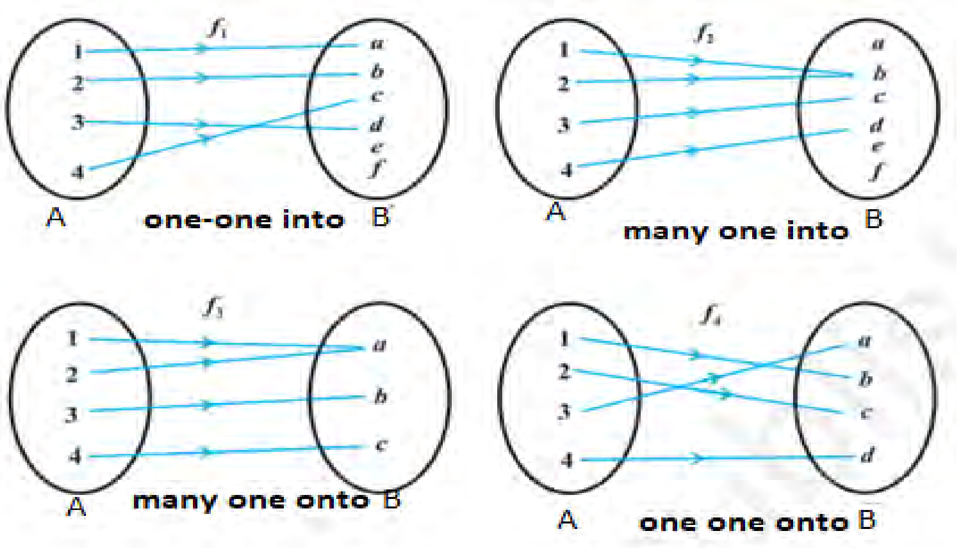
A function which is both one one and onto is called bijective function

- To find out whether a function is one-one or not, we use the following method
  (i) Draw the graph of the function
  (ii) Draw lines parallel to x-axis
  (iii) If any of the above line **intersects** the function at **more than one point**, it is not a one-one function.
  (iv) If **all the lines intersect** the curve in **at least one point**, the function is on-to

Inverse of a function

If $f: A \to B$ is a bijective function, then $f^{-1}: B \to A$ is the inverse of the f defined by $f^{-1}(y) = x$ if and only if $f(x) = y$

- Functions having inverse are called invertible functions
- A function is **invertible if and only if it is bijective**



QUESTIONS AND ANSWERS

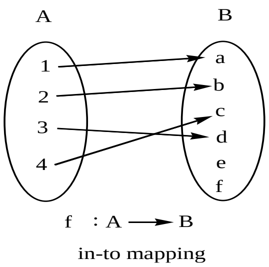
1. Consider the set A = {1, 2, 3,4} B= {a, b, c, d, e, f} Let $f: A \rightarrow B$ be a function defined by $f:\{(1,a),(2,b),(3,d),(4,c)\}$

(a) Is f one one or onto

(b) Does f^{-1} exists? Explain

Sol:

(a) Let us draw the arrow diagram of the function



Here, every element in A has different image in B. Hence the mapping is one-one

Range of $f = \{a,b,c,d\}$

Co domain = $\{a,b,c,d,e,f\}$

Range of $f \neq$ Co domain

∴ The mapping is not onto

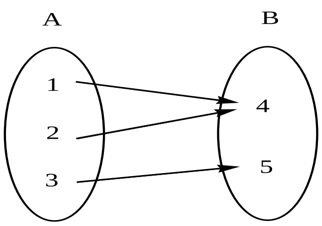
(b) f^{-1}does not exists ∵f is not bijective

2. Consider the function $f:\{1,2,3\} \rightarrow \{4,5\}$ defined by $f = \{(1,4),(2,4),(3,5)\}$.

(a) Is f one one or onto

(b) Does f^{-1} exists? Explain

Sol:



(a) Here $f(1) = 4$ *and* $f(2) = 4$

Hence f is not one one.

Co domain = $\{4,5\}$ = Range

∴ f is onto

(b) f^{-1}does not exists ∵f is not bijective

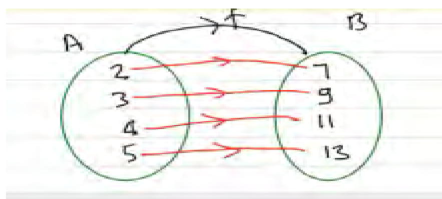
3. Let A = { 2, 3, 4,5} , B = {7,9,11,13}. Consider the function $f: A \rightarrow B$ defined by $f(2) = 7, f(3) = 9,$ $f(4) = 11$ *and* $f(5) = 13$

(a) Show that f is one- one and onto

(b) Find f^{-1}

Sol:

$f = \{(2,7),(3,9),(4,11),(5,13)\}$.



(a) Here, every element in A has different image in B. Hence the mapping is one-one

Co domain ={7,9,11,13}=Range

Range of f = Co domain

∴ f is onto ⇒f is bijective

(b) $f^{-1} = \{(7,2),(9,3),(11,4),(13,5)\}$

4. Show that the function f : N→ N defined by $f(x) = 2x$ is one one but not onto

Sol:

Let $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2 \Rightarrow f$ is one one

Let $y = 2x \Rightarrow x = \frac{y}{2}$

When $y = 1, x = \frac{1}{2} \notin N\ (Domain)$

ie 1∈ Co domain has no pre image in Domain

∴ f is not onto

5. Let f : R → R be a function defined by

$f(x) = 4x + 3.$

(a) Show that f is bijective

(b) Find f^{-1}.

Sol:

(a) Let $f(x_1) = f(x_2)$

$\Rightarrow 4x_1 + 3 = 4x_2 + 3$

$\Rightarrow 4x_1 = 4x_2$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one one

Let $y = 4x + 3$

$4x = y - 3$

$x = \frac{y-3}{4} \in R$(Domain)

for all $y \in R$(Co domain)

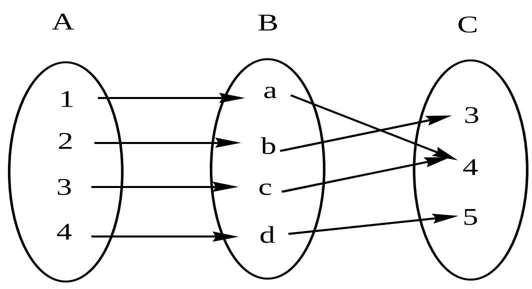
∴ Range of f = Co domain

$\Rightarrow f$ is bijective

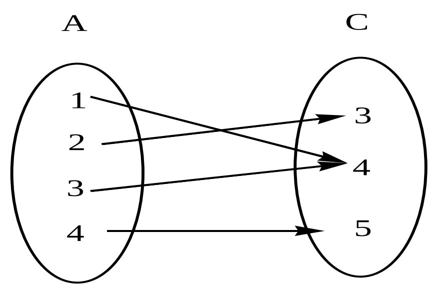
(b) $\therefore f^{-1}(y) = \frac{y-3}{4}$

❖ Composition of function

Let $f: A \to B$ and $g: B \to C$ be two functions then the composition of f and g denoted by $gof: A \to C$ defined as $\boldsymbol{gof(x) = g(f(x))}$



f : A ⟶ B g : B ⟶ C



gof : A ⟶ C

- $\boldsymbol{fog(x) = f(g(x))}$
- $\boldsymbol{fog(x) = gof(x) = x \Rightarrow f^{-1}(x) = g(x)}$ **and** $\boldsymbol{g^{-1}(x) = f(x)}$
- $\boldsymbol{fof(x) = x \Rightarrow f^{-1} = f}$

QUESTIONS AND ANSWERS

1. Find fog and gof of the following

(a) $f(x) = \sin x$, $g(x) = |x|$

(b) $f(x) = \cos x$, $g(x) = 3x^2$

(c) $f(x) = 8x^3$, $g(x) = x^{\frac{1}{3}}$

Sol :

(a) $fog(x) = f(g(x)) = f(|x|) = \sin|x|$

$gof(x) = g(f(x) = g(\sin x)| = |\sin x|$

(b) $fog(x) = f\big(g(x)\big)$

$= f(3x^2) = \cos(3x^2)$

$gof(x) = g\big(f(x)\big)$

$= g(\cos x) = 3\cos^2 x$

(c) $fog(x) = f\big(g(x)\big)$

$= f\left(x^{1/3}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$

$gof(x) = g\big(f(x)\big)$

$= g(8x^3) = (8x^3)^{\frac{1}{3}} = 8^{\frac{1}{3}}(x^3)^{\frac{1}{3}}$

$= 2x$

2. If $f(x) = 2x + 1$, find $fof(x)$

Sol : $f(x) = 2x + 1$

$fof(x) = f(f(x))$

$= f(2x + 1)$

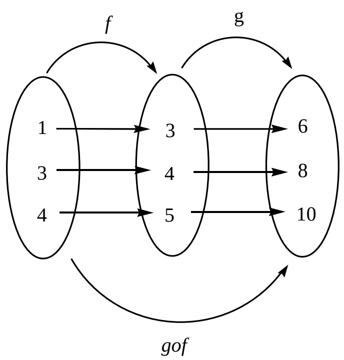
$= 2(2x + 1) + 1$

$= 4x + 2 + 1$

$= 4x + 3$

3. Let $f : \{1,3,4\} \rightarrow \{3,4,5\}$ and $g : \{3,4,5\} \rightarrow \{6,8,10\}$ be functions defined by $f(1) = 3, f(3) = 4, f(4) = 5$, $g(3) = 6, g(4) = 8,\ g(5) = 10$, Find gof

Sol :



$gof = \{(1,6), (3,8), (4,10)\}$

[$gof(1) = g\big(f(1)\big) = g(3) = 6$;

$gof(3) = g\big(f(4)\big) = g(4) = 8$;

$gof(4) = g\big(f(5)\big) = g(5) = 10$]

PRACTICE PROBLEMS

1. Consider the sets A ={1,2,3,4,5 } , B = { 1, 4, 9, 16, 25 } and a function $f: A \rightarrow B$ defined by $f(1) = 1, f(2) = 4$, $f(3) = 9$, $f(4) = 16$ and $f(5) = 25$

(a) Show that f is bijective

(b) Find f^{-1}

2. Show that the following functions are invertible. Also find f^{-1}

(a) $f: R \rightarrow R$ given by $f(x) = 5x + 2$

(b) $f: R \rightarrow R$ given by $f(x) = 2x + 1$

(c) $f: R \rightarrow R$ given by $f(x) = \frac{3x+4}{2}$

3. Show that $f: [-1,1] \rightarrow R$ given by $f(x) = \frac{x}{x+2}$ is one-one

4. If $f(x) = \frac{x}{x-1}$, $x \neq 1$

(a) Find $fof(x)$

(b) Find the inverse of f

5. Find fog and gof of the following

(a) $f(x) = |x|$, $g(x) = 5x + 2$

(b) $f(x) = \sin x$, $g(x) = \cos x$

(c) $f(x) = 27x^3, g(x) = x^{1/3}$

INVERSE TRIGONOMETRIC FUNCTIONS

KEY NOTES

- If $\sin x = \theta$, then $x = sin^{-1}\theta$

Properties :

✓ $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x$; $\text{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1} x$

✓ $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$; $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1} x$

✓ $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$; $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$

1. $\sin^{-1}(-x) = -\sin^{-1} x$; $x \in [-1, 1]$
2. $\tan^{-1}(-x) = -\tan^{-1} x$; $x \in R$
3. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$; $|x| \geq 1$

1. $\cos^{-1}(-x) = \pi - \cos^{-1} x$; $x \in [-1, 1]$
2. $\sec^{-1}(-x) = \pi - \sec^{-1} x$; $|x| \geq 1$
3. $\cot^{-1}(-x) = \pi - \cot^{-1} x$; $x \in R$

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$; $x \in [-1, 1]$
2. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$; $x \in R$
3. $\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$; $|x| \geq 1$

1. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; $xy < 1$
2. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$; $xy > 1$

1. $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$; $|x| < 1$

2. $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$; $x \geq 0$

3. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$; $|x| < 1$

QUESTIONS AND ANSWERS

1. Prove that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Sol :

Let $\sin^{-1}x = \theta$ ...... (1)

$$x = \sin\theta$$

$$x = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos^{-1}x = \frac{\pi}{2} - \theta$$

$$\cos^{-1}x + \theta = \frac{\pi}{2}$$

$$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2} \text{ (from eqn (1))}$$

2. Prove that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Sol :

Let $\tan^{-1}x = \theta$ and $\tan^{-1}y = \emptyset$

Then $x = \tan\theta$ and $y = \tan\emptyset$

We know that

$$\tan(\theta + \emptyset) = \frac{\tan\theta + \tan\emptyset}{1 - \tan\theta \cdot \tan\emptyset}$$

$$= \frac{x+y}{1-xy}$$

$$\theta + \emptyset = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

3. Prove that $tan^{-1}\frac{1}{2} + tan^{-1}\frac{2}{11} = tan^{-1}\frac{3}{4}$

Sol :

$$\tan^{-1}\frac{1}{2} + tan^{-1}\frac{2}{11} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{2}{11}}{1-\frac{1}{2}\times\frac{2}{11}}\right)$$

$$= \tan^{-1}\left(\frac{1\times11+2\times2}{2\times11-1\times2}\right)$$

$$= \tan^{-1}\left(\frac{15}{20}\right) = tan^{-1}\frac{3}{4}$$

4. Solve : $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

Sol :

$$\tan^{-1}2x + tan^{-1}3x = \frac{\pi}{4}$$

$$\Rightarrow tan^{-1}\left(\frac{2x+3x}{1-2x.3x}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2} = tan\frac{\pi}{4} = 1$$

$$\Rightarrow 5x = 1 - 6x^2$$

$$\Rightarrow 6x^2 + 5x - 1 = 0$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4\times6\times-1}}{2\times6}$$

$$= \frac{-5\pm\sqrt{49}}{12} = \frac{-5\pm7}{12}$$

$$\Rightarrow x = \frac{2}{12} \text{ or } x = -\frac{12}{12}$$

$$\Rightarrow x = \frac{1}{6} \text{ or } x = -1$$

$x = -1$ does not satisfy the equation as LHS of the equation becomes negative.

$$\therefore x = \frac{1}{6}$$

5. Find the value of $\cos(\sec^{-1}x + \operatorname{cosec}^{-1}x)$

Sol :

$$\cos(\sec^{-1}x + \operatorname{cosec}^{-1}x) = \cos\frac{\pi}{2} = 0$$

(By property)

6. If $\sin\left[\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}x\right] = 1$, find the value of x.

Sol :

$$\sin\left[\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}x\right] = 1$$

$$\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}x = \sin^{-1}1 = \frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}x = \frac{\pi}{2}$$

$$\therefore\ x = \frac{1}{5}$$

since $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

7. Prove that $2\,tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\frac{31}{17}$

Sol:

$$2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\times\frac{1}{2}}\right) + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{1\times2+1\times2}{2\times2-1\times1}\right) + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{4\times7+3\times1}{3\times7-4\times1}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right)$$

PRACTICE PROBLEMS

1. Prove that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
2. Prove that $tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24} = tan^{-1}\frac{1}{2}$
3. Prove that $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$
4. Express $tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0$ in the simplest form
5. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x,\ x > 0.$
6. Show that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

MATRICES

KEY NOTES

- Matrix is an array of objects arranged in rows and columns.
- If a matrix has 'm' rows and 'n' columns, its order is $m \times n$ (read as m by n)
- General form of an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots & a_{1j} \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots & a_{2j} \ldots\ldots\ldots & a_{2n} \\ \vdots & & & & \vdots \\ a_{i1} & a_{i2} & a_{i3} \ldots\ldots & a_{ij} \ldots\ldots\ldots & a_{in} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} \ldots\ldots & a_{mj} \ldots\ldots\ldots & a_{mn} \end{bmatrix}$$

- Types of matrices
  - (i) Row matrix : Matrix having only one row.
    Eg : $A = [1 \quad 2 \quad 3 \quad 4]$
  - (ii) Column matrix : Matrix having only one column.
    Eg : $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
  - (iii) Square matrix : Matrix in which number of rows = Number of columns
    Eg : $C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
    Here, a_{11}, a_{22}, a_{33} are called diagonal elements
  - (iv) Diagonal matrix : A square matrix in which all non-diagonal elements are Zero
  - (v) Scalar matrix : A diagonal matrix in which all diagonal elements are same
  - (vi) Identity matrix : A diagonal matrix in which all diagonal elements are unity. It is denoted by I
    $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - (vii) Zero matrix : A matrix in which all elements are zero.
    It is denoted by O
- Equality of matrices : Two matrices are said to be equal if
  (a) They are of the same order
  (b) Each element of A = Corresponding element of B

- Operations on matrices :

Addition $(A+B)$	A and B should be of the same order $A+B$ is obtained by adding the corresponding elements of A and B
Subtraction $(A-B)$	A and B should be of the same order A-B is obtained by subtracting the corresponding elements of A and B
Scalar multiplication (kA)	For any scalar k, kA is obtained by multiplying each element of A by k
Multiplication (AB)	AB exists only when number of columns of $A =$ Number of rows of B (i,j)th element of AB is the sum of the product of elements in i^{th} row of A and i^{th} column of B. $A_{m\times p}\times B_{p\times n}=(AB)_{m\times n}$

.

- $A^2 = A\cdot A$, $A^3 = A^2\cdot A = A\cdot A\cdot A$ and so on.
- $I^n = I$
- Transpose of a matrix : It is the matrix obtained by interchanging the rows and columns of the original matrix. Transpose of A is denoted by A' or A^T
- If order of A is $m\times n$, then order of A' will be $n\times m$
- $(A')' = A$, $(A+B)' = A'+B'$, $(A-B)' = A'-B'$, $(kA)' = kA'$, $(AB)' = B'A'$
- Symmetric matrix : A square matrix is said to be symmetric if $A' = A$
- Skew-symmetric matrix : A square matrix is said to be skew symmetric if $A' = -A$
- All diagonal elements of a skew symmetric matrix are zero
- Every square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

$$\boldsymbol{A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')}$$

Symmetric part of A $= \boldsymbol{\frac{1}{2}(A+A')}$

Skew symmetric part of A $= \boldsymbol{\frac{1}{2}(A-A')}$

QUESTIONS AND ANSWERS

1. (a) Construct a 2x2 matrix $A = [a_{ij}]$ whose elements are given by

$a_{ij} = 2i + j$

(b) Find A^2.

Sol :

(a) Consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Given $a_{ij} = 2i + j$

$$a_{11} = 2 \times 1 + 1 = 3; \; a_{12} = 2 \times 1 + 2 = 4$$

$$a_{21} = 2 \times 2 + 1 = 5; \; a_{22} = 2 \times 2 + 2 = 6$$

$$\therefore A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

(b) $A^2 = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$= \begin{bmatrix} (3 \times 3) + (4 \times 5) & (3 \times 4) + (4 \times 6) \\ (5 \times 3) + (6 \times 5) & (5 \times 4) + (6 \times 6) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 29 & 36 \\ 45 & 56 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ Find $2A - B$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 2 & 2 \times 3 \\ 2 \times 2 & 2 \times 3 & 2 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

$$2A - B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4-^{-}1 & 6-3 \\ 4-^{-}1 & 6-0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

3. If $= \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$. Find AB and BA. Show that $AB \neq BA$

Sol :

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}.$$

$$\boldsymbol{AB} = \begin{bmatrix} (1 \times 2) + (-2 \times 4) + (3 \times 2) & (1 \times 3) + (-2 \times 5) + (3 \times 1) \\ (-4 \times 2) + (2 \times 4) + (5 \times 2) & (-4 \times 3) + (2 \times 5) + (5 \times 1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 8 + 6 & 3 - 10 + 3 \\ -8 + 8 + 10 & -12 + 10 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \times 1) + (3 \times -4) & (2 \times -2) + (3 \times 2) & (2 \times 3) + (3 \times 5) \\ (4 \times 1) + (5 \times -4) & (4 \times -2) + (5 \times 2) & (4 \times 3) + (5 \times 5) \\ (2 \times 1) + (1 \times -4) & (2 \times -2) + (1 \times 2) & (2 \times 3) + (1 \times 5) \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 12 & -4 + 6 & 6 + 15 \\ 4 - 20 & -8 + 10 & 12 + 25 \\ 2 - 4 & -4 + 2 & 6 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly $AB \neq BA$

4. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that

$$A^2 - 5A + 7I = O$$

Sol :

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A \times A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 \times 3 + 1 \times -1 & 3 \times 1 + 1 \times 2 \\ -1 \times 3 + 2 \times -1 & -1 \times 1 + 2 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$\therefore A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

5. Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

(a) Find A^2

(b) Find *k* so that $A^2 = KA - 2I$

Sol :

(a) $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 3 + -2\times 4 & 3\times -2 + -2\times -2 \\ 4\times 3 + -2\times 4 & 4\times -2 + -2\times -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

(b) $A^2 = KA - 2I$

$$KA = A^2 + 2I$$

$$K\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3K & -2K \\ 4K & -2K \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$\Rightarrow \quad 3K = 3$

$\Rightarrow \quad K = 1$

6. If $\begin{bmatrix} x+3 & 2 \\ 2z & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 6 & y-1 \end{bmatrix}$, find x, y and z

Sol:

Given $\begin{bmatrix} x+3 & 2 \\ 2z & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 6 & y-1 \end{bmatrix}$

$\therefore$ the corresponding elements are equal

$\Rightarrow x + 3 = 0,\ 2z = 6,\ and\ 4 = y - 1$

$\Rightarrow x = 0 - 3 = -3$

$z = \frac{6}{2} = 3$

$y = 4 + 1 = 5$

7. (a) Find X and Y if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and

$$X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

(b) Find x and y if $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

Sol :

(a) $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ …… (1)

$X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ …… (2)

(1) + (2) $\Longrightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$

$\Longrightarrow X = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Substituting in (1) ,

$$\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix} + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$$

$$Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

(b) $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$$\begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$2x - y = 10 \ldots\ldots (1)$$
$$3x + y = 15 \ldots\ldots.. (2)$$
$$(1) + (2) \Longrightarrow 5x = 25 \Longrightarrow x = 5$$
$$(1) \Longrightarrow 10 - y = 10 \Longrightarrow y = 0$$

8.(a) If a matrix has 24 elements, what are the possible orders it can have.

(b) Find the value of x, y and z if

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

Sol :

(a) The possible orders are 1×24 , 24×1 2×12 , 12×2 , 3×8 , 8×3, 4×6 , 6×4

(b)

$$\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$$

Equating the corresponding elements,

$x + y + z = 9$ ….. (1)

$x + z = 5$ ….. (2)

$y + z = 7$ ….. (3)

(1) - (2) $\Longrightarrow y = 9 - 5 = 4$

Substituting in (3) , $4 + z = 7$

$z = 7 - 4 = 3$

(1) $\Longrightarrow$ $x + 4 + 3 = 9$

$x = 9 - 7 = 2$

$x = 2, y = 4, z = 3$

9. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$ find x& y

$$\begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$2x - y = 10$$
$$3x + y = 5$$
$$\overline{5x + 0 = 15}$$

$x = \frac{15}{5}$; $x = 3$

$$2x - y = 10 \Rightarrow 2 \times 3 - y = 10$$
$$6 - y = 10 \Rightarrow 6 - 10 = y$$
$$y = -4$$

10. (a) If A is a matrix of order 2×3 , then A^T will be of the order ……….

(b) If $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix}$ Find $A^T$and show that $(A^T)^T = A$

Sol :

(a) 3×2

(b) $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix}$.

$$A^T = \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 7 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 7 \end{bmatrix} = A$$

11. For the symmetric matrix $A = \begin{bmatrix} 2 & x & 4 \\ 5 & 3 & 8 \\ 4 & y & 9 \end{bmatrix}$, find the values of x and y

Sol :

For a symmetric matrix, $A = A'$

Here $A = \begin{bmatrix} 2 & x & 4 \\ 5 & 3 & 8 \\ 4 & y & 9 \end{bmatrix}$; $A' = \begin{bmatrix} 2 & 5 & 4 \\ x & 3 & y \\ 4 & 8 & 9 \end{bmatrix}$

$A = A' \Rightarrow x = 5$ and $y = 8$

12. Consider a 2×2 matrix $A = [a_{ij}]$ with $a_{ij} = 2i + j$.

(a) Construct A

(b) Find $A + A'$ and $A - A'$

(c) Express A as the sum of a symmetric and skew-symmetric matrix.

Sol :

(a) Consider a 2×2 matrix, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Given $a_{ij} = 2i + j$

$a_{11} = 2 \times 1 + 1 = 3 \,;\, a_{12} = 2 \times 1 + 2 = 4$

$a_{21} = 2 \times 2 + 1 = 5 \,;\, a_{22} = 2 \times 2 + 2 = 6$

$\therefore A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

(b) $A' = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$A + A' = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$= \begin{bmatrix} 3+3 & 4+5 \\ 5+4 & 6+6 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 9 & 12 \end{bmatrix}$

$A - A' = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$= \begin{bmatrix} 3-3 & 4-5 \\ 5-4 & 6-6 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $\frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 6 & 9 \\ 9 & 12 \end{bmatrix}$

$\Rightarrow$ It is Symmetric

$\frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow$ It is skew-symmetric

$\frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$= \frac{1}{2}\begin{bmatrix} 6 & 9 \\ 9 & 12 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} = A$

13. **(a)** If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then find BA

(b) Write $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Sol :

(a) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

$\frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix}$

$\Rightarrow$ It is symmetric

$\frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

$\Rightarrow$ It is skew symmetric

$\frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$= \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$

14. (a) Let $A = [a_{ij}]_{2\times3}$, where $a_{ij} = i + j$. Construct A.

(b) Find AA' and hence prove that AA' is symmetric.

(c) For any square matrix A, prove that $A + A'$ is symmetric.

Sol :

(a) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

Given $a_{ij} = i + j$

$a_{11} = 1 + 1 = 2 \quad a_{12} = 1 + 2 = 3 \quad a_{13} = 1 + 3 = 4$

$a_{21} = 2 + 1 = 3 \quad a_{22} = 2 + 2 = 4 \quad a_{23} = 2 + 3 = 5$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, A' = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$

$$AA' = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 29 & 38 \\ 38 & 50 \end{bmatrix}$$

$$(AA')' = \begin{bmatrix} 29 & 38 \\ 38 & 50 \end{bmatrix} = AA'$$

$\therefore$ AA' is symmetric.

(c) $(A + A')' = A' + (A')'$

$= A' + A = A + A'$

$\therefore$ $A + A'$ is symmetric

15. (a) Construct a 3 x 3 matrix A, whose $(i,j)^{th}$ element is $a_{ij} = 2i - j$

(b) Express A as the sum of a symmetric and a skew-symmetric matrix.

Sol :

(a) Consider 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Given $a_{ij} = 2i - j$

$a_{11} = 2 \times 1 - 1 = 1$; $a_{12} = 2 \times 1 - 2 = 0$;

$a_{13} = 2 \times 1 - 3 = -1$; $a_{21} = 2 \times 2 - 1 = 3$

$a_{22} = 2 \times 2 - 2 = 2$; $a_{23} = 2 \times 2 - 3 = 1$

$a_{31} = 2 \times 3 - 1 = 5$; $a_{32} = 2 \times 3 - 2 = 4$

$a_{33} = 2 \times 3 - 3 = 3$

$$\therefore A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix}$

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix}\right)$$

$= \frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ is Symmetric matrix

$$\frac{1}{2}(A - A)^T = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ -1 & 1 & 3 \end{bmatrix}\right)$$

$= \frac{1}{2}\begin{bmatrix} 0 & -3 & -6 \\ 3 & 0 & -3 \\ 6 & 3 & 0 \end{bmatrix}$ is skew symmetric

Now , $\frac{1}{2}(A + A)^T + \frac{1}{2}(A - A)^T$

$$= \frac{1}{2}\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -3 & -6 \\ 3 & 0 & -3 \\ 6 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix} = A$$

16. (a) The value of k such that the matrix $\begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}$ is symmetric is

(i) 0 (ii) 1

(iii) -1 (iv) 2

(b) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then

prove that $A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$

Sol :

(a) (i) or 0

(b) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$A^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\theta + \sin\theta^{-}\sin\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta^{-}\sin\theta & -\sin\theta\sin\theta + \cos\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta \, . \cos\theta \\ -2\sin\theta \, . \cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

Hence Proved.

17. If $A=\begin{bmatrix}-2\\4\\5\end{bmatrix}$ and $B=[1 \quad 3 \quad -6]$

Verify that $(AB)'=B'A'$

Sol :

$$A=\begin{bmatrix}-2\\4\\5\end{bmatrix},\ B=[1 \quad 3 \quad -6]$$

$$AB=\begin{bmatrix}-2\\4\\5\end{bmatrix}[1 \quad 3 \quad -6]=\begin{bmatrix}-2 & -6 & 12\\4 & 12 & -24\\5 & 15 & -30\end{bmatrix}$$

$$(AB)'=\begin{bmatrix}-2 & 4 & 5\\-6 & 12 & 15\\12 & -24 & -30\end{bmatrix}$$

Now

$$A'=[-2 \quad 4 \quad 5] \text{ and } B'=\begin{bmatrix}1\\3\\-6\end{bmatrix}$$

$$B'A'=\begin{bmatrix}1\\3\\-6\end{bmatrix}[-2 \quad 4 \quad 5]=\begin{bmatrix}-2 & 4 & 5\\-6 & 12 & 15\\12 & -24 & -30\end{bmatrix}$$

$=(AB)'$. hence verified

PRACTICE PROBLEMS

1. (a) Construct a 2 x 2 matrix $A=[a_{ij}]$ whose elements are given by $a_{ij}=i+j$

(b) Find A^2

2. Express the matrix $A=\begin{bmatrix}2 & -2 & -4\\-1 & 3 & 4\\1 & -2 & -3\end{bmatrix}$ as the sum of a symmetric and skew symmetric matrices.

3. If $X+Y=\begin{bmatrix}7 & 0\\2 & 5\end{bmatrix}$, $X-Y=\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$ then

(a) Find X and Y

(b) Find $2X+Y$

4. Let $A=\begin{bmatrix}2 & 4\\1 & -3\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 5\\0 & 2 & 6\end{bmatrix}$

(a) Find AB

(b) Is BA defined? Justify your answer

5. If A and B are symmetric matrices of the same order, show that $AB+BA$ is symmetric and $AB-BA$ is skew symmetric.

6. Find the values of a, b, c and d from the equation :

$$\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$$

7. Consider $A=\begin{bmatrix}3 & 4\\5 & 6\end{bmatrix}$ and $B=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}$

(a) Find AB and BA

(b) Verify that $(AB)'=B'A'$

DETERMINANTS

4

- Determinant of $A = [a_{11}]_{1\times 1}$ is $|a_{11}| = a_{11}$
- Determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$
- Determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- Minor :

  Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then , minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

  minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

  minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- Co factor :

  Signed minor is called cofactor

  If $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then cofactor of $a_{11} = A_{11} = M_{11}$

  cofactor of $a_{12} = A_{12} = -M_{12}$

  cofactor of $a_{13} = A_{13} = M_{13}$

- Adjoint of a square matrix is the transpose of the cofactor matrix, denoted by adj A
- $A(adjA) = (adjA)A = |A|I$
- A square matrix is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$
- If $AB = BA = I$, then B is called the inverse of A and we write $B = A^{-1}$
- A square matrix A has inverse if and only if $|A| \neq 0$
- $A^{-1} = \frac{adj\ A}{|A|}, |A| \neq 0$

,

.

- The system of linear equations,

  $a_1x + b_1y + c_1z = d_1$

  $a_2x + b_2y + c_2z = d_2$

  $a_3x + b_3y + c_3z = d_3$ can be written as $AX = B$

  Where $= \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \boldsymbol{d_1} \\ \boldsymbol{d_2} \\ \boldsymbol{d_3} \end{bmatrix}$
- Unique solution of $AX = B$ is given by $\boldsymbol{X = A^{-1}B}$, where $|A| \neq 0$
- A system of equations is consistent or inconsistent according as its solution exists or not.
- **Short cut to find adjA for a 2 × 2 matrix**.

  Interchange the position of the diagonal elements and change the sign of the off diagonal elements.

  If $= \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$**, then** $\boldsymbol{adj\ A} = \begin{bmatrix} \boldsymbol{d} & \boldsymbol{-b} \\ \boldsymbol{-c} & \boldsymbol{a} \end{bmatrix}$

QUESTIONS AND ANSWERS

1. **(a)** Evaluate $\begin{vmatrix} 3 & 3 \\ 4 & -1 \end{vmatrix}$

(b) If $\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = 15$, then find values of x

Sol :

(a)

$$\begin{vmatrix} 3 & 3 \\ 4 & -1 \end{vmatrix} = (3 \times -1) - (4 \times 3) = -15$$

(b) $\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = 15$

$\Rightarrow x^2 - 1 = 15$

$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

2. Evaluate $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Sol :

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = cos^2\theta - (-sin^2\theta)$$

$$= cos^2\theta + sin^2\theta = 1$$

3. Find the value of x if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Sol :

$$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$$

$$2 - 20 = 2x^2 - 24$$

$$-18 = 2x^2 - 24$$

$$2x^2 = 6 \Rightarrow x^2 = \frac{6}{2} = 3$$

$$x = \pm\sqrt{3}$$

4. If $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$,

(a) Find |A|

(b) Find adjA

(c) verify that $A(adjA) = |\,A\,|\,I$

Sol :

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix};$$

(a) $|A| = (2 \times -1) - (4 \times 3) = -14$

(b) $adjA = \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$

[Diagonal elements interchanged and sign of off diagonal elements changed]

(c) R.H.S = |A| I $= -14 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

L.H.S $= A(adj\,A) = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2 \times -1 + 3 \times -4 & 2 \times -3 + 3 \times 2 \\ 4 \times -1 + -1 \times -4 & 4 \times -3 + -1 \times 2 \end{bmatrix}$$

$= \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$ = R.H.S

L. H. S = R. H. S.

Hence Proved

5. Find the inverse of $A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$

Sol :

$$A = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

$$|A| = -4 -^{-}6 = 2 \neq 0$$

$$adjA = \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, Find $|A|$

Sol :

$$|A| = 1\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} -^{-}1\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(1+3) + 1(2+3) + 1(2-1) = 10$$

7. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$,

(a) Find $|A|$.

(b) Find cofactor matrix of A

(c) Find A^{-1}

(d) Use A^{-1} from part (c) and solve system of equations.

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Sol :

(a) $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = 2(-4 -^{-}4) -^{-}3(-6 -^{-}4) + 5(3-2)$$

$$= 2 \times 0 + 3 \times^{-} 2 + 5 \times 1 = -1$$

(b) $A_{11} = 0,\quad A_{12} = 2,\quad A_{13} = 1,$

$A_{21} = -1,\quad A_{22} = -9,\quad A_{23} = -5,$

$A_{31} = 2,\quad A_{32} = 23,\quad A_{33} = 13$

Cofactor matrix $= \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$

(c) $AdjA = (\text{cofactor matrix})^T$

$$adjA = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

(d) The system can be written as $AX = B$

Where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From part (c) $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

Solution is $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 11 + 1 \times -5 + -2 \times -3 \\ -2 \times 11 + 9 \times -5 + -23 \times -3 \\ -1 \times 11 + 5 \times -5 + -13 \times -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = 1, y = 2, z = 3$$

8. Using matrix method solve following system of linear equations.

$$x + y + 2z = 4$$

$$2x - y + 3z = 9$$

$$3x - y - z = 2$$

Sol :

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}; B = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$|A| = 1(1 - -3) - 1(-2 - 9) + 2(-2 - -3)$$
$$= 17$$

$$\text{Cofactor matrix} = \begin{bmatrix} 4 & 11 & 1 \\ -1 & -7 & 4 \\ 5 & 1 & -3 \end{bmatrix}$$

$$adjA = \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{17} \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{17} \begin{bmatrix} 4 & -1 & 5 \\ 11 & -7 & 1 \\ 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 4 \times 4 + -1 \times 9 + 5 \times 2 \\ 11 \times 4 + -7 \times 9 + 1 \times 2 \\ 1 \times 4 + 4 \times 9 + -3 \times 2 \end{bmatrix}$$

$$= \frac{1}{17} \begin{bmatrix} 17 \\ -17 \\ 34 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$x = 1, y = -1, z = 2$$

9. Examine the consistency of the system of equations $x + 2y = 2$, $2x + 3y = 3$

Sol :

$$x + 2y = 2$$
$$2x + 3y = 3$$

Here, $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$,

$$|A| = 3 - 4 = -1 \neq 0$$

$\therefore A^{-1}$ exists. Hence the given system is consistent

10. If $A = \begin{bmatrix} -1 & 2 \\ x & 4 \end{bmatrix}$ is singular, find x.

Sol :

A is singular $\Longrightarrow |A| = 0$

$$-4 - 2x = 0$$
$$2x = -4$$
$$x = -2$$

PRACTICE PROBLEMS

1. The value of $\begin{vmatrix} x & x-1 \\ x+1 & x \end{vmatrix}$ is

2. If $\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} = 15$, then find the values of x.

3. Find the inverse of the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

4. $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$

(a) Find adj A

(b) Find A^{-1}

(c) Using A^{-1} , solve the system of equations

$$2x + 5y = 1 , 3x + 2y = 7$$

5. Let $A = \begin{bmatrix} -1 & 2 & 4 \\ 1 & 1 & 3 \\ 3 & 2 & 3 \end{bmatrix}$

(a) Find |A|

(b) Find adj A

(c) Verify that $A.\ adj.A = |A|I$

6. Solve the following system of equations

$$3x - 2y + 3z = 8 \ ;$$
$$2x + y - z = 1 \ ;$$
$$4x - 3y + 2z = 4$$

7. Solve the system of equations using the matrix method :

$$x + y + z = 1$$
$$2x + 3y - z = 6$$
$$x - y + z = -1$$

8. If $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ then

(a) show that $A^2 - 5A + 10I = 0$

(b) Hence find A^{-1}

9. Using matrix method solve following system of linear equations.

$$x + y + 2z = 4$$
$$2x - y + 3z = 9$$
$$3x - y - z = 2$$

CONTINUITY AND DIFFERENTIABILITY

5

❖ CONTINUITY

- Continuity at a point : *f* is a real function on a subset of the real numbers and let *c* be a point in the domain of *f*. Then *f* is continuous at *c* if

$$\lim_{x \to c} f(x) = f(c)$$

- If *f* is discontinuous at *c* then *c* is called a point of discontinuity of *f*.
- Continuous function : A real function *f* is said to be continuous if it is continuous at every point in the domain of *f*.
- Standard continuous function :
  - (i) Constant function
  - (ii) Polynomial function
  - (iii) Modulus function
  - (iv) Trigonometric functions in its domain

QUESTIONS AND ANSWERS

1. Check the continuity of the function *f* given by $f\ (x) = 2x + 3$ at $x = 1$.

Sol :

$$\operatorname{Lim}_{x \to 1} f(x) = \lim_{x \to 1} (2x+3)$$

$$= (2 \times 1) + 3 = 5$$

$$f(1) = (2 \times 1) + 3 = 5$$

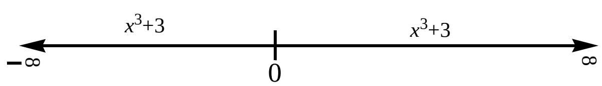
Here $\lim_{x \to 1} f(x) = f(1)$

Hence, f is continuous at $x = 1$.

2. Show that the function *f* given by

$f\ (x) = \begin{cases} x^3 + 3 & if \quad x \neq 0 \\ 1, & if \quad x = 0 \end{cases}$ is not continuous at $x = 0$.

Sol :



$$\operatorname{Lim}_{x \to 0} f(x) = \lim_{x \to 0} (x^3+3)$$

$$= 0 + 3 = 3$$

$$f(0) = 1$$

Here $\lim_{x \to 0} f(x) \neq f(0)$

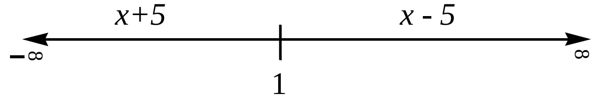
Since the limit of *f* at $x = 0$ does not coincide with *f* (0), the function is not continuous at $x = 0$

3. Examine whether the function defined by

$$f(x) = \begin{cases} x+5, & if \ \ x \leq 1 \\ x-5, & if \ \ x > 1 \end{cases}$$

is continuous or not.

Sol :



$$f(x) = \begin{cases} x+5\,, & if\ x \leq 1 \\ x-5\,, & if\ x > 1 \end{cases}$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (x-5) = 1 - 5$$

$$= -4$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1}(x+5)$$

$$= 1 + 5 = 6$$

$$\lim_{x\to1^+} f(x) \neq \lim_{x\to1^-} f(x)$$

i.e., f is not continuous at $x = 1$

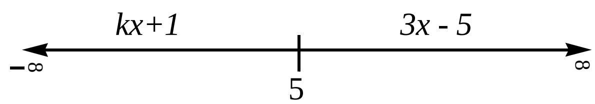
Also, f is a polynomial in x at all other points and hence continuous

4. Find the value of k if the function

$$f(x)=\begin{cases} kx+1 & \text{if } x \leq 5 \\ 3x-5 & if\ x > 5 \end{cases}$$

is continuous at $x = 5$.

Sol :



Given $f(x)$ is continuous at $x = 5$

$\Rightarrow \lim_{x\to5^+} f(x) = \lim_{x\to5^-} f(x)$

$\Rightarrow \lim_{x\to5}(3x-5) = \lim_{x\to5}(kx+1)$

$\Rightarrow (3\times5) - 5 = 5\times k + 1$

$\Rightarrow 10 = 5k + 1$

$\Rightarrow 5k = 9$

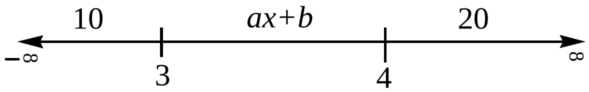
$\Rightarrow k = \frac{9}{5}$

5. Find 'a' and 'b' such that the function defined by

$$f(x) = \begin{cases} 10\ , & x \leq 3 \\ ax+b\ , & 3 < x < 4 \\ 20\ , & x \geq 4 \end{cases}$$

is a continuous function

Sol :



$$\lim_{x\to3^-} f(x) = \lim_{x\to3} 10 = 10$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3}(ax+b)$$

$$= 3a + b$$

Since f is a continuous function,

RHL = LHL

$3a + b = 10$ ………. (1)

$$\lim_{x\to4^-} f(x) = \lim_{x\to4}(ax+b)$$

$$= 4a + b$$

$$\lim_{x\to4^+} f(x) = \lim_{x\to4} 20 = 20$$

LHL = RHL

$4a + b = 20$ …… … (2)

$3a + b = 10$ ……. … (1)

$(2) - (1) \Rightarrow \quad a = 10$

$3a + b = 10$

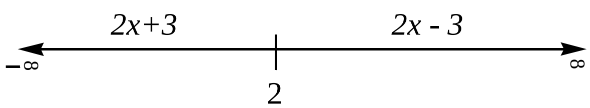
$30 + b = 10$

$b = -20$

6. Find all points of discontinuity of f where f is defined by

$$f(x) = \begin{cases} 2x+3, & if\ x \leq 2 \\ 2x-3, & if\ x > 2 \end{cases}$$

Sol :



$$f(x) = \begin{cases} 2x+3, & x \leq 2 \\ 2x-3, & x > 2 \end{cases}$$

If < 2, $f(x) = 2x + 3$, which is a polynomial and hence continuous.

If > 2, $f(x) = 2x - 3$, which is also a polynomial and hence continuous.

Let $x = 2$

$$\text{Lim}_{x\to2^+} f(x) = \lim_{x\to2}(2x-3)$$

$$= 2 \times 2 - 3 = 1$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (2x + 3)$$

$$= 2 \times 2 + 3 = 7$$

i.e., $\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$

$\therefore f$ is not continuous at $x = 2$

$\therefore f$ is continuous at all points

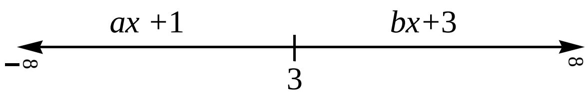
except at $x = 2$

7. Find a relationship between $a \; and \; b$ if the function f defined by

$$f(x) = \begin{cases} ax + 1, if \; x \leq 3 \\ bx + 3, \;\; if \; x > 3 \end{cases}$$

is continuous

Sol :



$$f(x) = \begin{cases} ax + 1, & x \leq 3 \\ bx + 3, & x > 3 \end{cases}$$

Given that f is continuous

$\Longrightarrow$ f is continuous at $x = 3$

$\Longrightarrow$ $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$

$\Longrightarrow$ $\lim_{x \to 3} (bx + 3) = \lim_{x \to 3} (ax + 1)$

$\Longrightarrow$ $3b + 3 = 3a + 1$

$3a - 3b = 2$

8. Prove that the function defined by $f(x) = cos(x^2)$ is continuous function

Sol :

Let $(x) = \cos x$, $h(x) = x^2$

Both $g(x)$ and $h(x)$ are continuous functions.

$\therefore$ their composite function

$g\big(h(x)\big) = \cos(x^2)$ is also continuous

PRACTICE PROBLEMS

1. Consider

$f(x)$=$\begin{cases} 3x - 8 & if \; x \; \leq \; 5 \\ 2k & if \; x > 5 \end{cases}$. Find the value of k if $f(x)$ is continuous at $x = 5$

2. The function

$$f(x) = \begin{cases} 5 & x \leq 2 \\ ax + b & 2 < x < 10 \\ 21 & x \geq 10 \end{cases}$$ is

continuous. Find a and b

3. Prove that the function defined by

$f(x) = |\sin x|$ is a continuous function.

4. Find all points of discontinuity of f , where f is defined by

$$f(x) = \begin{cases} 2x + 3, & x \leq 2 \\ 2x - 3, & x > 2 \end{cases}$$

❖ DIFFERENTIABILITY

SOME STANDARD RESULTS

(1) $\frac{d}{dx}(c) = 0$ **(2)** $\frac{d}{dx}(x) = 1$ **(3)** $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

(4) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ **(5)** $\frac{d}{dx}(x^n) = nx^{n-1}$ **(6)** $\frac{d}{dx}(\sin x) = \cos x$

(7) $\frac{d}{dx}(\cos x) = -\sin x$ **(8)** $\frac{d}{dx}(\tan x) = sec^{\,2}\, x$ **(9)** $\frac{d}{dx}(\cot x) = -cosec^2 x$

(10) $\frac{d}{dx}(\sec x) = \sec x \tan x$ **(11)** $\frac{d}{dx}(cosec\ x) = -cosec\ xcot\ x$ **(12)** $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(13) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ **(14)** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ **(15)** $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(16) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ **(17)** $\frac{d}{dx}(cosec^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$ **(18)** $\frac{d}{dx}(a^x) = a^x \log a$

(19) $\frac{d}{dx}(e^x) = e^x$ **(20)** $\frac{d}{dx}(\log x) = \frac{1}{x}$

Chain Rule (Function of a function rule)

If $y = f(g(x))$ then $\frac{dy}{dx} = f'\big(g(x)\big)\frac{d}{dx}\big(g(x)\big)$

Derivatives Of Implicit Functions

Consider one of the following relationship between x and y :

$$x - y - \pi = 0$$

$$x + \sin(xy) - y = 0$$

In the first case, the relationship between x and y is expressed in a way that it is easy to solve for y and write y = f (x). Thus we say that y is given as an explicit function of x . In the second case, it is implicit that y is a function of x and we say that the relationship of the second type , above given function implicitly

Mean Value Theorem

Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b) . Then there exists some c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Rolle 'S Theorem

Let $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b), such that $f(a) = f(b)$, where a and b are some real numbers. Then there exists some c in (a, b) such that $f'(c) = 0$.

QUESTIONS AND ANSWERS

1. Find derivative of $y = \sqrt{\tan x}$

Sol :

$$y = \sqrt{tanx}$$

$$\frac{dy}{dx} = \frac{1}{2.\sqrt{\tan x}}.\sec^2 x$$

2. Find $\frac{dy}{dx}$ if $y = \log(\cos(e^x))$

Sol :

$$y = \log(\cos(e^x))$$

$$y' = \frac{1}{\cos(e^x)}\frac{\mathrm{d}}{\mathrm{dx}}(\cos(e^x))$$

$$y' = \frac{1}{\cos(e^x)} \cdot -\sin(e^x) \cdot \frac{d}{dx}(e^x)$$

$$y' = -\tan(e^x) \cdot e^x$$

3. Find $\frac{dy}{dx}$ if $y = e^{\sin x}$

Sol :

$$\frac{dy}{dx} = e^{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \cos x$$

4. Find $\frac{dy}{dx}$ if $y = \cos\sqrt{x}$

Sol :

$$\frac{dy}{dx} = -\sin\sqrt{x}\frac{d}{dx}(\sqrt{x})$$

$$= -\sin\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)$$

5. Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

Sol :

We differentiate the relationship directly with respect to x,

i.e.,$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$

$$\frac{dy}{dx} + \cos y\frac{dy}{dx} = -\sin x$$

$$\frac{dy}{dx}(1 + \cos y) = -\sin x$$

$$\frac{dy}{dx} = \frac{-\sin x}{1+\cos y}$$

6. Find $\frac{dy}{dx}$, if $x^2 + y^2 + xy = 100$

Sol :

$$x^2 + y^2 + xy = 100$$

$$2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$(2y + x)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = \frac{-(2x+y)}{2y+x}$$

7. Find $\frac{dy}{dx}$ of $sin^2x + cos^2y = 1$

Sol:

$$sin^2x + cos^2y = 1$$

Differentiating with respect to x

$$2\sin x\cos x + 2\cos y\,(-\sin y)\frac{dy}{dx} = 0$$

$$\sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$$\sin 2x = \sin 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\sin 2x}{\sin 2y}$$

8. Verify Mean Value theorem for the function $f(x) = x^2 - 4x - 3$ in the interval [1,4]

Sol :

$$f(x) = x^2 - 4x - 3\,,\ x \in [1, 4]$$

(i) being a polynomial $f(x)$ is continuous in [1,4]

(ii) $f'(x) = 2x - 4$ exist in (1, 4)

$$f(1) = 1 - 4 - 3 = -6$$

$$f(4) = 16 - 16 - 3 = -3$$

Then by MVT there exist a point $c \in (1,4)$ such that

$$f'(c) = \frac{f(4)-f(1)}{4-1}$$
$$2c - 4 = \frac{(-3)-(-6)}{4-1}$$
$$2c - 4 = \frac{3}{3}$$
$$2c - 4 = 1$$
$$2c = 5$$
$$c = \frac{5}{2} \in (1,4)$$

Hence MVT is verified

9. Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8$, $x \epsilon [-4,2]$

Sol :

$f(x) = x^2 + 2x - 8\,,\ x \in [-4,2]$

being a polynomial $f(x)$ is continuous in $[-4,2]$

$f'(x) = 2x + 2$ exist in $(-4,2)$

$f(-4) = 16 - 8 - 8 = 0$

$f(2) = 4 + 4 - 8 = 0$

$f(-4) = f(2)$

then by Rolle's theorem there exists a point $c \in (-4,2)$ such that $f'(c) = 0$

$$2c + 2 = 0$$
$$2c = -2$$
$$c = -1 \in (-4,2)$$

Hence Rolle's theorem is verified.

PRACTICE PROBLEMS

1. Find $\frac{dy}{dx}$, if $y = logx, x > 0$

2. Find $\frac{dy}{dx}$ if $y = a^x$

3. Find $\frac{dy}{dx}$ if $y = \sin(x^3 + 7)$

4. Find $\frac{dy}{dx}$ if $x^{2/3} + y^{2/3} = 2$

5. Find $\frac{dy}{dx}$ if $x^2 + 2xy + y^2 = 5$

6. Verify Mean Value Theorem for the function $f(x) = x + \frac{1}{x}$ in the interval [1,3]

7. Verify Rolle's Theorem for the function $f(x) = x^2 - 6x + 8$ in the interval [2, 4]

APPLICATION OF DERIVATIVE

6

KEY NOTES

❖ **RATE OF CHANGE OF QUANTITIES**

$\frac{dy}{dx} =$ **Rate of change of y with respect to x .**

❖ **INCREASING AND DECREASING FUNCTIONS**

A function f is said to be

(a) Increasing on an interval (a, b) if $f'(x) \geq 0$ for each x in (a, b)

(b) Decreasing on (a, b) if $f'(x) \leq 0$ for each x in (a, b)

(c) Strictly increasing on an interval (a, b) if $f'(x) > 0$ for each x in (a, b)

(d) Strictly decreasing on (a, b) if $f'(x) < 0$ for each x in (a, b)

❖ **TANGENTS AND NORMALS**

The equation of the tangent at (x_0 , y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0 , y_0)} (x - x_0)$$

Equation of the normal to the curve $y = f(x)$ at a point (x_0 , y_0) is given by

$$y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0 , y_0)}} (x - x_0)$$

QUESTIONS AND ANSWERS

1. Find the rate of change of the area of a circle per second with respect to its radius *r* when *r* = 5 cm.

Sol :

The area A of a circle with radius *r* is given by $A = \pi r^2$. Therefore, the rate of change of the area A with respect to its radius *r* is given by $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$

When *r* = 5 cm, $\frac{dA}{dr} = 10\pi\ cm^2/cm$

2. A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Sol :

The area A of a circle with radius *r* is given by A = πr^2 Therefore, the rate of change of area A with respect to time *t* is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = 2\pi r \frac{dr}{dt}$$

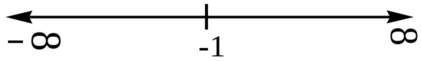
It is given that $\frac{dr}{dt}$= 4cm/s

$$\frac{dA}{dt} = 2\pi r \times 4 = 8\pi r\ cm^2/s$$

3. Find the interval in which the function $f(x) = x^2 + 2x - 5$ is strictly increasing or decreasing

Sol :

$f(x) = x^2 + 2x - 5$
$f'(x) = 2x + 2$
$f'(x) = 0 \Rightarrow 2x + 2 = 0$
$\Rightarrow \quad x = -1$



$\therefore$ The intervals are $(-\infty, -1)$and $(-1, \infty)$

Interval	$f'(x)$	f(x)
$(-\infty, -1)$	<0	Strictly decreasing
$(-1, \infty)$	>0	Strictly increasing

4. Show that the function *f* is given by

$$f(x) = x^3 - 3x^2 + 4x, x \in \mathbb{R}$$

is strictly increasing?

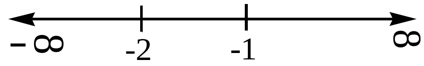
Sol :

$$f(x) = x^3 - 3x^2 + 4x$$
$$f'(x) = 3x^2 - 6x + 4$$
$$= 3x^2 - 6x + 3 + 1$$
$$= 3(x^2 - 2x + 1) + 1$$
$$= 3(x-1)^2 + 1 > 0$$
$\therefore f(x)$ is strictly increasing in $\mathbb{R}$

5. Find the interval in which $2x^3 + 9x^2 + 12x - 1$ is strictly increasing?

Sol :

$$f(x) = 2x^3 + 9x^2 + 12x - 1$$
$$f'(x) = 6x^2 + 18x + 12$$
$$= 6(x^2 + 3x + 2)$$
$$= 6(x+1)(x+2)$$
$$f'(x) = 0 \Rightarrow 6(x+1)(x+2) = 0$$
$$(x+1)(x+2) = 0$$
$\Rightarrow \quad x = -1, -2$



$\therefore$ The intervals are

$(-\infty, -2), (-2, -1) \& (-1, \infty)$

Interval	$\boldsymbol{f'(x) = 6x^2 + 18x + 12}$	$\boldsymbol{f(x)}$
$(-\infty, -2)$	> 0	Strictly increasing
$(-2, -1)$	< 0	Strictly decreasing
$(-1, \infty)$	> 0	Strictly increasing

6. Prove that the function given by $f(x) = \cos x$ is

(a) Strictly decreasing in $(0, \pi)$
(b) Strictly increasing in $(\pi, 2\pi)$, and
(c) Neither increasing nor decreasing in $(0, 2\pi)$.

Sol :

$$f'(x) = -\sin x$$

(a) Since for each $x \in (0, \pi), \sin x > 0$, we have $f'(x) < 0$ and so f is strictly decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi), \sin x < 0$, we have $f'(x) > 0$ and so f is strictly increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, *f* is neither increasing nor decreasing in $(0, 2\pi)$.

7. Find the slope of the tangent line to the curve $y = x^2 - 2x + 1$?

Sol :

$$y = x^2 - 2x + 1,$$
$$\frac{dy}{dx} = 2x - 2$$
Slope of tangent $= 2x - 2$

8. Find the equation of tangent and normal to the curve $y = x^3 at\ (1,1)$

Sol : $y = x^3$

$$\frac{dy}{dx} = 3x^2$$

At (1,1), slope of tangent(m) $= \frac{dy}{dx} = 3$

Equation of tangent is

$$y - y_1 = \text{m}\ (x - x_1)$$
$$y - 1 = 3(x - 1)$$
$$y - 1 = 3x - 3$$
$$3x - y - 2 = 0$$

Slope of normal $= -\frac{1}{m} = -\frac{1}{3}$

Equation of normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$
$$y - 1 = -\frac{1}{3}(x - 1)$$
$$3y - 3 = -x + 1$$
$$x + 3y - 4 = 0$$

9. Find the equation of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1,1) ?

Sol :

Given $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$

Differentiating w.r.t. x,

$$\Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$
$$\Rightarrow \quad \frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$$
$$\Rightarrow \quad y^{-1/3}\frac{dy}{dx} = -x^{-1/3}$$
$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Slope of tangent,

$$m = \frac{dy}{dx} \text{at } (1,1) = -1$$

At (1, 1), Equation of tangent is

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow y - 1 = -1(x - 1) \Rightarrow x + y = 2$$

PRACTICE PROBLEMS

1. The radius of a circle is increasing at the rate of 2 cm/s. Find the rate at which area of the circle is increasing when radius is 6 cm.
2. A spherical bubble is decreasing in volume at the rate of 2cm^3/sec. Find the rate of which the surface area is diminishing when the radius is 3 cm
3. The length x of a rectangle is decreasing at the rate of 5 cm/min. and the width y is increasing at the rate of 4 cm/min. when x = 8 cm and y = 6 cm. Find the rate of change of
   (a) Perimeter
   (b) Area of the rectangle
4. Find the intervals in which the function $x^2 - 2x + 5$ is strictly increasing
5. Show that the function $x^3 - 6x^2 + 15x + 4$ is strictly increasing in R.
6. Prove that the function $f(x) = \log sinx$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$
7. Find the slope of the tangent to the curve $y = (x - 2)^2$ at $x = 1$
8. The slope of the tangent to the curve $y = x^3 - 1$at $x = 2$ is ……….
9. Find the slope of the normal to the curve $y = \sin\theta$ at $\theta = \frac{\pi}{4}$
10. Find the equation of tangent and normal to the parabola $y^2 = 4x$ at (1,1)
11. Find the equation of the tangent to the curve $y = 3x^2$ at (1,2)

INTEGRALS

7

KEY NOTES

- **Indefinite Integrals**

If $\frac{d}{dx}f(x) = g(x)$, then $g(x)$ is called an antiderivative or primitive of $f(x)$.

$\therefore \int g(x)dx = f(x) + C$, where C is an arbitrary constant.

- **Important Integrals**

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1}$+C $(n \neq -1)$: $\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C$ if $n \neq 1$
2. $\int \frac{1}{x} \, dx = \log|x| + \mathrm{C}$: $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + C$
3. $\int e^x \, dx = e^x + \mathrm{C}$: $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
4. $\int a^x \, dx = \frac{a^x}{\log a}$+C :
5. $\int \sin x \, dx = -cosx + \mathrm{C}$: $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + \mathrm{C}$
6. $\int \cos x \, dx = \sin x + \mathrm{C}$: $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + \mathrm{C}$
7. $\int \tan x \, dx = \log|\sec x| + \mathrm{C}$: $\int \tan(ax+b)\, dx = \frac{1}{a}\log|\sec(ax+b)| + \mathrm{C}$
8. $\int \cot x \, dx = \log|\sin x| + \mathrm{C}$: $\int \cot(ax+b)\, dx = \frac{1}{a}\log|\sin(ax+b)| + \mathrm{C}$
9. $\int \sec x \, dx = \log|\sec x + \tan x| + \mathrm{C}$: $\int \sec(ax+b)\, dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + \mathrm{C}$
10. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + \mathrm{C}$: $\int \operatorname{cosec}(ax+b)\, dx = \frac{1}{a}\log|\operatorname{cosec}(ax+b) - \cot(ax+b)| + \mathrm{C}$
11. $\int sec^2 x \, dx = \tan x$+C : $\int sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b)$+C
12. $\int cosec^2 x \, dx = -\cot x$+C : $\int cosec^2(ax+b)\, dx = -\frac{1}{a}\cot(ax+b)$ +C
13. $\int \sec x \tan x \, dx = \sec x + \mathrm{C}$: $\int \sec(ax+b)\tan(ax+b)\, dx = \frac{1}{a}\sec(ax+b) + \mathrm{C}$
14. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + \mathrm{C}$: $\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + \mathrm{C}$

SPECIAL FUNCTIONS

15. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
16. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
17. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right| + C$
20. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$
21. $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1} x + \mathrm{C}$
22. $\int \frac{-dx}{\sqrt{1-x^2}} = cos^{-1} x + \mathrm{C}$
23. $\int \frac{dx}{1+x^2} = tan^{-1} x + \mathrm{C}$
24. $\int \frac{-dx}{1+x^2} = cot^{-1} x + \mathrm{C}$
25. $\int \frac{dx}{x\sqrt{x^2-1}} = sec^{-1} x + \mathrm{C}$
26. $\int \frac{-dx}{x\sqrt{x^2-1}} = cosec^{-1} x + \mathrm{C}$

- **Basic theorems on integration**

$\int[f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

$\int C\, f(x) dx = C \int f(x)\, dx$, where C is a constant

- **Most frequently used results**

1. $\int dx = x + C$
2. $\int x dx = \frac{x^2}{2} + C$
3. $\int x^2 dx = \frac{x^3}{3} + C$
4. $\int \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} + C$
5. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$
6. $\int \frac{1}{x^2} dx = \frac{-1}{x} + C$

- **Integration by substitution**

  - $\int \boldsymbol{f(ax+b)dx.}$ **[Put $\boldsymbol{ax+b=t}$]**
  - $\int \boldsymbol{f\big(g(x)\big)g'(x)\, dx}$ **[Put $\boldsymbol{g(x)=t}$]**
  - $\int \boldsymbol{(f(x))^n f'(x)\, dx}$ **[Put $\boldsymbol{f(x)=t}$]**
  - $\int \boldsymbol{f(x)f'(x)\, dx = \frac{[f(x)]^2}{2} + C}$
  - $\int \boldsymbol{\frac{f\prime(x)}{f(x)} = \log|f(x)| + C}$
  - $\int \boldsymbol{\frac{f\prime(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C}$

QUESTIONS AND ANSWERS

Evaluate the following :

1. $\int (x^4 + 3x^2 - 8x + 7)\, dx$

Sol :

$\int (x^4 + 3x^2 - 8x + 7)\, dx$

$= \int x^4\, dx + \int 3x^2 dx - \int 8x\, dx + \int 7\, dx$

$= \int x^4\, dx + 3\int x^2 dx - 8\int x\, dx + 7\int 1\, dx$

$= \frac{x^5}{5} + 3\frac{x^3}{3} - 8\frac{x^2}{2} + 7x + C$

$= \frac{x^5}{5} + x^3 - 4x^2 + 7x + C$

2. $\int (2x - 3\cos x + e^x)\, dx$

Sol :

$\int (2x - 3\cos x + e^x)\, dx$

$$= 2 \times \frac{x^2}{2} - 3\sin x + e^x + C$$
$$= x^2 - 3\sin x + e^x + C$$

3. $\int (x+1)(x+2)dx$

Sol :

$$\int (x+1)(x+2)dx$$
$$= \int (x^2 + 2x + 1x + 2)\, dx$$
$$= \int (x^2 + 3x + 2)\, dx$$
$$= \frac{x^3}{3} + \frac{3x^2}{2} + 2x + C$$

4. $\int tan^2 x\, dx$

Sol :

$$\int tan^2 x\, dx = \int (sec^2 x - 1)dx$$
$$= \tan x - x + C$$

5. $\int \sec x\,(\sec x + \tan x)\, dx$

Sol :

$$\int \sec x\,(\sec x + \tan x)\, dx$$
$$= \int (sec^2 x + \sec x \tan x)dx$$
$$= \tan x + \sec x + C$$

6. $\int e^{2x+3}\, dx$

Sol :

$$\int e^{2x+3}\, dx = \frac{1}{2} e^{2x+3} + C$$

Since $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

7. Integrate $\frac{\sin(\tan^{-1} x)}{1+x^2}$

Sol:

Put $t = \tan^{-1} x \Rightarrow dt = \frac{1}{1+x^2} dx$

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx = \int \sin t\, dt$$
$$= -\cos t + C$$
$$= -\cos(\tan^{-1} x) + C$$

8. Integrate $\frac{(1+\log x)^2}{x}$

Put $t = 1 + \log x \Rightarrow dt = \frac{1}{x} dx$

$$\int \frac{(1+\log x)^2}{x} dx = \int t^2 dt = \frac{t^3}{3} + C$$
$$= \frac{(1+\log x)^3}{3} + C$$

9. Integrate $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$

Put $t = \sin^{-1} x \Rightarrow dt = \frac{1}{\sqrt{1-x^2}} dx$

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t\, dt = \frac{t^2}{2} + C$$
$$= \frac{(\sin^{-1} x)^2}{2} + C$$

10. Evaluate $\int \frac{2x}{1+x^2} dx$

Sol :

$$\int \frac{2x}{1+x^2} dx = \log|1 + x^2| + C$$

Since $\int \frac{f\prime(x)}{f(x)} = \log|f(x)| + C$

11. Evaluate $\int \frac{\cos x}{1+\sin x} dx$

Sol :

$$\int \frac{\cos x}{1+\sin x} dx = \log|1 + \sin x| + C$$

Since $\int \frac{f\prime(x)}{f(x)} = \log|f(x)| + C$

12. Evaluate $\int \frac{\sin x}{\sqrt{1+\cos x}} dx$

Sol:

$$\int \frac{\text{Sin}\, x}{\sqrt{1+\cos x}} dx = \frac{1}{-1} \int \frac{-\sin x}{\sqrt{1+\cos x}} dx$$
$$= -2\sqrt{1+\cos x} + C$$

Since $\int \frac{f\prime(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$

13. Evaluate $\int \frac{dx}{x^2+4}$

Sol :

$$\int \frac{dx}{x^2+4} = \int \frac{dx}{x^2+2^2}$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Since $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

14. Evaluate $\int \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x+4}}dx$

Sol :

Put $t = \tan x \Rightarrow dt = \sec^2 x\, dx$

$$\int \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$
$$= \log\left|t + \sqrt{t^2+2^2}\right| + C$$
$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

15. Evaluate $\int \frac{1}{x^2+2x+2}dx$

$$x^2 + 2x + 2 = (x+1)^2 - 1^2 + 2$$
$$= (x+1)^2 + 1^2$$

$$\int \frac{1}{x^2+2x+2}dx = \int \frac{1}{(x+1)^2+1^2}dx$$
$$= \frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + C$$
$$= \tan^{-1}(x+1) + C$$

16. Evaluate $\int \frac{dx}{\sqrt{2x-x^2}}dx$

$$2x - x^2 = -(x^2 - 2x)$$
$$= -(x^2 - 2x + 1 - 1)$$
$$= -((x-1)^2 - 1)$$
$$= 1 - (x-1)^2$$

$$\therefore \int \frac{1}{\sqrt{2x-x^2}}dx = \int \frac{1}{\sqrt{1-(x-1)^2}}dx$$
$$= sin^{-1}(x-1) + C$$

PRACTICE PROBLEMS

Integrate the following :

1. $2x^3$
2. $x^3 - x^2 + 1$
3. $x^2 + 5x - 4 + \frac{1}{x} - \frac{2}{x^3}$
4. $\frac{1-\sin x}{\cos^2 x}$
5. $\sec 7x \tan 7x$
6. $\frac{e^{\tan^{-1} x}}{1+x^2}$
7. $x\cos(x^2)$
8. $\frac{(\log x)^2}{x}$
9. $\frac{1}{x+x\log x}$
10. $\frac{\sec^2 x}{\sqrt{tanx}}$
11. $\frac{1}{x^2-4}$
12. $\frac{1}{9-x^2}$
13. $\frac{1}{x^2+5}$
14. $\frac{1}{\sqrt{16-4x^2}}$
15. $\frac{1}{x^2-6x+13}dx$
16. $\frac{1}{\sqrt{3-2x-x^2}}$
17. $\frac{x+2}{2x^2+6x+5}$

❖ Definite integral

$$\int f(x)dx = F(x), \text{then } \int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Eg : $\int_2^3 x^2\, dx = \left[\frac{x^3}{3}\right]_2^3 = \left[\frac{3^3}{3} - \frac{2^3}{3}\right] = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$

QUESTIONS AND ANSWERS

1. $\int_4^5 e^x dx$

Sol :

$\int_4^5 e^x dx = [e^x]_4^5 \quad = e^5 - e^4$

2. $\int_2^3 \frac{1}{x} dx$

Sol :

$\int_2^3 \frac{1}{x} dx = [\log|x|]_2^3$

$= \log 3 - \log 2 = \log\frac{3}{2}$

3. $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$

Sol :

$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \frac{\pi}{2} - 0 \quad = \frac{\pi}{2}$

4. $\int_2^3 \frac{dx}{x^2-1}$

Sol :

$\int_2^3 \frac{dx}{x^2-1} = \frac{1}{2\times1}\left[\log\left|\frac{x-1}{x+1}\right|\right]_2^3$

$= \frac{1}{2}\left[\log\frac{3-1}{3+1} - \log\frac{2-1}{2+1}\right]$

$= \frac{1}{2}\left[\log\frac{2}{4} - \log\frac{1}{3}\right]$

$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right] = \frac{1}{2}\left[\log\left(\frac{\frac{1}{2}}{\frac{1}{3}}\right)\right]$

$= \frac{1}{2}\left[\log\left(\frac{3}{2}\right)\right]$

5. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+cos^2 x} dx$

Sol :

Put $t = \cos x$

$dt = -\sin x\, dx$

When $x = 0, \quad t = \cos 0 = 1$

When $x = \frac{\pi}{2}, \quad t = \cos\frac{\pi}{2} = 0$

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+cos^2 x} dx = \int_1^0 \frac{-dt}{1+t^2} dx$

$= \int_0^1 \frac{dy}{1+t^2} dx$

$= [\tan^{-1} t]_0^1$

$= \tan^{-1} 1 - \tan^{-1} 0$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

PRACTICE PROBLEMS

1. $\int_1^2 (x^3 - x^2 + 1)\, dx$
2. $\int_0^{\frac{\pi}{4}} tan^2 x\, dx$
3. $\int_0^1 \frac{1}{1+x^2} dx$
4. $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

APPLICATION OF INTEGRALS

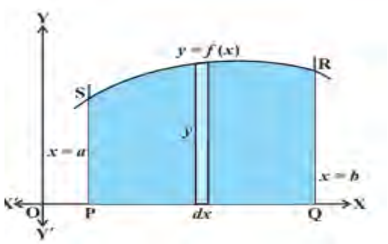
8

KEY NOTES

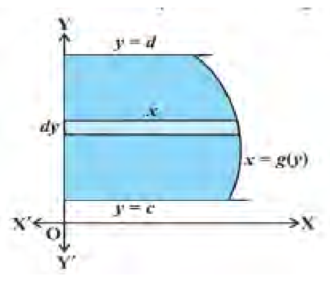
- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$
- $\int (ax+b)\, dx = \frac{(ax+b)^2}{2a} + C$
- $\int \sqrt{x}\, dx = \frac{2}{3} x\sqrt{x}$+C
- $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \sin x\; dx = -\cos x + C$
- $\int \cos x\, dx = \sin x\; + C$

- **Area under simple curves**

The area of the region bounded by the curve $= f(x)$, x- axis and the line $x = a$ and $x = b$ $(b > a)$ is given by **: Area** $= \int_a^b y\, dx\; = \int_a^b f(x)\, dx$



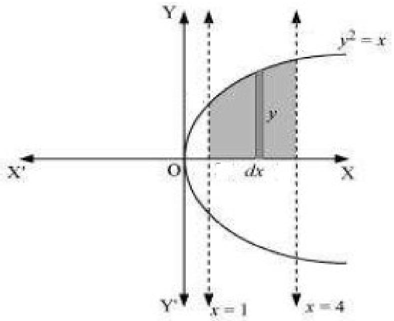
The area of the region bounded by the curve $= g(y)$, y- axis and the line $y = c$ and $y = d$ $(d > c)$ is given by : **Area** $= \int_c^d x\, dy\; = \int_c^d g(y) dy$



QUESTIONS AND ANSWERS

1. Find the area of the region bounded by the curve $y^2 = x$, x-axis and the lines $x = 1, x = 4$ in the first quadrant

Sol :



Given $y^2 = x \Rightarrow y = \sqrt{x}$

Area of the region bounded by the curve $y^2 = x$, lines $x = 1$ & $x = 4$ in the 1st quadrant is = Area of shaded region

$= \int_1^4 y\, dx$

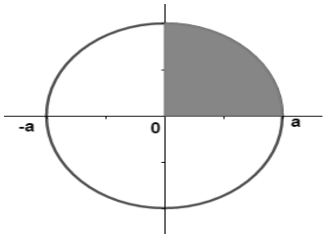
$= \int_1^4 \sqrt{x} dx = \left[\frac{2}{3} x\sqrt{x}\right]_1^4 = \frac{2}{3}\left[x\sqrt{x}\right]_1^4$

$= \frac{2}{3}[4\sqrt{4} - 1\sqrt{1}] = \frac{2}{3}[8 - 1]$

$= \frac{14}{3}$ sq. units

Note : Area of the region bounded by the curve $y^2 = x$, lines $x = 1$ & $x = 4$ is $2 \times \frac{14}{3} = \frac{28}{3}$ sq. units. [$\because$ the curve is symmetric with respect x - axis, the area below and above x axis are equal]

2. Find the area enclosed by the curve $x^2 + y^2 = a^2$



Given, $x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$

$\Rightarrow y = \sqrt{a^2 - x^2}$

Required Area = 4 area of shaded region

$= 4\int_0^a y\, dx$

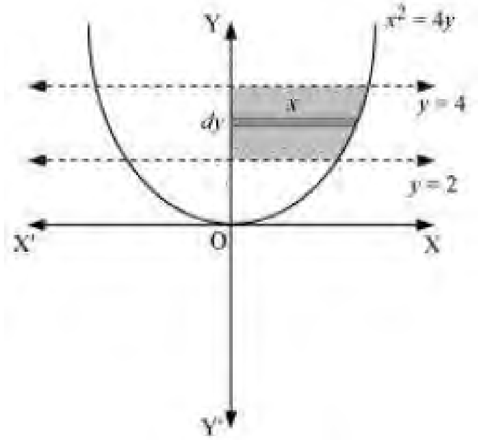
$= 4\int_0^a \sqrt{a^2 - x^2}\, dx$

$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= 4\left[\left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - 0\right]$

$= 4 \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi a^2$ sq units.

3. Find the area of the region bounded by the curve $x^2 = 4y$ and the lines $y = 2,\ y = 4$ and the y- axis in the 1st quadrant



Given $x^2 = 4y \Rightarrow x = \sqrt{4y} = 2\sqrt{y}$ and the lines are $y = 2$ & $y = 4$

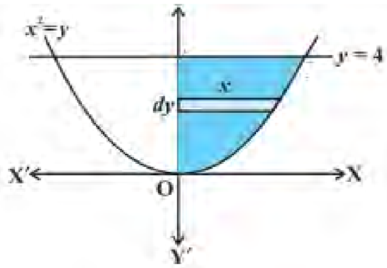
Required area $= \int_2^4 x\, dy$

$= \int_2^4 2\sqrt{y}\, dy = 2\left[\frac{2}{3} y\sqrt{y}\right]_2^4$

$= \frac{4}{3}\left[y\sqrt{y}\right]_2^4 = \frac{4}{3}[4\sqrt{4} - 2\sqrt{2}]$

$= \frac{4}{3}[8 - 2\sqrt{2}]$ sq units

4. Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$



Given $y = x^2 \Rightarrow x = \sqrt{y}$

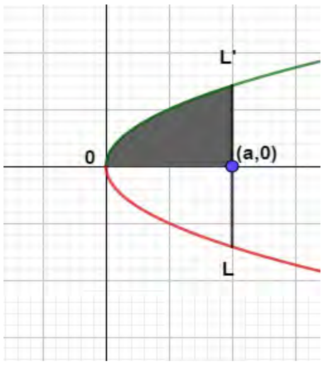
Required area $= 2\int_0^4 x\,dy$

$= 2\int_0^4 \sqrt{y}\,dy$

$= 2\left[\frac{2}{3}y\sqrt{y}\right]_0^4 = \frac{4}{3}[4\sqrt{4} - 0]$

$= \frac{4}{3}[8 - 0] = \frac{32}{3}\,sq\ units$

5. Find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum



Given curve is $y^2 = 4ax$

$\Rightarrow\ y = \sqrt{4ax} = 2\sqrt{a}\sqrt{x}$

The required area $2\int_0^a y\,dx$

$= 2\int_0^a 2\sqrt{a}\sqrt{x}\ dx$

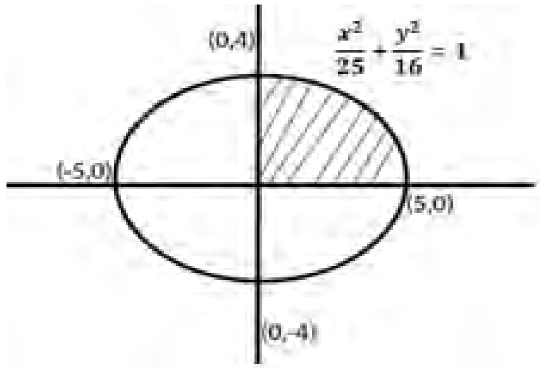
$= 4\sqrt{a}\left[\frac{2}{3}x\sqrt{x}\right]_0^a$

$= \frac{8}{3}\sqrt{a}\left[x\sqrt{x}\right]_0^a$

$= \frac{8}{3}\sqrt{a}[a\sqrt{a} - 0]$

$= \frac{8a^2}{3}$ sq units

6. Find the area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$



$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\frac{y^2}{16} = 1 - \frac{x^2}{25}$$

$$y^2 = 16\left(\frac{25 - x^2}{25}\right)$$

$$y = \frac{4}{5}\sqrt{25 - x^2}$$

Area of the region bounded by ellipse

= 4 × Area of shaded region

$= 4\int_0^5 \frac{4}{5}\sqrt{25 - x^2}\,dx$

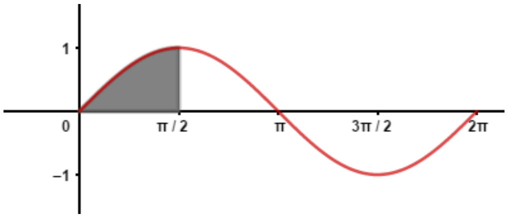
$= \frac{16}{5}\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^5$

$= \frac{16}{5}\left[\left(0 + \frac{25}{2}(\sin^{-1}1)\right) - \left(0 + \frac{25}{2}(\sin^{-1}0)\right)\right]$

$= \frac{16}{5}\left[\frac{25}{2} \times \frac{\pi}{2}\right]$

$= 4 \times 5\pi = 20\,\pi$ sq. units

7. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$



Required area = 4 area of shaded region

$$= 4\int_0^{\frac{\pi}{2}} y\ dx = 4\int_0^{\frac{\pi}{2}} \sin x\ dx$$

$$= 4[-cosx\ ]_0^{\frac{\pi}{2}}$$

$$= -4[\cos\frac{\pi}{2} - \cos 0]$$

$$= -4[0 - 1] = 4 \text{ sq units}$$

PRACTICE PROBLEMS

1. Find the area bounded by the curve $y^2 = 9x$ and the lines $x = 2, x = 4$ and the x axis in the 1^{st} quadrant

2. Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

3. Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$

4. Find the area bounded by the curve $y = |x|$ between $x = -2$ and $x = 2$

5. Find the area of the region in the 1^{st} quadrant enclosed by the x-axis , the line $y = x$, and the circle $x^2 + y^2 = 32$

DIFFERENTIAL EQUATIONS

9

KEY NOTES

$\int \frac{f'(x)}{f(x)} dx = log\,|f(x)| + C$

$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$

$\int e^x\, dx = e^x + C$

Slope of tangent at any point $(x, y) = \frac{dy}{dx}$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

Differential Equations

An equation involving derivatives of dependent variables with respect to independent variables is known as a differential equation

Order and Degree Of a Differential Equation

Order of a differential equation is the **order of the highest order derivative** occurring in the differential equation

Degree of a differential equation is the highest **power of the highest order derivative** in it.

Note: ***Degree of a differential equation is defined if it is a polynomial equation in its derivative***

Eg: Consider $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

Here highest order derivative is $\frac{d^2y}{dx^2}$, which is 2nd order.

∴ Order of the d.e is 2

Power of highest order derivative is 1. ∴ Degree of the d.e is 1

Solution of a Differential Equation

A function which satisfies the given differential equation is called its solution. The solution which contains **as many arbitrary constants as the order of the differential equation** is called a **general solution** and the solution **free from arbitrary constant** is called **particular solution.**

Variable Separable Form

- Write the given differential equation in the form $\mathbf{N}\, dy = \mathbf{M}\, dx$, where **N** is function of y and **M** is a function of of x.
- Integrate both sides and add an **arbitrary constant** on **one side**, gives the general solution.

Eg: Consider $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$$dy = (1+x^2)(1+y^2)dx$$

$$\frac{\boldsymbol{dy}}{(\mathbf{1}+\boldsymbol{y}^{\mathbf{2}})} = (\mathbf{1}+\boldsymbol{x}^{\mathbf{2}})\boldsymbol{dx}$$ [**N** $\boldsymbol{dy}$ = **M** $\boldsymbol{dx}$ form]

Now integrating both sides , $\int \frac{dy}{(1+y^2)} = \int (1+x^2)dx$

$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C$ which is the general solution

Note: To get particular solution from the general solution , find the value of arbitrary constant C using given values of x and y .

QUESTIONS AND ANSWERS

1. Find the order and degree of the following differential equations

(a) $xy\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^3 - y\frac{dy}{dx} = 0$

(b) $\left(\frac{ds}{dt}\right)^4 + 3s\frac{d^2s}{dt^2} = 0$

(c) $x^4\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^3$

(d) $\left(\frac{d^3y}{dx^3}\right)^2 + cos\left(\frac{dy}{dx}\right) = 0$

(e) $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

(f) $y''' + y^2 + e^{y'} = 0$

Sol :

(a) Highest order derivative in d.e is $\frac{d^2y}{dx^2}$.

$\therefore$ order =2

Power of highest order derivative is 2

$\therefore$ degree =2

(b) Highest order derivative in d.e is $\frac{d^2s}{dt^2}$.

$\therefore$ order =2

Power of highest order derivative is 1.

$\therefore$ degree =1

(c) Highest order derivative in d.e is $\frac{d^2y}{dx^2}$.

$\therefore$ order =2

Power of highest order derivative is 1.

$\therefore$ degree =1

(d) Highest order derivative in d.e is $\frac{d^3y}{dx^3}$

$\therefore$ order =3

Here given d.e is not a polynomial equation ($\because cos\left(\frac{dy}{dx}\right)$ is present)

$\therefore$ Degree is not defined

(e) Highest order derivative in d.e is y'''

$\therefore$ order =3

Power of highest order derivative is 2 .

$\therefore$ degree =2

(f) Highest order derivative in d.e is y'''.

$\therefore$ order =3

Here given d.e is not a polynomial equation ($\because e^{y'}$ is present)

$\therefore$ Degree is not defined.

2. Solve the following differential equations.

(a) $\frac{dy}{dx} = \frac{x+1}{2-y}$

(b) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

(c) $\frac{dy}{dx} = e^{x-y}$

(d) $\sec^2 x \ \tan y \ dx + \sec^2 y \ \tan x \, dy = 0$

(e) $y \log y \ dx - xdy = 0$

Sol :

(a) $\frac{dy}{dx} = \frac{x+1}{2-y}$

$(2-y)dy = (x+1)dx$

Integrating both sides

$\int (2-y)\, dy = \int (x+1)\, dx$

$2y - \frac{y^2}{2} = \frac{x^2}{2} + x + \text{C}$,

which is the general solution

(b) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both sides

$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$\tan^{-1} y = \tan^{-1} x + C$, which is the general solution

(c) $\frac{dy}{dx} = e^{x-y}$

$\frac{dy}{dx} = \frac{e^x}{e^y}$

$e^y dy = e^x \ dx$

Integrating both sides

$e^y = e^x + C$, which is the general solution

(d) $sec^2 \, x \, tan \, y \, dx + sec^2 \, y \, tan \, x \, dy = 0$

$\Rightarrow sec^2 \, x \, tan \, y \, dx = - sec^2 \, y \, tan \, x \, dy$

$\Rightarrow \quad \frac{sec^2 \, x}{\tan x} dx = \frac{- sec^2 \, y}{\tan y} dy$

Integrating both sides,

$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy$

$\Rightarrow \log|\tan x| = -\log|\tan y| + \log \ |C|$

$\Rightarrow \log|tan \, x| + \log|\tan y| = \log \ |C|$

$\Rightarrow \ \log|\tan x \tan y| = \log|C|$

$\Rightarrow \quad tan \, x \, tan \, y = C$

(e) $y \ \text{logy} \ dx - xdy = 0$

$y \log y \ \ dx = x \ dy$

$\frac{dy}{y \, log \, y} = \frac{dx}{x}$

Integrating both sides, $\int \frac{\frac{1}{y} dy}{\log y} = \int \frac{dx}{x}$

$\Rightarrow log|\log y| = \log|x| + log|c|$

$\Rightarrow \ \log|\log y| = \log|Cx|$

$\Rightarrow \quad \log y = Cx$

$\therefore y = e^{Cx}$ is the general solution

3. Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$,given that $y = 1$, when $x = 0$

Sol:

$\frac{dy}{dx} = -4xy^2$

$\frac{dy}{y^2} = -4x \ dx$

Integrating both sides,

$\int -\frac{dy}{y^2} = \int 4x\, dx$

$\frac{1}{y} = 4(\frac{x^2}{2}) + C$

$\frac{1}{y} = 2x^2 + C$

Given that $y = 1$, when $x = 0$

$\Rightarrow \frac{1}{1} = 2 \times 0^2 + C \;\Rightarrow C = 1$

$\therefore$ particular solution is

$\frac{1}{y} = 2x^2 + 1$

$\Rightarrow \quad y = \frac{1}{2x^2+1}$

PRACTICE PROBLEMS

1. Find the order and degree of the following

(a) $2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5 = 0$

(b) $\frac{d^2y}{dx^2} + y = 0$

(c) $y'' + (y')^2 + 2y = 0$

(d) $\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - \sin^2 y \;= 0$

2. Solve $x^2 \frac{dy}{dx} - 2\, xy = 0$

3. Solve

$e^x \tan y \;dx + (1 - e^x)\sec^2 y \;dy = 0$

4. Solve $\frac{dy}{dx} = (xy + y + x + 1)$

5. Find the equation of the curve passing through the point $(-2,3)$, given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$

6. Find the equation of curve passing through the point $(0,2)$given that at any point (x, y) on the curve, the product of slope of its tangent and y coordinate of the point is equal to the x coordinate of the point

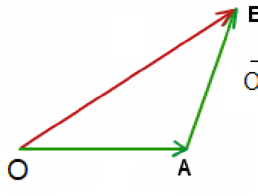
10

VECTOR ALGEBRA

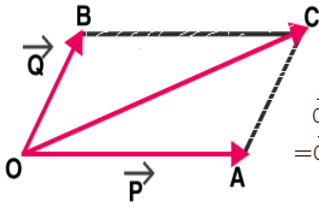
KEY NOTES

❖ **Addition of vectors**

(i) Triangular law



(ii) Parallelogram law



Let $\bar{a} = a_1 i + b_1 j + c_1 k$; $\bar{b} = a_2 i + b_2 j + c_2 k$

$\bar{a} + \bar{b} = (a_1 + a_2)i + (b_1 + b_2)j + (c_1 + c_2)k$

$\bar{a} - \bar{b} = (a_1 - a_2)i + (b_1 - b_2)j + (c_1 - c_2)k$

❖ **Scalar multiplication**

Let $\bar{a}$ be any vector and k any scalar

Case – I : If $k > 0$

$k\bar{a}$ is a vector whose magnitude is k times $|\bar{a}|$ and direction along $\bar{a}$

Case – II : If $k < 0$

$k\bar{a}$ is a vector whose magnitude is k times $|\bar{a}|$ and direction opposite of $\bar{a}$

$\bar{a}$ is parallel to $\bar{b} \Longrightarrow \bar{a} = \lambda\bar{b}$, λ being a scalar

Let $\bar{a} = a_1 i + b_1 j + c_1 k$, then $k\bar{a} = (ka_1)i + (kb_1)j + (kc_1)k$

$|\bar{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

- **Unit vector in the direction of** $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$
- **Vector in the direction of $\bar{a}$ having magnitude k** $= k\left(\frac{\bar{a}}{|\bar{a}|}\right)$

❖ **Vector joining A$(x_1, y_1, z_1)$and B(x_2, y_2, z_2) is**

$\overrightarrow{AB} = \textit{Position vector of } B - \textit{Position vector of } A$

Or

$\overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

❖ Product of vectors

Dot product/ Scalar product	Cross product/ Vector product
• $\bar{a}\cdot\bar{b} = \lvert\bar{a}\rvert\lvert\bar{b}\rvert\cos\theta$ • $\bar{a}\cdot\bar{b} = a_1a_2 + b_1b_2 + c_1c_2$ • $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{\lvert a\rvert\lvert b\rvert}$ • $i\cdot i = j\cdot j = k\cdot k = 1$ • $i\cdot j = j\cdot k = k\cdot i = 0$ • $\bar{a} \perp \bar{b} \Longrightarrow \bar{a}\cdot\bar{b} = 0$ i.e., $a_1a_2 + b_1b_2 + c_1c_2 = 0$ • $\bar{a} \parallel \bar{b} \Longrightarrow \bar{a} = \lambda\bar{b} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ • **Projection of** $\bar{a}$ **on** $\bar{b} = \frac{\bar{a}\cdot\bar{b}}{\lvert b\rvert}$	• $\bar{a}\times\bar{b} = \lvert a\rvert\lvert b\rvert\sin\theta\cdot\hat{n}$, where $\hat{n}$, a unit vector $\perp$ to both $\bar{a}$ and $\bar{b}$ • $\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ • $\lvert\bar{a}\times\bar{b}\rvert = \lvert\bar{a}\rvert\lvert\bar{b}\rvert\sin\theta$ • $\sin\theta = \frac{\lvert\bar{a}\times\bar{b}\rvert}{\lvert a\rvert\lvert b\rvert}$ • vector perpendicular to both $\bar{a}\&\bar{b} = \bar{a}\times\bar{b}$ • Unit vector perpendicular to both $\bar{a}\&\bar{b}$, $\hat{n} = \frac{\bar{a}\times\bar{b}}{\lvert\bar{a}\times\bar{b}\rvert}$ • **The area of a parallelogram with adjacent sides** $\bar{a}$ **and** $\bar{b} = \lvert\bar{a}\times\bar{b}\rvert$ • **Area of Δ with adjacent sides** $\bar{a}$ **and** $\bar{b}$ $= \frac{1}{2}\lvert\bar{a}\times\bar{b}\rvert$

QUESTIONS AND ANSWERS

1. Consider $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

(a) Find the magnitude $\vec{a}$

(b) Find a unit vector in the direction of $\vec{a}$

(c) Find a vector in the direction of $\vec{a}$ having magnitude 10

Sol:

(a) $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.

$|\vec{a}| = \sqrt{a^2 + b^2 + c^2}$

$= \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$

(b) Unit vector in the direction of $\bar{a} = \frac{\bar{a}}{|\bar{a}|}$

$$= \frac{3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}}{\sqrt{17}}$$

(c) Vector in the direction of $\bar{a}$ having magnitude $10 = 10\left(\frac{\bar{a}}{|\bar{a}|}\right) = 10\left(\frac{3\hat{\imath}+2\hat{\jmath}+2\hat{k}}{\sqrt{17}}\right)$

2. If $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$, Find

(a) $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$

(b) Find a unit vector in the direction of $\vec{a} + \vec{b}$

Sol:

(a) $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$

$\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$

$\vec{a} + \vec{b} = (2+3)\hat{\imath} + (1+^{-}2)\hat{\jmath} + (3+1)\hat{k}$

$= 5\hat{\imath} - \hat{\jmath} + 4\hat{k}$

$\vec{a} - \vec{b} = (2-3)\hat{\imath} + (1-^{-}2)\hat{\jmath} + (3-1)\hat{k}$

$= -\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$

(b) Unit vector in the direction of

$\vec{a} + \vec{b} = \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{5\hat{\imath}-\hat{\jmath}+4\hat{k}}{\sqrt{5^2+(-1)^2+4^2}}$

$= \frac{5\hat{\imath}-\hat{\jmath}+4\hat{k}}{\sqrt{30}}$

3. Consider $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$ & $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$.

(a) Find $\vec{a}.\vec{b}$

(b) Find the angle between the $\vec{a}$ and $\vec{b}$.

Sol:

(a) $\vec{a} = \hat{\imath} + \hat{\jmath} - \hat{k}$, $\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\vec{a}.\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$

$= 1 \times 1 + 1 \times^{-} 1 +^{-}1 \times 1 = -1$

(b) $|\vec{a}| = \sqrt{1^2 + 1^2 + (^{-}1)^2} = \sqrt{3}$

$|\vec{b}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\vec{a}.\vec{b} = -1$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{-1}{3}$

$\therefore \theta = \cos^{-1}\left(-\frac{1}{3}\right)$

4. The position vectors of three points A, B, C are given to be $\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$, $4\hat{\imath} + 4\hat{k}$ and $-2\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$ respectively

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(c) Find $\overrightarrow{AB} \times \overrightarrow{AC}$

(d) Find a vector which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having magnitude 9 units

Sol:

(a) $\overrightarrow{AB} = p.v\,(B) - p.v(A)$

$= (4\,\hat{\imath} + 4\,\hat{k}) - (\hat{\imath} + 3\,\hat{\jmath} + 3\,\hat{k})$

$= 3\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}$

$\overrightarrow{AC} = p.v\,(C) - p.v(A)$

$= (-2\,\hat{\imath} + 4\,\hat{\jmath} + 2\,\hat{k}) - (\hat{\imath} + 3\,\hat{\jmath} + 3\,\hat{k})$

$= -3\,\hat{\imath} + \hat{\jmath} - \hat{k}$

(b) $\cos\theta = \frac{\overrightarrow{AB}\,.\,\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}$

$= \frac{(3\,\hat{\imath} - 3\,\hat{\jmath} + \hat{k}).(-3\,\hat{\imath} + \hat{\jmath} - \hat{k})}{\sqrt{9+9+1}\,\sqrt{9+1+1}}$

$= \frac{-9-3-1}{\sqrt{9+9+1}\,\sqrt{9+1+1}}$

$= \frac{-13}{\sqrt{19}\,\sqrt{11}}$

$\therefore \theta = \cos^{-1}\left(\frac{-13}{\sqrt{19}\,\sqrt{11}}\right)$

(c) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -3 & 1 \\ -3 & 1 & -1 \end{vmatrix}$

$= \hat{\imath}(3-1) - \hat{\jmath}(-3+3) + \hat{k}(3-9)$

$= 2\,\hat{\imath} - 6\,\hat{k}$

(d) Vector having magnitude 9 units and perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$= 9\left(\frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}\right)$

$= 9\left(\frac{2\,\hat{\imath} - 6\,\hat{k}}{\sqrt{2^2+(-6)^2}}\right)$

$= \frac{18\,\hat{\imath} - 54\,\hat{k}}{\sqrt{40}}$

5. Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = 2\hat{\imath} + m\hat{\jmath} + 3\hat{k}$, if $\vec{a}$ is perpendicular to $\vec{b}$, find *m*

Sol :

Given vectors are $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$

and $\vec{b} = 2\,\hat{\imath} + m\,\hat{\jmath} + 3\,\hat{k}$

Given that $\vec{a} \perp \vec{b}$

$\therefore \vec{a} \cdot \vec{b} = 0$

$\Rightarrow \quad 1 \times 2 + 1 \times m + 1 \times 3 = 0$

$\Rightarrow \quad 5 + m = 0$

$\Rightarrow \quad m = -5$

6. If $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 2\hat{k}, \vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

(a) Find $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

(b) Find a unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Sol :

(a) $\vec{a} + \vec{b} = 4\hat{\imath} + 4\hat{\jmath} + 0\hat{k}$

$\vec{a} - \vec{b} = 2\hat{\imath} + 0\hat{\jmath} + 4\hat{k}$

(b) Vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= 16\hat{\imath} - 16\hat{\jmath} - 8\hat{k}$$

$$|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|$$

$$= \sqrt{(16)^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{576}$$

$$= 24$$

∴ Unit vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is

$$= \frac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$$

$$= \frac{16\hat{\imath}-16\hat{\jmath}-8\hat{k}}{24}$$

7. If $\vec{a} = \hat{\imath} + 3\hat{\jmath}$ and $\vec{b} = 3\hat{\jmath} + \hat{k}$, then

(a) Find $\vec{a}.\vec{b}$ and $\vec{a} \times \vec{b}$

(b) Find the projection of $\vec{a}$ *on* $\vec{b}$

(c) Find a unit vector which is perpendicular to both $\vec{a}$ and $\vec{b}$

Sol:

(a) $\vec{a} = \hat{\imath} + 3\hat{\jmath} + 0\hat{k}$, $\vec{b} = 0\hat{\imath} + 3\hat{\jmath} + \hat{k}$

$$\vec{a}.\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$$
$$= 1 \times 0 + 3 \times 3 + 0 \times 1 = 9$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{vmatrix} = 3\hat{\imath} - \hat{\jmath} + 3\hat{k}$$

(b) The projection of $\vec{a}$ *on* $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{9}{\sqrt{0^2 + 3^2 + 1^2}} = \frac{9}{\sqrt{10}}$$

(c) $\vec{a} \times \vec{b} = 3\hat{\imath} - \hat{\jmath} + 3\hat{k}$

$$|\vec{a} \times \vec{b}| = \sqrt{9 + 1 + 9} = \sqrt{19}$$

∴ Unit vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$= \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{3\hat{\imath}-\hat{\jmath}+3\hat{k}}{\sqrt{19}}$$

8. Consider the triangle ABC with vertices A (1,2,3), B (−1,0,4), C(0,1,2)

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) Find the area of triangle ABC

Sol:

(a) A (1,2,3), B (−1,0,4), C (0,1,2)

We have

$$\overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

$$\overrightarrow{AB} = (-1 - 1)i + (0 - 2)j + (4 - 3)k$$

$$\overrightarrow{AB} = -2\hat{\imath} - 2\hat{\jmath} + k$$

$$\overrightarrow{AC} = (0 - 1)i + (1 - 2)j + (2 - 3)k$$

$$\overrightarrow{AC} = -\hat{\imath} - \hat{\jmath} - k$$

(b) $$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & -2 & 1 \\ -1 & -1 & -1 \end{vmatrix}$$

$$= 3\hat{\imath} - 3\hat{\jmath} + 0k$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$$

Area of Δ ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{\sqrt{18}}{2}$ sq. units

9. Consider $\vec{a} = 2\hat{\imath} + \hat{\jmath}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$

(a) Find $\vec{a} \times \vec{b}$

(b) Find the area of triangle for which $\vec{a} \& \vec{b}$ are adjacent sides

(c) Find the area of parallelogram for which the vectors $\vec{a} \& \vec{b}$ are adjacent sides

Sol:

(a) $\vec{a} = 2\hat{\imath} + \hat{\jmath}$ and $\vec{b} = 3\hat{\imath} + \hat{\jmath} + 4\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 0 \\ 3 & 1 & 4 \end{vmatrix} = 4\hat{\imath} - 8\,\hat{\jmath} - \hat{k}$$

(b) $|\vec{a} \times \vec{b}| = \sqrt{4^2 + (-8)^2 + (-1)^2}$

$= \sqrt{81} = 9$

Area of triangle $= \frac{1}{2}|\vec{a} \times \vec{b}|$

$= \frac{9}{2}$ sq. units

(c) Area of parallelogram $= |\vec{a} \times \vec{b}|$

$= 9$ sq units

10. If $\vec{a}$ and $\vec{b}$ are any two vectors, then prove that $|\vec{a} \times \vec{b}|^2 = a^2b^2 - (a \cdot b)^2$

Sol :

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

$$|\vec{a} \times \vec{b}|^2 = a^2b^2 sin^2\theta$$

$$= a^2b^2(1 - cos^2\theta)$$

$$= a^2b^2 - a^2b^2cos^2\theta$$

$$= a^2b^2 - (abcos\theta)^2$$

$$= a^2b^2 - (a \cdot b)^2$$

11. If $\vec{a} \cdot \vec{b} = 12$, $|\vec{a}| = 10$, $|\vec{b}| = 2$, then find $|\vec{a} \times \vec{b}|$

$|\vec{a}| = 10$, $|\vec{b}| = 2$ $\quad \vec{a} \cdot \vec{b} = 12$

$$|\vec{a} \times \vec{b}|^2 = a^2b^2 - (a \cdot b)^2$$

$$= 10^2 \times 2^2 - 12^2$$

$$= 400 - 144 = 256$$

$$|\vec{a} \times \vec{b}| = \sqrt{256} = 16$$

PRACTICE PROBLEMS

1. Consider $\vec{a} = 3\hat{\imath} + 4\hat{\jmath} + \hat{k}$

(a) Find the magnitude $\vec{a}$

(b) Find a unit vector in the direction of $\vec{a}$

(c) Find a vector in the direction of $\vec{a}$ having magnitude 8

2. Consider the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\vec{b} = \hat{\imath} + 4\hat{\jmath} - \hat{k}$

(a) Find $\vec{a}.\vec{b}$ and $\vec{a} \times \vec{b}$

(b) Find the projection of $\vec{a}$ and $\vec{b}$

(c) Find the area parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$

(d) Find the area of triangle with adjacent sides $\vec{a}$ and $\vec{b}$

3. If $\vec{a} = 3\hat{\imath} - \hat{\jmath} - 5\hat{k}$; $\vec{b} = \hat{\imath} - 5\hat{\jmath} + 3\hat{k}$,

(a) $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$

(b) Find a unit vector in the direction of $\vec{a} + \vec{b}$

(c) Find a vector perpendicular to both $\vec{a} \& \vec{b}$

4. Consider the triangle ABC with vertices A (1,1,1) B (1,2,3) and C (2,3,1)

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(c) Find the area of triangle ABC

THREE DIMENSIONAL GEOMETRY

11

KEY NOTES

- Equation of a line passing through a given point and parallel to a given vector

  Vector equation : $\boldsymbol{\bar{r} = \bar{a} + \lambda\bar{b}}$

  Cartesian equation : $\boldsymbol{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}}$

  Here, $\bar{a}$: Position vector of point on the line

  $\bar{b}$: vector parallel to the line

  (x_1, y_1, z_1) : Point on the line

  (a, b, c) : DR's of the line

- Equation of line passing through 2 points

  Vector equation : $\boldsymbol{\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})}$

  Cartesian equation : $\boldsymbol{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$

  Here, $\bar{a}$ and $\bar{b}$: Position vectors of point on the line

  (x_1, y_1, z_1) and (x_2, y_2, z_2) : Points on the line

- Distance between two skew lines

  Consider two lines in space,

  $$\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$$
  $$\bar{r} = \bar{a}_2 + \mu\bar{b}_2$$

  The shortest distance between these lines is given by

  $$\textbf{S.D} = \left|\boldsymbol{\frac{(\bar{a}_2-\bar{a}_1)\cdot(\bar{b}_1\times\bar{b}_2)}{|\bar{b}_1\times\bar{b}_2|}}\right|$$

- Equation of a plane passing through a point and perpendicular to a vector.

  Vector equation : $\boldsymbol{(\bar{r} - \bar{a})\cdot\bar{m} = 0}$

  Here, $\bar{a}$: Position vectors of point on the plane

  $\bar{m}$: vector perpendicular to the plane

  Cartesian equation : $\boldsymbol{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$

  Here, (x_1, y_1, z_1) : Point on the plane

  (a, b, c) : DR's of the normal to the plane

- Equation of a plane passing through the non collinear points

  Vector equation : $\boldsymbol{(\bar{r} - \bar{a})\cdot\left[(\bar{b} - \bar{a})\times(\bar{c} - \bar{a})\right] = 0}$

  Here, $\bar{a}, \boldsymbol{\bar{b}, \bar{c}}$: Position vectors of point on the plane

  Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

  Where, (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are the points on the plane.

QUESTIONS AND ANSWERS

1. Find the vector and cartesian equation of the line through the point (5, 2, −4) and is parallel to the vector $3\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$

Sol :

Given $(x_1, y_1, z_1) = (5, 2, -4)$

$\bar{b} = 3\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$

Vector equation, $\bar{r} = \bar{a} + \lambda\bar{b}$

$= 5\hat{\imath} + 2\hat{\jmath} - 4\hat{k} + \lambda(3\hat{\imath} + 2\hat{\jmath} - 8\hat{k})$

Cartesian equation,

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z-(-4)}{-8}$$

$$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$$

2. Find the equation of line in vector form and cartesian form that passes through the point with position vector $2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and is in the direction of $\hat{\imath} + 2\hat{\jmath} - \hat{k}$.

Sol :

Given

$\bar{a} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k} \Rightarrow (x_1, y_1, z_1) = (2, -1, 4)$

$\bar{b} = \hat{\imath} + 2\hat{\jmath} - \hat{k} \Rightarrow (a, b, c) = (1, 2, -1)$

Vector equation, $\bar{r} = \bar{a} + \lambda\bar{b}$

$= 2\hat{\imath} - \hat{\jmath} + 4\hat{k} + \lambda(\hat{\imath} + 2\hat{\jmath} - \hat{k})$

Cartesian equation,

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

3. Find the vector and cartesian equation of the line passing through the points (−1, 0, 2) and (3, 4, 6)

Sol :

Given

$(x_1, y_1, z_1) = (-1, 0, 2)$

$\Rightarrow \bar{a} = -\hat{\imath} + 0\hat{\jmath} + 2\hat{k}$

$(x_2, y_2, z_2) = (3, 4, 6)$

$\Rightarrow \bar{b} = 3\hat{\imath} + 4\hat{\jmath} + 6\hat{k}$

Vector equation,

$\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$= -\hat{\imath} + 0\hat{\jmath} + 2\hat{k} + \lambda\left[(3\hat{\imath} + 4\hat{\jmath} + 6\hat{k}) - (-\hat{\imath} + 0\hat{\jmath} + 2\hat{k})\right]$

$= -\hat{\imath} + 2\hat{k} + \lambda\left[(4\hat{\imath} + 4\hat{\jmath} + 4\hat{k})\right]$

Cartesian equation,

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-(-1)}{3-(-1)} = \frac{y-0}{4-0} = \frac{z-2}{6-2}$$

$$\frac{x+1}{4} = \frac{y}{4} = \frac{z-2}{4} \text{ Or } \frac{x+1}{1} = \frac{y}{1} = \frac{z-2}{1}$$

4. The cartesian equation of a line is $\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$. Find its vector equation.

Sol :

Given

$$\frac{x+3}{2} = \frac{y-5}{4} = \frac{z+6}{2}$$

$$\frac{x-(-3)}{2} = \frac{y-5}{4} = \frac{z-(-6)}{2}$$

∴ vector equation is $\bar{r} = \bar{a} + \lambda\bar{b}$

$= -3\hat{\imath} + 5\hat{\jmath} - 6\hat{k} + \lambda(2\hat{\imath} + 4\hat{\jmath} + 2\hat{k})$

5. The vector equation of a line is

$\bar{r} = 5\hat{\imath} - 4\hat{\jmath} + 6\hat{k} + \lambda(3\hat{\imath} + 7\hat{\jmath} + 2\hat{k})$.

Obtain its cartesian equation.

Sol :

Given

$\bar{r} = 5\hat{\imath} - 4\hat{\jmath} + 6\hat{k} + \lambda(3\hat{\imath} + 7\hat{\jmath} + 2\hat{k})$

$(x_1, y_1, z_1) = (5, -4, 6)$

$(a, b, c) = (3, 7, 2)$

Cartesian equation is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\frac{x-5}{3} = \frac{y-(-4)}{7} = \frac{z-6}{2}$$

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$$

6. Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\bar{r} = \hat{\imath} + \hat{\jmath} + \lambda(2\hat{\imath} - \hat{\jmath} + \hat{k})$ and

$\bar{r} = 2\hat{\imath} + \hat{\jmath} - \hat{k} + \mu(3\hat{\imath} - 5\hat{\jmath} + 2\hat{k})$.

Sol :

Here,

$\bar{a}_1 = \hat{\imath} + \hat{\jmath}$

$\bar{b}_1 = 2\hat{\imath} - \hat{\jmath} + \hat{k}$

$\bar{a}_2 = 2\hat{\imath} + \hat{\jmath} - \hat{k}$

$\bar{b}_2 = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

$\bar{a}_2 - \bar{a}_1 = \left(2\hat{\imath} + \hat{\jmath} - \hat{k}\right) - (\hat{\imath} + \hat{\jmath})$

$= i - k$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$= i(-2+5) - j(4-3) + k(-10+3)$

$= 3\hat{\imath} - \hat{\jmath} - 7\hat{k}$

$\left|\bar{b}_1 \times \bar{b}_2\right| = \sqrt{3^2 + (-1)^2 + (-7)^2}$

$= \sqrt{9 + 1 + 49}$

$= \sqrt{59}$

$(\bar{a}_2 - \bar{a}_1) \cdot \left(\bar{b}_1 \times \bar{b}_2\right) = (i - k) \cdot \left(3\hat{\imath} - \hat{\jmath} - 7\hat{k}\right)$

$= (1 \times 3) + (0 \times -1) + (-1 \times -7)$

$= 10$

$\therefore$ S.D $= \left|\frac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$

$= \frac{10}{\sqrt{59}}$

7. The cartesian equations of two lines are

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

(a) Write the vector equations

(b) Find the shortest distance between these lines.

Sol :

(a) $\vec{r} = \left(-\hat{\imath} - \hat{\jmath} - \hat{k}\right) + \lambda(7\hat{\imath} - 6\hat{\jmath} + \hat{k})$

$\vec{r} = \left(3\hat{\imath} + 5\hat{\jmath} + 7\hat{k}\right) + \mu(\hat{\imath} - 2\hat{\jmath} + \hat{k})$

(b) $S.D = \left|\frac{(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_1} \times \overrightarrow{b_2})}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}\right|$ ……. (1)

$\overrightarrow{a_1} = -\hat{\imath} - \hat{\jmath} - \hat{k}$; $\overrightarrow{b_1} = 7\hat{\imath} - 6\hat{\jmath} + \hat{k}$

$\overrightarrow{a_2} = 3\hat{\imath} + 5\hat{\jmath} + 7\hat{k}$; $\overrightarrow{b_2} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$\overrightarrow{a_2} - \overrightarrow{a_1} = 4\hat{\imath} + 6\hat{\jmath} + 8\hat{k}$

$$\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$= \hat{\imath}(-6+2) - \hat{\jmath}(7-1) + \hat{k}(-14+6)$

$= -4\hat{\imath} - 6\hat{\jmath} - 8\hat{k}$

$(1) \Rightarrow$

$S.D = \left|\frac{\left(4\hat{\imath}+6\hat{\jmath}+8\hat{k}\right).(-4i-6j-8k)}{\sqrt{16+36+64}}\right|$

$= \left|\frac{-16-36-64}{\sqrt{116}}\right|$

$= \frac{116}{\sqrt{116}}$

$= \sqrt{116}$

8. Find the vector and cartesian equations of the plane which passes through the point (5, 2, -4) and perpendicular to the line with direction ratios (2, 3, -1)

Sol :

Given

$$(x_1, y_1, z_1) = (5, 2, -4)$$
$$\bar{a} = 5\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$$
$$\bar{m} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$$

Vector equation,

$$(\bar{r} - \bar{a}) \cdot \bar{m} = 0$$
$$\left(\bar{r} - \left(5\hat{\imath} + 2\hat{\jmath} - 4\hat{k}\ \right)\right) \cdot \left(2\hat{\imath} + 3\hat{\jmath} - \hat{k}\ \right) = 0$$

Cartesian equation,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
$$2(x - 5) + 3(y - 2) - 1(z + 4) = 0$$
$$2x - 10 + 3y - 6 - z - 4 = 0$$
$$2x + 3y - z = 20$$

9. Find the vector and cartesian equations of the plane passing through (1, 0, -2) and normal to the vector $\hat{\imath} + \hat{\jmath} - \hat{k}$.

Sol :

Point $(1, 0, -2) \Rightarrow \vec{a} = \hat{\imath} - 2\hat{k}$

Normal vector $\vec{n} = \hat{\imath} + \hat{\jmath} - \hat{k}$

Vector form : $(\vec{r} - \vec{a}).\vec{n} = 0$

$$\Rightarrow\ \vec{r}.\vec{n} - \vec{a}.\vec{n} = 0$$
$$\vec{r}.(\hat{\imath} + \hat{\jmath} - \hat{k}) -. (\hat{\imath} - 2\hat{k}) \cdot (\hat{\imath} + \hat{\jmath} - \hat{k}) = 0$$
$$\vec{r}.(\hat{\imath} + \hat{\jmath} - \hat{k}) - (1 \times 1\ +\ 0 \times 1\ +\ (-2)(-1)) = 0$$
$$\vec{r}.(\hat{\imath} + \hat{\jmath} - \hat{k}) - 3 = 0$$

Cartesian form : $x + y - z - 3 = 0$

10. Find the equation of the plane passing through the points (2, 5, -3) , (-2, -3, 5) and (5, 3, -3)

Sol :

Given

$$(x_1, y_1, z_1) = (2, 5, -3)$$
$$(x_2, y_2, z_2) = (-2, -3, 5)$$
$$(x_3, y_3, z_3) = (5, 3, -3)$$

Equation of plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

i.e., $\begin{vmatrix} x - 2 & y - 5 & z + 3 \\ -2 - 2 & -3 - 5 & 5 + 3 \\ 5 - 2 & 3 - 5 & -3 + 3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 5 & z + 3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$(x - 2)[0 + 16] - (y - 5)[0 - 24] + (z + 3)[8 + 24] = 0$$
$$16(x - 2) + 24(y - 5) + 32(z + 3) = 0$$
$$2(x - 2) + 3(y - 5) + 4(z + 3) = 0$$
$$2x - 4 + 3y - 15 + 4z + 12 = 0$$
$$2x + 3y + 4z - 7 = 0$$

PRACTICE PROBLEMS

1. Find vector and cartesian equations of the line passes through (1,2,3) and is parallel to the vector $3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

2. Find vector and cartesian equations of the line that passes through origin and (5,-2,3)

3. Find the vector equation of a line whose cartesian equation is $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

4. Find the shortest distance between the lines

$$\vec{r} = \left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$$
$$\vec{r} = \left(2\hat{\imath} - \hat{\jmath} - \hat{k}\right) + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$$

5. Cartesian equations of two lines are $\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z-3}{2}$ and $\frac{x-4}{2} = \frac{y-5}{3} = \frac{z-6}{1}$

(a) Write their vector equations

(b) Find the shortest distance between these lines

6. Find vector and cartesian equations of plane passing through (1,4,6) and the normal vector to the plane is $\hat{\imath} - 2\hat{\jmath} + \hat{k}$

7. Find the equation of plane passing through the points $(3,-1,2)$, $(5,2,4)$ and $(-1,-1,6)$

12

LINEAR PROGRAMMING PROBLEM

KEY NOTES

- **Objective function**

  Linear function $z = ax + by$ which has to be maximised or minimised is called objective function

- **Constraints**

  The linear inequalities in a LPP are called constraints.

- Solving an LPP is to find values of x and y which maximise or minimise the objective function

- **Steps to solve LPP graphically**

  1. **Draw the graphs of the inequalities and identify the intersection region which is called feasible region**
  2. **Determine the coordinates of the corner points of the feasible region**
  3. **Substitute the values of corner points in objective function, find the maximum or minimum values of $z = ax + by$**

QUESTIONS AND ANSWERS

1. Solve the L.P.P

Maximize $z = 3x + 2y$

Subject to $x + 2y \leq 10$

$3x + y \leq 15$

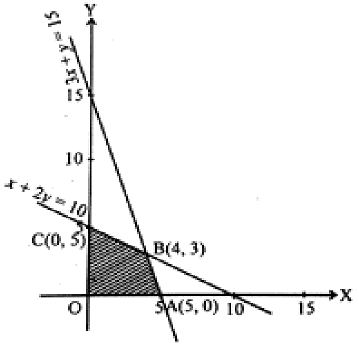
$x, y \geq 0$

Sol: $x + 2y = 10$

x	0	10
y	5	0

$3x + y = 15$

x	0	5
y	15	0



The feasible region is shaded in the figure

B is the point of intersection of lines

$3x + y = 15 \ldots\ldots (1)$ &

$x + 2y = 10 \ldots\ldots (2)$

$(2) \times 3 \Rightarrow 3x + 6y = 30 \ldots\ldots (3)$

$(3) - (1) \Rightarrow 5y = 15 \Rightarrow y = 3$

When $y = 3, (2) \Rightarrow \; x + 2 \times 3 = 10$

$x = 10 - 6 = 4$

B is $(4, 3)$

Corner points are O (0,0), A (5,0), B (4,3), and C (0,5)

Corner Points	$z = 3x + 2y$
O(0,0)	$z = 3(0) + 2(0) = 0$
A(5,0)	$z = 3(5) + 2(0) = 15$
B(4,3)	$z = 3(4) + 2(3) = 18$
C(0,5)	$z = 3(0) + 2(5) = 10$

Maximum value of Z is 18 at B (4, 3)

2. Solve the LPP graphically

Maximize $Z = 3x + 5y$

The constraints are

$$x + 3y \leq 3$$
$$x + y \leq 2$$
$$x, y \geq 0$$

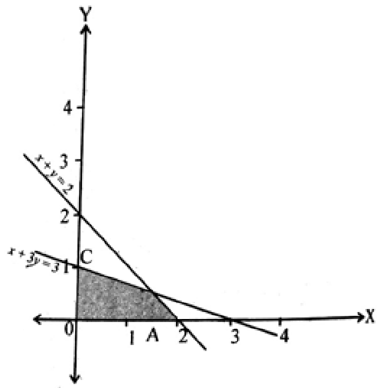
Sol:

$x + 3y = 3$

x	0	3
y	1	0

$x + y = 2$

x	0	2
y	2	0



Shaded region OABC is the feasible region B is the intersecting point of the lines

$x + 3y = 3 \ldots\ldots(1)$

$x + y \ = 2 \ldots\ldots(2)$

$(1) - (2) \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}$

When $y = \frac{1}{2}, (2) \Rightarrow \ x + \frac{1}{2} = 2$

$\Rightarrow x = 2 - \frac{1}{2} = \frac{3}{2}$

$\therefore$ B is $\left(\frac{3}{2}, \frac{1}{2}\right)$

The vertices of the feasible region are

$O(0,0)$, $A(2,0), B\left(\frac{3}{2}, \frac{1}{2}\right), C(0,1)$

Corner point	$Z = 3x + 5y$
$O(0,0)$	$Z = 3 \times 0 + 5 \times 0 = 0$
$A(2,0)$	$Z = 3 \times 2 + 5 \times 0 = 6$
$B(\frac{3}{2}, \frac{1}{2})$	$Z = 3 \times \frac{3}{2} + 5 \times \frac{1}{2} = \frac{14}{2} = 7$
$C(0,1)$	$Z = 3 \times 0 + 5 \times 1 = 5$

Maximum value of Z is 7 at B $\left(\frac{3}{2}, \frac{1}{2}\right)$

3. Solve the LPP given below graphically

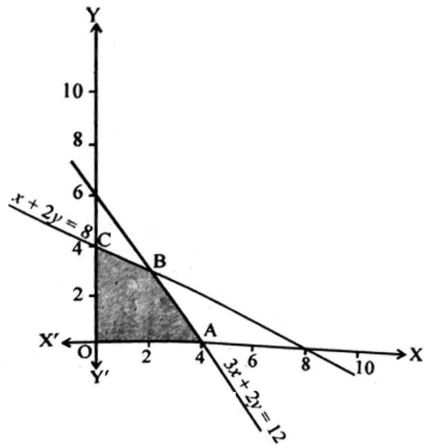
Minimize $z = -3x + 4y$ subject to

$$x + 2y \leq 8$$
$$3x + 2y \leq 12$$
$$x \geq 0, y \geq 0$$

Sol: $x + 2y = 8$ $\quad$ $3x + 2y = 12$

x	0	8
y	4	0

x	0	4
y	6	0



Feasible region is OABC,

B is the intersecting point of the lines

$x + 2y = 8 \ldots\ldots(1)$

$3x + 2y = 12 \ldots\ldots(2).$

$(2) - (1) \Rightarrow 2x = 4 \Rightarrow x = 2$

When $x = 2, (1) \Rightarrow \ 2 + 2y = 8$

$\Rightarrow 2y = 8 - 2 = 6$

$\Rightarrow y = 3$

$\therefore$ B is $(2,3)$

The corner points of the feasible region are: $O(0,0), A(4,0), B(2,3)$ and $C(0,4)$

Corner point	$Z = -3x + 4y$
$O(0,0)$	$Z = -3 \times 0 + 4 \times 0 = 0$
$A(4,0)$	$Z = -3 \times 4 + 4 \times 0 = -12$
$B(2,3)$	$Z = -3 \times 2 + 4 \times 3 = 6$
$C(0,4)$	$Z = -3 \times 0 + 4 \times 4 = 16$

$\therefore$ Minimum value of $Z = -12$ obtained at $A(4,0)$

4. Solve the LPP given below graphically
Minimize and maximize $Z = 3x + 9y$
subject to the constraints

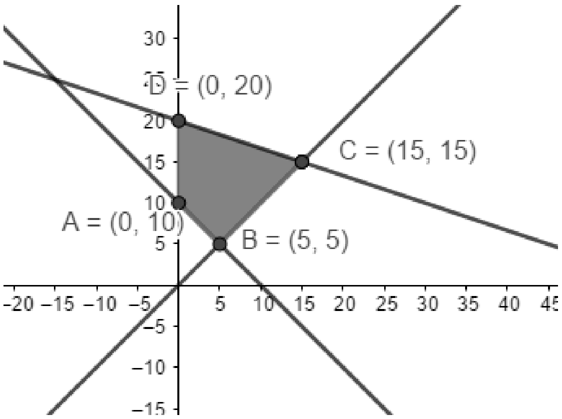
$$x + 3y \leq 60$$
$$x + y \geq 10$$
$$x \leq y$$
$$x, y \geq 0$$

$x + 3y = 90$

x	0	8
y	4	0

$x + y = 10$

x	0	8
y	4	0



The corner points of the feasible region are

$A(0,10), (5,5), B(15,15)$ and $D(0,20)$

Corner point	$Z = 3x + 9y$
A(0,10)	90
B(5,5)	**60**
C(15,15)	**180**
D(0,20)	**180**

Minimum value of Z 60 at B(5,5)

Maximum value of Z is 180 **at all the points on the line segment joining C(15,15) and D(0,20)**

PRACTICE PROBLEMS

1. Solve the following LPP graphically
Maximize Z=4x+y subject to the constraints
$x + y \leq 50$
$3x + y \leq 90$
$x \geq 0,\ y \geq 0$

2. Minimize Z=200x+500y subject to the constraints
$x + 2y \geq 10$
$3x + 4y \leq 24$
$x \geq 0,\ y \geq 0$

3. Minimize and maximize
$Z = 5x + 10y$ subject to the constraints
$x + 2y \leq 120$
$x + y \geq 60$
$x - 2y \geq 0$
$x, y \geq 0$

13

PROBABILITY

KEY NOTES

- $\mathbf{P(A) = \frac{n(A)}{n(S)}}$ $\quad\quad$ $\mathbf{P(A') = 1 - P(A)}$
- $\mathbf{P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)}$
- $\mathbf{P(A \text{ and } B) = P(A \cap B) = P(A) + P(B) - P(A \cup B)}$
- $\mathbf{P(A \text{ and not } B) = P(A \cap B') = P(A - B) = P(A) - P(A \cap B)}$
- $\mathbf{P(\text{not } A \text{ and not } B) = P(A' \cap B') = 1 - P(A \cup B)}$
- $\mathbf{P(\text{not } A \text{ or not } B) = P(A' \cup B') = 1 - P(A \cap B)}$
- $\mathbf{P(\text{exactly one of } A \text{ and } B) = P(A \cup B) - P(A \cap B)}$

- **Conditional Probability**
  If A and B are two events, then the conditional probability of A given B is given by
  $$\mathbf{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$
  Note :
  We can use the formula $\mathbf{P(A/B) = \frac{n(A \cap B)}{n(B)}}$
  where $n(A \cap B)$ is the number of elements in $A \cap B$ and
  $n(B)$ is the number of elements in B

- **Independent Events**
  Two events are said to be independent if the occurrence or non-occurrence of one event does not affect the occurrence of other.

  ie $\mathbf{P(A/B) = P(A)}$ **and** $\mathbf{P(B/A) = P(B)}$
  Or
  Two events A and B are **independent** if $\mathbf{P(A \cap B) = P(A) \times P(B)}$

Note:

If A and B are independent events, then

- **A'and B** are independent events
- **A and B'** are independent events
- **A' and B'** are independent events.

QUESTIONS AND ANSWERS

1. If $P(A) = \frac{7}{13}, P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, then find

(a) P(A/B)
(b) P(B/A)
(c) P(A∪B)

Sol: $P(A) = \frac{7}{13}; P(B) = \frac{9}{13}; P(A \cap B) = \frac{4}{13}$

(a) $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{4}{13}}{\frac{9}{13}} = \frac{4}{9}$

(b) $P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{4}{13}}{\frac{7}{13}} = \frac{4}{7}$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{7}{13} + \frac{9}{13} - \frac{4}{13} = \frac{12}{13}$

2. If P(A) = 0.6, P(B)= 0.7 and $P(A \cup B) = 0.9$, then find

(a) P (A∩B)

(b) $P(A/B)$

(c) $P(B/A)$

Sol :

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.9 = 0.6 + 0.7 - P(A \cap B)$

$P(A \cap B) = 0.6 + 0.7 - 0.9 = 0.4$

(b) $P\left({}^{A}/{}_{B}\right) = \frac{P(A\cap B)}{P(B)} = \frac{0.4}{0.7} = \frac{4}{7}$

(c) $P\left({}^{B}/{}_{A}\right) = \frac{P(A\cap B)}{P(A)} = \frac{0.4}{0.6} = \frac{4}{6} = \frac{2}{3}$

3. If $P(A) = 0.8, P(B) = 0.5$ and $P(B|A) = 0.4$, find

(i) $P(A \cap B)$
(ii) $P(A|B)$
(iii) $P(A \cup B)$

Sol :

(i) $P(A) = 0.8, P(B) = 0.5$

$P(B|A) = 0.4,$

$\Rightarrow \frac{P(A\cap B)}{P(A)} = 0.4$

$P(A \cap B) = 0.4 \times P(A)$

$= 0.4 \times 0.8 = 0.32$

(ii) $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{0.32}{0.5} = \frac{32}{50} = \frac{16}{25}$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.8 + 0.5 - 0.32 = 0.98$

4. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?

Sol :

Let **A** be the event '**the number on the card drawn is even'** and **B** be the event **'the number on the card drawn is greater than 3'.**

Then $A = \{2, 4, 6, 8, 10\}$,&
$B = \{4, 5, 6, 7, 8, 9, 10\}$ and
$A \cap B = \{4, 6, 8, 10\}$

$n(A) = 5,\ n(B) = 7\ \&\ n(A \cap B) = 4$

Here we have to find $P(A/B)$

$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{4}{7}$

5. A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Sol:

Let **A** be the event that **'number 4 appears at least once'** and **B** be the event that **'the sum of the numbers appearing is 6'**.

Then,
A= {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),(1,4), (2,4), (3,4), (5,4), (6,4)}

B = {(1,5), (2,4), (3,3), (4,2), (5,1)}

$A \cap B = \{(2,4), (4,2)\}$

$n(A) = 11,\ n(B) = 5$
$n(A \cap B) = 2$

Here we have to find $P(A/B)$

$P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{5}$

6. Let A and B be events with $P(A) = \frac{3}{5}$ $P(B) = \frac{3}{10}$ $and\ P(A \cap B) = \frac{1}{5}$. Are A and B independent?

Sol:

$P(A) = \frac{3}{5}\ P(B) = \frac{3}{10}\ , P(A \cap B) = \frac{1}{5}$

$P(A) \times P(B) = \frac{\mathbf{3}}{\mathbf{5}} \times \frac{\mathbf{3}}{\mathbf{10}}$

$= \frac{\mathbf{9}}{\mathbf{50}} \neq P(A \cap B)$

∴ A and B are not independent

7. Given two independent events A and B such that P(A)=0.3, P(B)=0.6 find

(i) P(A and B)

(ii) P(A and not B)

(iii) P(A or B)

(iv) P(neither A nor B)

Sol :

Given $P(A) = 0.3\ P(B) = 0.6$

$P(A') = 1 - P(A) = 1 - 0.3 = 0.7,$

$P(B') = 1 - P(B) = 1 - 0.6 = 0.4$

(i) P(A and B) $= P(A \cap B)$

$= P(A).P(B) = 0.3 \times 0.6$

$= 0.18$

(ii) P(A and not B) $= P(A \cap B')$

$= P(A).P(B')$

$= 0.3 \times 0.4 = 0.12$

$[Or\ \ P(A \cap B') = P(A - B) = P(A) - P(A \cap B)$

$= 0.3 - 0.18 = 0.12$]

(iii) P(A or B)= $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.6 - 0.18 = 0.72$

(iv) P(neither A nor B) $= P(A' \cap B')$

$= P(A').P(B')$

$= 0.7 \times 0.4 = 0.28$

[Or $P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - 0.72 = 0.28$]

8. Probability of solving a specific problem independently by A and B are 1/2 and 1/3 respectively. If both try to solve the problem independently then,

(i) Find the probability that both of them solves the problem

(ii) Find the probability that problem is solved

(iii) Find the probability that exactly one of them solves the problem

(iv) Find the probability that none of them solves the problem

Sol:

Let A and B denote event that the problem is solved by A and B respectively.

$P(A) = \frac{1}{2}; P(B) = \frac{1}{3}$

Since A and B are independent

$P(A \cap B) = P(A) \times P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

(i) P(Both of them solves) $= P(A \cap B)$

$= P(A) \times P(B)$[∵A &B are independent]

$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

(ii) P (Problem solved) = $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{5}{6} - \frac{1}{6} = \frac{4}{6}$

(iii) P (exactly one of them solves)

$= P(A \cup B) - P(A \cap B)$

$= \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$

(iv) P(none of them solved)= $P(A \cup B)'$

$= 1 - P(A \cup B) = 1 - \frac{4}{6} = \frac{2}{6} = \frac{1}{3}$

9. A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent?

Sol :

$S = \{1, 2, 3, 4, 5, 6\}$

Now $E = \{3, 6\}, F = \{2, 4, 6\}$ and

$E \cap F = \{6\}$

Then $(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$;

$P(F) = \frac{n(F)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

$P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{6}$

$P(E) \times P(F) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} = P(E \cap F)$

Hence E and F are independent events

10. If A and B are two independent events, then, prove that the probability of occurrence of at least one of A and B is given by $1 - P(A)' P(B')$

Sol :

We have

P (at least one of A and B) = $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A) P(B)$

$= P(A) + P(B) [1 - P(A)]$

$= P(A) + P(B). P(A')$

$= 1 - P(A') + P(B) P(A')$

$= 1 - P(A') [1 - P(B)]$

$= 1 - P(A') P(B')$

PRACTICE PROBLEMS

1. Evaluate $P(A \cup B)$, if

$2P(A) = P(B) = \frac{5}{13}$ and $P(A|B) = \frac{2}{5}$

2. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and

$P(A \cup B) = \frac{7}{11}$, find

(i) $P(A \cap B)$

(ii) $P(A|B)$

(iii) $P(B|A)$

3. Let A and B be independent events with P (A) = 0.3 and P(B) = 0.4. Find

(i) $P(A \cap B)$

(ii) $P(A \cup B)$

(iii) $P(A|B)$

(iv) $P(B|A)$

4. Events A and B are such that P(A) = 1/2, P(B) = 7/12 and P(not A or not B) =¼ State whether A and B are independent ?

5. If A and B are two independent events, then prove that $A \text{ and } B'$ are independent events.

6. The probability that A solves a problem is $\frac{1}{5}$ and the probability that B solve the problem is $\frac{1}{3}$. If both try to solve the problem independently. Find the probability, that :

(i) The problem is solved.

(ii) None of them solve the problem.

(iii) Exactly one of them solves the problem.

7. Two events E and F are such that $P(E) = 0.6, P(F) = 0.2$ and $P(E \cup F) = 0.68$. Are E and F independent?