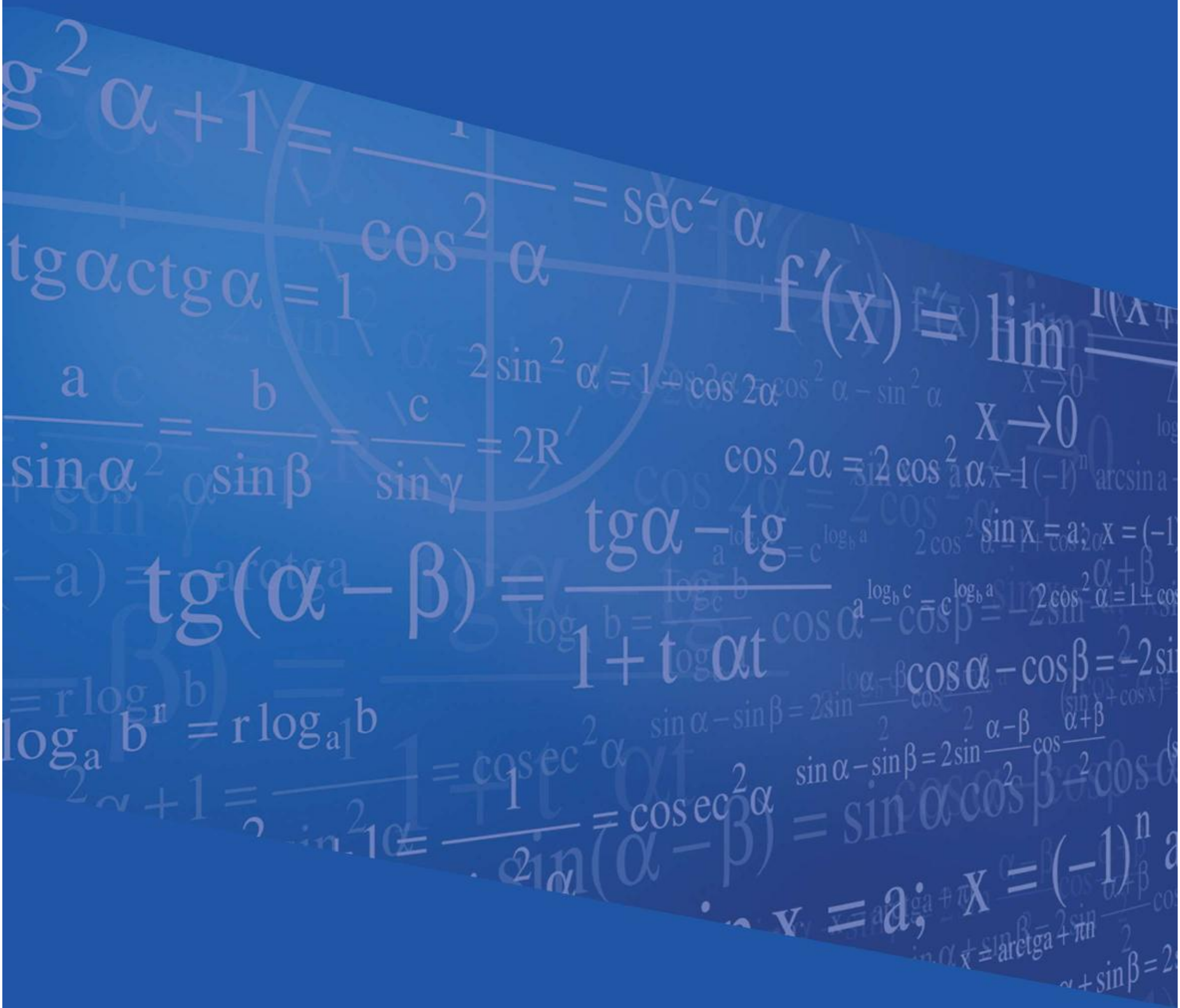
Class-12

MATHEMATICS



GENERAL EDUCATION DEPARTMENT

SAMAGRA SHIKSHA, KERALA

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Chapter - 1

Relations and functions

Worksheet 1

Focus Area – 1 . 3

At a Glance

- * A function f:A→B is said to be one-one function or an injective function if distinct elements of A have distinct images in B. Otherwise f is said to be many-one.
- * To prove a function f is one-one we have to show that · $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- · A fuction f:A→B is said to be an onto function (surjective) if each element of B is the image of some element of A under f. Otherwise f is said to be into function.
- · A function f is onto then range of f= Codmain of f
- · To prove f:A→B is onto, take some element y ∈ B, such that their exist an element x∈A such that $f(x) = y$
- · A function $f : A \to B$ is a bijection if it is both one-one and onto
- · The number of onto functions from the set to itself is n!

<u>Activity - 1</u>

Let A=$\{1,2,3\}, B = \{4,5,6,7\}$ and let $f = \{(1,4), (2,5), (3,6)\}$ be a fucntion from A to B. Check whether f is one-one, onto or bijective

Ans:

f(1)=..............................., f(2)=......................................, f(3)=..................................
Different elements in A have different images in..

f is ....’

The element 7 has no pre-image

Hence f is ....

Since f is not onto, f is ....

Activity - 2

Show that f :N→N given by $f(x) = 2x$, is one-one but not onto

Ans:

$f(x_1) = f(x_2)$

....

....

$\therefore$ f is..

Let y ∈ N and let f(x)=y

....

....

$x = \frac{y}{2} \notin$ _________

Hence f is not onto

Activity - 3

Consider f : R→R given by f(x)=5x+2, show that f is bijective.

Ans:

Let $f(x_1) = f(x_2)$

....

....

$x_1 = x_2$

$\therefore$ f is one - one

Let $y \in R$ *and* $f(x) = y$

....

....

$x = \frac{y-2}{5} \in R$

$\therefore$ f is ....

Since f is is both one-one and onto it is bijective.

Activity - 4

Consider the functions

(i) f : R→R given by $f(x)=x^2$

(ii) f : R→R given by $f(x)=x^3$

(iii) f : R→R given by $f(x)=|x|$

check the injectivity and surjectivity of the above functions.

(i) f(-1) = ...

f(1) = ...

∴ f is ....

-2 has no pre-image

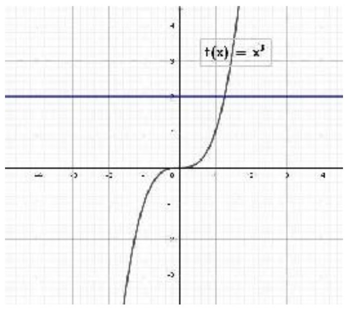
f is ....

from horizonal line test it is clear that f is....................................

Horizontal line test

1. If we draw horizontal lines in a graph and if they intersect the graph only one point then the function is 1-1, otherwise many one.
2. If we draw horizontal lines and if they intersect the graph at atleast one point then the function is onto.

(ii) $f(x)=x^3$

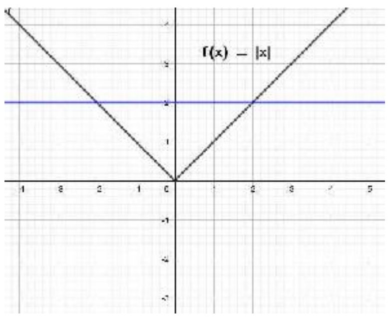


using horizontal line test it is clear

that f is......................

f is onto since ....

(iii) f(x) = |x|



The horizontal line meets the curve

at more than one point hence f is -------

f is not onto since...................................

Focus Area 1.4

At a glance

- Let $f:A \rightarrow B$, $g : B \rightarrow C$ be two functions then composition of these functions denoted by gof is a function from $A \rightarrow C$ such that

  $gof(x) = g(f(x))$

  Also $fog(x) = f(g(x)$

  A function $f : A \rightarrow B$ is invertibile if there exist $g:B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$

- A function f is invertibile iff f is bijective

Activity - 1

Find fog and gof, if $f:R \rightarrow R$ and $g: R \rightarrow R$ given by $f(x) = \cos x$, $g(x) = 3x^2$

$fog(x) = f[g(x)]$

=.......=.......

$gof(x) = g[f(x)]$

=.......=.......

Activity - 2

Find fog and gof if $f(x) = 8x^3$, $g(x)=x^{1/3}$

Ans:

$fog(x) = f[g(x)]$

=...........=............

$gof(x) = g[f(x)]$

=...........=............

Activity - 3

Let $f:\{1,3,4\} \rightarrow \{1,2,5\}$ and $g:\{1,2,5\} \rightarrow \{1,3\}$ given by $f=\{(1,2), (3,5), (4,1)\}$, $g=\{1,3), (2,3), (5,1)\}$ write down gof

Ans:

gof(1) = g[f(1)] = g(2)=................

gof(3) = g[f(3)] =........................

gof (4) =.............................

gof = {(1,.....), (3,....), (4,.....)}

Activity - 4

Consider f: R $\rightarrow$ R given by f(x) = 4x+3, show that f is invertibile. Also find f^{-1}

Ans:

Let $f(x_1) = f(x_2)$

....

$x_1 = x_2$

$\therefore$ f is ....

Let $y \in R$ and let $f(x) = y$

....

....

$x =$ .... $\in R$

$\therefore$ f is ....

Since f is both one-one and onto f is....................

$f^{-1} =$ ....

Activity 5

If $f(x) = \frac{4x+3}{6x+4}, x \neq \frac{2}{3}$, find fof and hence find inverse off?

Ans:

fof(x) = f[f(x)]

=..........

=.........

= x

Since fof (x) = x, f^{-1}=...

Activity - 6

Let f: {2,3,4,5} $\rightarrow$ {7,9,11,13} be a function defined by f={(2,7), (3,11), (4,9), (5,13)}

Is f is invertibile. If invertibile find f^{-1}

Ans:

$f(2) = 7$, $f(3) =$ ...., $f(4)=$................., $f(5)$.........................

$\therefore$ f is...........................

Since range of f = ...

f is onto

Since f is both one - one and onto it is ...., hence f is invertibile

$\therefore$ f^{-1}= {(7,2), (11,.....), (9,.....), (13,......)}

Chapter - 2

Inverse Trigonometric Functions

Worksheet 1

Focus Area 2.3

At a glance

* $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}, x\in[-1,1]$

* $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}, x\in R$

* $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2}, |x|\geq 1$

* $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), xy<1$

* $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right), xy>-1$

* $\sin^{-1}(-x)=-\sin^{-1}x, x\in[-1,1]$

* $\tan^{-1}(-x)=-\tan^{-1}x, x\in R$

* $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x, |x|\geq 1$

* $\cos^{-1}(-x)=\pi-\cos^{-1}x, x\in[-1,1]$

* $\sec^{-1}(-x)=\pi-\sec^{-1}x, |x|\geq 1$

* $\cot^{-1}(-x)=\pi-\cot^{-1}x, x\in R$

* $\sin^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}x, x\geq 1 \quad or \quad x\leq -1$

* $\cos^{-1}\frac{1}{x}=\sec^{-1}x, x\geq 1 \quad or \quad x\leq -1$

* $\tan^{-1}\frac{1}{x}=\cot^{-1}x, x>0$

Activity 1

Show that $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Ans:

$$LHS=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$$
$$=\tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right)$$
$$=\tan^{-1}\left(\qquad\right)$$
$$=\tan^{-1}\left(\frac{1}{2}\right)=RHS$$

Activity -2

Solve $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

Ans: $$\tan^{-1}\left(\qquad\right)=\frac{\pi}{4}$$

$$\frac{5x}{1-6x^2}=\tan\frac{\pi}{4}=1$$
$$6x^2+5x-1=0$$
$$x=$$

Activity -3

show that $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\left(\frac{84}{85}\right)$

Ans: Let $\sin^{-1}\frac{3}{5}=x$ and $\sin^{-1}\frac{8}{17}=y$

$\sin x=\frac{3}{5}$ and $\sin y=\frac{8}{17}$

$$\cos x = \sqrt{1-\sin^2 x} =$$
$$\cos y = \sqrt{1-\sin^2 y} =$$

We have $\cos(x-y) = \cos x.coy + \sin x.\sin y$

$$=$$

$$=\frac{84}{85}$$

$$\therefore x - y = \cos^{-1}\left(\frac{84}{85}\right)$$

$$\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{84}{85}\right)$$

<u>Acitity -4</u>

Find value of (i) $\cos\left[\sec^{-1} x + \cos ec^{-1} x\right]$

(ii) $\cos^{-1}\left(\frac{^{-}1}{2}\right)$

(iii) $\cot^{-1}\sqrt{3}$

Ans:

(i) $\cos\left[\sec^{-1} x + \cos ec^{-1} x\right] = \cos\frac{\pi}{2} =$ ______________

(ii) $\cos^{-1}\left(\frac{^{-}1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right) =$ _________________

(iii) $\cot^{-1}\sqrt{3} = \tan^{-1}\frac{1}{\sqrt{3}} =$ _____________

Worksheet 2

Focus Area 2.3

At a glance

* $\sin^{-1}(\sin x) = x, x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

* $\cos^{-1}(\cos x) = x, x \in [0, \pi]$

* $\tan^{-1}(\tan x) = x, x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

* $\cot^{-1}(\cot x) = x, x \in (0, \pi)$

* $\sec^{-1}(\sec x) = x, x \in [0, \pi] - \{\pi/2\}$

* $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x, x \in [-\pi/2, \pi/2] - \{0\}$

* $2\tan^{-1} x = \sin^{-1}\left[\frac{2x}{1+x^2}\right], |x| \le 1$

* $2\tan^{-1} x = \cos^{-1}\left[\frac{1-x^2}{1+x^2}\right], x \ge, 0$

* $2\tan^{-1} x = \tan^{-1}\left[\frac{2x}{1-x^2}\right], -1 < x < 1$

Activity -1

Find value of (i) $\sin^{-1}\left(\sin 3\pi/5\right)$

(ii) $\tan^{-1}\left(\tan 3\pi/4\right)$

Ans:

1. $\sin^{-1}\left(\sin 3\frac{\pi}{5}\right) = 3\frac{\pi}{5} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{3\pi}{5}\right)\right)$$

$= ....$

$= ....$

2. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\frac{3\pi}{4}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left(\tan\left(\frac{3\pi}{4}-\pi\right)\right)$$
$$=\frac{3\pi}{4}-\pi$$
$$=............$$

Activity -2

Simplity (i) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}, x<\pi$

(ii) $\tan^{-1}\sqrt{\frac{\cos x-\sin x}{\cos x+\sin x}}, 0<x<\pi$

Ans:

(i) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}=\tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2.......}}$

$$=\tan^{-1}\sqrt{........}$$
$$=..........$$
$$=\frac{x}{2}$$

(ii)

$\tan^{-1}\left[\frac{\cos x-\sin x}{\text{cosx+sinx}}\right]=$................(Dividing Numerator and Denominator by cosx)

$$=\tan^{-1}\left[\frac{1-\tan x}{1+\tan x}\right]$$
$$=...................$$
$$=\frac{\pi}{4}-x$$

Activity - 3

If $\sin\left[\sin^{-1}\frac{1}{5}+\cos^{-1}x\right]=1$ then find x

$$\sin\left[\sin^{-1}\frac{1}{5}+\cos^{-1}x\right]=1$$

$$\sin^{-1}\frac{1}{5}+\cos^{-1}x=\text{................}$$

$$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

x = ...

x = ...

Activity - 4

Simplify $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right), x \neq 0$

Ans:

Put $x = \tan\theta$

$\theta = \tan^{-1} x$

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right] = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\ldots\ldots\ldots\ldots}-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\ldots\ldots\ldots\ldots-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\ldots\ldots\ldots\ldots-1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left[\frac{1-\ldots\ldots\ldots}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\ldots\ldots\ldots}{2\sin\frac{\theta}{2}\ldots\ldots\ldots}\right]$$

$$= \tan^{-1}[\ldots\ldots\ldots\ldots]$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

Chapter - 3

Matrices

Worksheet-1

Activity - 1

(i) Construct a 3x2 matrix $A=\left[a_{ij}\right]$ for which $a_{ij}=2i-j$

Ans: (ii) If $B=\begin{bmatrix}3 & 2\\ 11 & 4\\ 14 & 10\end{bmatrix}$, find the matrix X such that $2A+X=B$

(i) a_{11}=2x1-1=.........., a_{12}=2x1-2=..........

a_{21}=2x2-1=.........., a_{22}=2x2-2=..........

a_{31}=...................., a_{32}=....................

$$A=\begin{bmatrix}1 & 0\\ 3 & 2\\ 5 & 4\end{bmatrix}$$

(ii) $2A+X=B$

$$X=B-2A=\begin{bmatrix}3 & 2\\ 11 & 4\\ 14 & 10\end{bmatrix}-2\text{ ....}$$

$$=\begin{bmatrix}3 & 2\\ 11 & 4\\ 14 & 10\end{bmatrix}-\begin{bmatrix}2 & 0\\ 6 & 4\\ 10 & 8\end{bmatrix}=\text{..............}$$

Activity - 2

Find the matrix X and Y if $X+Y=\begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix}$ and $X-Y=\begin{bmatrix}-1 & -1\\ 7 & 2\end{bmatrix}$

Ans: $X+Y=\begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix} \rightarrow(1)$

$X-Y=\begin{bmatrix}-1 & -1\\ 7 & 2\end{bmatrix} \rightarrow(2)$

$(1)+(2) \Rightarrow \quad 2X=\begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix}+\begin{bmatrix}-1 & -1\\ 7 & 2\end{bmatrix}=\begin{bmatrix}.... & ....\\ .... & ....\end{bmatrix}$

$X=\frac{1}{2}\begin{bmatrix}4 & 2\\ 10 & 6\end{bmatrix}=\begin{bmatrix}.... & ....\\ .... & ....\end{bmatrix}$

From (1) $Y=\begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix}-X=\begin{bmatrix}5 & 3\\ 3 & 4\end{bmatrix}-..........=..........$

Activity - 3

If $2\begin{bmatrix}x & y\\ z & w\end{bmatrix}+\begin{bmatrix}x & y+x\\ 4 & 2\end{bmatrix}=\begin{bmatrix}3 & 7\\ 5y & 4z\end{bmatrix}$ find the values of x,y,z and w.

Ans: $\begin{bmatrix}2x & 2y\\ 2z & 2w\end{bmatrix}+\begin{bmatrix}x & y+x\\ 4 & 2\end{bmatrix}=\begin{bmatrix}3 & 7\\ 5y & 4z\end{bmatrix}$

$\begin{bmatrix}....... & 3y+x\\ 2z+4 & ....\end{bmatrix}=\begin{bmatrix}3 & 7\\ 5y & 4z\end{bmatrix}$

3x =3, $x=\frac{3}{3}=1$

$3y+x=........\Rightarrow 3y+1=7\Rightarrow 3y=7-1=........\Rightarrow y=\frac{6}{3}=........$

$2z+4=5y \Rightarrow 2z+4=5\times..... \Rightarrow 2z+4=10$

$\Rightarrow 2z=10-4=.....\Rightarrow z=\frac{6}{2}=......$

$2w+2=4z \Rightarrow 2w+2=4\times....... \Rightarrow 2w+2=12$

$\Rightarrow 2w=12-2=..... \Rightarrow w=\frac{10}{2}=.....$

Worksheet - 2

Activity - 1

(i) If A is a 3x2 matrix and AB is a 3x4 matrix, then order of B =.......

(ii) If $A=\begin{bmatrix}2 & 1\\ 3 & -2\end{bmatrix}$ and $B=\begin{bmatrix}1 & -2 & 3\\ 5 & 2 & 0\end{bmatrix}$

Ans: (a) Fnd AB (b) Is BA defined ? justify.

(i) Order of $B=2\times 4$

because if A has order $m\times p$ and B has order $p\times n$, then AB has order $m\times n$

(ii) (a) $AB=\begin{bmatrix}2 & 1\\ 3 & -2\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ 5 & 2 & 0\end{bmatrix}$

$$=\begin{bmatrix}....... & .... & ....\\ .... & .... & ....\end{bmatrix}=\begin{bmatrix}7 & -2 & 6\\ -7 & -10 & 9\end{bmatrix}$$

(b) BA is not defined because number of columns of B $\neq$ number of .... of A

Activity - 2

If $A=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}, B=\begin{bmatrix}-1 & 4\\ 2 & 1\end{bmatrix}, C=\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}$, then

prove that (A B) C = A (B C)

Ans: $AB=\begin{bmatrix}2 & 3\\ 1 & 5\end{bmatrix}\begin{bmatrix}-1 & 4\\ 2 & 1\end{bmatrix}=\begin{bmatrix}2\times -1+3\times 2 & ....\\ .... & 1\times 4+5\times 1\end{bmatrix}$

$$=\begin{bmatrix}........ & ....\\ .... & ....\end{bmatrix}=\begin{bmatrix}4 & 11\\ 9 & 9\end{bmatrix}$$

$$(AB)C=\begin{bmatrix}4 & 11\\ 9 & 9\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 2\end{bmatrix}=\begin{bmatrix}........ & ....\\ .... & ....\end{bmatrix}$$

$$=\begin{bmatrix}12+0 & 4+22\\ 27+0 & 9+18\end{bmatrix}=\begin{bmatrix}... & ...\\ ... & ...\end{bmatrix}$$

$$BC = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}\begin{bmatrix} -3 & 7 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} -6+18 & 14+12 \\ -3+30 & 7+20 \end{bmatrix} = \begin{bmatrix} 12 & 26 \\ 27 & 27 \end{bmatrix}$$

Hence (A B) C = A(BC)

Activity - 3

Consider the Matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

(a) Find A^2

(b) Find the scalar K , such that $A^2 = KA - 7I$

Ans: (a) $A^2 = A \times A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3\times3+1\times-1 & 3\times1+1\times2 \\ -1\times3+2\times-1 & -1\times1+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

(b) $KA = K\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3K & K \\ -K & 2K \end{bmatrix}$

$$7I = 7\times\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$KA - 7I = \begin{bmatrix} 3K & K \\ -K & 2K \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 3K-7 & K \\ \ldots\ldots & \ldots\ldots\ldots \end{bmatrix}$$

$$A^2 = KA - 7I$$

$$\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 3K-7 & K \\ -K & 2K-7 \end{bmatrix}$$

$K = \ldots\ldots$

Activity - 4

(a) Consider the matrix $A=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$, then

prove that $A^2=\begin{bmatrix}\cos(2x) & -\sin(2x)\\ \sin(2x) & \cos(2x)\end{bmatrix}$

(b) If $A(x)=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$, then

prove that A(x) A(y) = A(x+y)

Ans:

(a) $A=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$

$$A^2=A\times A=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2 x+-\sin^2 x & \dots\dots\dots\dots\\ \sin x\cos x+\cos x\sin x & \dots\dots\dots\dots\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2 x-\sin^2 x & -\cos x\sin x-\sin x\cos x\\ 2\sin x\cos x & \cos^2 x-\sin^2 x\end{bmatrix}=\begin{bmatrix}\dots\dots & \dots\dots\\ \dots\dots & \dots\dots\end{bmatrix}$$

(b) $$A(x)\times A(y)=\begin{bmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{bmatrix}\begin{bmatrix}\dots\dots & \dots\dots\\ \dots\dots & \dots\dots\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y+-\sin x\sin y & \dots\dots\dots\dots\dots\dots\\ \dots\dots\dots\dots\dots\dots & -\sin x\sin y+\cos x\cos y\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\cos y-\sin x\sin y & -(\sin x\cos y+\cos x\sin y)\\ \sin x\cos y+\cos x\sin y & \cos x\cos y-\sin x\sin y\end{bmatrix}$$

$$=\begin{bmatrix}\cos(x+y) & -\sin(x+y)\\ \dots\dots\dots & \dots\dots\dots\end{bmatrix}=A(x+y)$$

Worksheet-3

Activity - 1

1. (i) If the matrix $\begin{bmatrix} u & x+y & z \\ -5 & v & -3 \\ 4 & x-y & w+z \end{bmatrix} is$

skew-symmetric find u, v, w, x, y, z.

(ii) $A=\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 3 \\ 5 & 4 \\ 1 & 6 \end{bmatrix}$ prove that $(AB)'=B'A'$

Ans: (i) For a skew-symmetric matrix daigonal elements are 0

Hence u=0, v=o, w+z=0----(1)

also $a_{ij}=-a_{ji}$

Henace $x+y=-(-5)=5 \quad \rightarrow (2)$

$z=..........$

$x-y=........... \quad \rightarrow (3)$

(2)+(3) gives

$2x=8$

$x=\frac{8}{2}=............$

from (2) $4+y=5$

$y=5-4=..........$

from (1) w + z = 0

w + - 4 = 0

w = ...

x = 4, y = 1, z = - 4, u = 0, v = 0, w = 4

(ii) $AB = \begin{bmatrix} 1 & 2 & ^-3 \\ 2 & 1 & ^-1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 4 \\ 1 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1\times2+2\times5+{}^-3\times1 & 1\times3+2\times4+{}^-3\times6 \\ 2\times2+1\times5+{}^-1\times1 & 2\times3+1\times4+{}^-1\times6 \end{bmatrix} = \begin{bmatrix} .... & .... \\ .... & .... \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 9 & 8 \\ ^-7 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ ^-3 & ^-1 \end{bmatrix}, \quad B' = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 2 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ ^-3 & ^-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+5\times2+1\times{}^-3 & 2\times2+5\times1+1\times{}^-1 \\ 3\times1+4\times2+6\times{}^-3 & 3\times2+4\times1+6\times{}^-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+10-3 & .... \\ .... & 6+4-6 \end{bmatrix} = \begin{bmatrix} .... & 8 \\ -7 & .... \end{bmatrix}$$

$$\therefore (AB)' = B'A'$$

<u>Activity - 2</u>

Consider the matrix $A = \begin{bmatrix} 1 & 3 & 6 \\ ^-1 & 2 & 4 \\ ^-2 & 2 & 0 \end{bmatrix}$

(i) Find $A + A^1$

(ii) Find $A - A^1$

(iii) Express A as the sum of a symmetric matrix and a skew-symmetric matrix

Ans: (i) $A = \begin{bmatrix} 1 & 3 & 6 \\ 7 & 2 & 4 \\ ^{-}2 & 2 & 0 \end{bmatrix}$

$A' = \begin{bmatrix} 1 & 7 & ^{-}2 \\ 3 & 2 & 2 \\ 6 & 4 & 0 \end{bmatrix}$

$A' + A = \begin{bmatrix} 1 & 3 & 6 \\ 7 & 2 & 4 \\ -2 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 7 & -2 \\ 3 & 2 & 2 \\ 6 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$

(ii) $A - A' = \begin{bmatrix} 1 & 3 & 6 \\ 7 & 2 & 4 \\ -2 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 7 & -2 \\ 3 & 2 & 2 \\ 6 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$

(iii) $A = \frac{1}{2}(A + A^1) + \frac{1}{2}(A - A^1) \quad \rightarrow (1)$

$= \frac{1}{2}\begin{bmatrix} 2 & 10 & 4 \\ 10 & 4 & 6 \\ 4 & 6 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -4 & 8 \\ 4 & 0 & 2 \\ -8 & -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 & 2 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} + \begin{bmatrix} 0 & -2 & 4 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix}$

Clearly first matrix is symmetric and second matrix is skew-symmetric

Activity - 3

(i) For the matrix $A = \begin{pmatrix} 2 & 3 \\ 5 & 2 \end{pmatrix}$

prove that AA' is symmetric matrix

(ii) If $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ prove that $AA' = I$

Ans: (i) $A=\begin{bmatrix}2 & 3\\5 & 2\end{bmatrix}$, $A'=\begin{bmatrix}\ldots & \ldots\\\ldots & \ldots\end{bmatrix}$

$$AA^1=\begin{bmatrix}2 & 3\\5 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\3 & 2\end{bmatrix}$$

$$=\begin{bmatrix}2\times2+3\times3 & 2\times5+3\times2\\5\times2+2\times3 & 5\times5+2\times2\end{bmatrix}$$

$$=\begin{bmatrix}\ldots\ldots & \ldots\ldots\\\ldots\ldots & \ldots\ldots\end{bmatrix}=\begin{bmatrix}\ldots & \ldots\\\ldots & \ldots\end{bmatrix}$$

$$(AA')'=\begin{bmatrix}13 & 16\\16 & 29\end{bmatrix}=AA'$$

Hence AA^1 is symmetric

(ii) $A=\begin{bmatrix}\cos x & -\sin x\\\sin x & \cos x\end{bmatrix}$, $A^1=\begin{bmatrix}\cos x & \sin x\\-\sin x & \cos x\end{bmatrix}$

$$AA'=\begin{bmatrix}\cos x & -\sin x\\\sin x & \cos x\end{bmatrix}\begin{bmatrix}\cos x & \sin x\\-\sin x & \cos x\end{bmatrix}$$

$$=\begin{bmatrix}\cos x\times\cos x+-\sin x\times-\sin x & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \sin x\sin x+\cos x\cos x\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2 x+\sin^2 x & \cos x\sin x-\cos x\sin x\\\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

Chapter - 4

Determinants

Worksheet - 1

Activity - 1

(i) Evaluate $\begin{vmatrix} 5 & 3 \\ -4 & 2 \end{vmatrix}$

(ii) If $\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -4 & 1 \end{vmatrix}$, find x

Ans: (i) $\begin{vmatrix} 5 & 3 \\ -4 & 2 \end{vmatrix} = 5 \times 2 - (-4) \times 3 = \ldots\ldots.$

(ii) $\begin{vmatrix} x & 8 \\ 2 & x \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -4 & 1 \end{vmatrix}$

$\Rightarrow x^2 - 2 \times 8 = 1 \times 1 - (-4) \times 2$

$\Rightarrow x^2 - 16 = \ldots\ldots\ldots$

$\Rightarrow x^2 = 9 + 16 = \ldots\ldots$

$\Rightarrow x = \sqrt{25} = \pm 5$

Activity - 2

(i) Let A= $\begin{vmatrix} 2 & -4 & 5 \\ 1 & 2 & 3 \\ -3 & 1 & 4 \end{vmatrix}$ find $|A|$

(ii) State whether A is singular or non-singular

Ans: (i) $|A| = \begin{vmatrix} 2 & -4 & 5 \\ 1 & 2 & 3 \\ -3 & 1 & 4 \end{vmatrix}$

$= 2(8-3) + 4(4+9) + 5(1+5)$

$= \ldots\ldots\ldots\ldots\ldots$

(ii) since $|A| \neq 0$, A is non-singular

Activity - 3

(i) Find $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(ii) Find $\begin{vmatrix} \cos 20 & \sin 20 \\ \sin 70 & \cos 70 \end{vmatrix}$

Ans: (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = ...$

(ii) $\begin{vmatrix} \cos 20 & \sin 20 \\ \sin 70 & \cos 70 \end{vmatrix} = \cos 20 \cos 70 - \sin 20 \sin 70 = ... = ...$

Activity - 4

Consider $A = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 5 & 2 \\ 0 & -3 & 6 \end{bmatrix}$ find minors and co-factors of the elements - 2, 5 and - 3

Ans: Minor of $-2 = M_{12} = \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix}$

$= 1 \times 6 - 0 \times 2 = ...$

Co-factor $-2 = A_{12} = (-1)^{1+2} M_{12} = (-1)^3 \times 6 = -6$

Minor of $5 = M_{22} = \begin{vmatrix} 4 & 3 \\ 0 & 6 \end{vmatrix}$

Co-factor $5 = f13^{2+2}(24)$

$= 24$

$= 4 \times 6 - 0 \times 3 = ...$

Minor of $-3 = M_{32} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix}$

$= 4 \times 2 - 1 \times 3 = 8 - 3 = ...$

Co-factor of $^-3 = A_{32} = (^-1)^{3+2} M_{32} = (-1)^5 \times 5 = -1 \times 5 = -5$

Worksheet - 2

Activity 1

(i) Consider $A=\begin{bmatrix}2 & -5\\3 & 1\end{bmatrix}$ find adj A

(ii) Find A^{-1}

Ans: (i) $A=\begin{bmatrix}2 & -5\\3 & 1\end{bmatrix}$

adj $A=\begin{bmatrix}1 & 5\\-3 & 2\end{bmatrix}$

(ii) $|A|=\begin{vmatrix}2 & -5\\3 & 1\end{vmatrix}=2\times1-3\times-5=..........$

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{17}\begin{bmatrix}1 & 5\\-3 & 2\end{bmatrix}=\begin{bmatrix}\dots & \dots\\\dots & \dots\end{bmatrix}$$

Activity 2

Consider $A=\begin{bmatrix}2 & 5\\-4 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & -3\\2 & -4\end{bmatrix}$

(i) Find A^{-1} and B^{-1}

(ii) Prove that $(AB)^{-1}=B^{-1}A^{-1}$

Ans: (i)

$|A|=\begin{vmatrix}2 & 5\\-4 & 3\end{vmatrix}=2\times3-(-4)\times5=.......$

$adjA=\begin{bmatrix}3 & -5\\4 & 2\end{bmatrix}$

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{26}\begin{bmatrix}\dots & \dots\\\dots & \dots\end{bmatrix}$$

$B=\begin{bmatrix}1 & -3\\2 & -4\end{bmatrix}$

$$|B| = \begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} = 1\times -4 - 2\times -3 = -4+6 = ...$$

$$adjB = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adjB = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix}$$

(ii) $$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \times \frac{1}{26}\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$= \frac{1}{52}\begin{bmatrix} -4\times 3 + 3\times 4 & -4\times -5 + 3\times 2 \\ -2\times 3 + 1\times 4 & -2\times -5 + 1\times 2 \end{bmatrix} = \frac{1}{52}\begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} .... & .... \\ .... & .... \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -26 \\ 2 & 0 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 12 & -26 \\ 2 & 0 \end{vmatrix} = 12\times 0 - 2\times(-26) = ...$$

$$adj(AB) = \begin{bmatrix} 0 & 26 \\ -2 & 12 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} adj(AB) = \frac{1}{52}\begin{bmatrix} 0 & 26 \\ -2 & 12 \end{bmatrix} = \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Activity 3

Consider $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

(i) Is A invertible? (ii) find A^{-1}

Ans: (i) $|A| = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$= 2(-4+4) + 3(-6+4) + 5(3-2) = ...$$

Since $|A| \neq 0, A$ is invertibale

(ii) co- factors are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = (-1)^2 (2\times-2-1\times-4) = 1(-4+4) = 1\times 0 = 0$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = (-1)^3 (3\times-2-1\times-4) = -1(-6+4) = -1\times-2 = 2$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (-1)^4 (3\times1-1\times2) = 1(3-2) = 1\times1 = 1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = (-1)^3 (3\times-2-1\times5) = -1(6-5) = -1\times1 = 1$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = (-1)^4 (2\times-2-1\times5) = 1(-4-5) = 1\times-9 = -9$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = (-1)^5 (2\times-1-1\times-3) = -1(2+3) = -1\times5 = -5$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 12-10 = 2$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & ^{-}4 \end{vmatrix} = (^{-}1)^5(2\times^{-}4 - 3\times 5) = 1(-8-15)$$

$$= {}^{-}1\times {}^{-}23 = 23$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & ^{-}3 \\ 3 & 2 \end{vmatrix} = (^{-}1)^6(2\times 2 - 3\times ^{-}3)$$

$$= 1(4+9) = 13$$

$$\text{Co-factor matrix} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Activity - 4

(i) If A is a square matrix of order 3 and |A|=4 , find |adjA|

(ii) If $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ prove that $A^2 - 5A + 10I = 0$ use this to find A^{-1}

(iii) $A = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$ verify that A (adj A)= (adj A)A=|A| I

Ans: (i) $|\text{adj } A| = |A|^{n-1} = |A|^2 = 4^2 =$.........

(ii) $A^2 = A\times A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$

$$= \begin{bmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{bmatrix} = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

$$5A = 5\times\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix} = \begin{bmatrix}\cdots & \cdots\\ \cdots & \cdots\end{bmatrix}$$

$$10I = 10\times\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}\cdots & \cdots\\ \cdots & \cdots\end{bmatrix}$$

$$A^2 - 5A + 10I = \begin{bmatrix}-5 & 15\\ ^-10 & 10\end{bmatrix} - \begin{bmatrix}5 & 15\\ ^-10 & 10\end{bmatrix} + \begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}$$

$$= \begin{bmatrix}\cdots & \cdots\\ \cdots & \cdots\end{bmatrix} = 0$$

$$A^2 - 5A + 10I = 0$$

Multiplying by A^{-1} on both sides

$$A^{-1}A^2 - 5A^{-1}A + 10A^{-1}I = 0$$

$$A - 5..... + 10A^{-1} = 0$$

$$10A^{-1} = -A + 5I$$

$$A^{-1} = \frac{1}{10}(-A + 5I)$$

$$= \frac{1}{10}\left\{-\begin{bmatrix}1 & 3\\ ^-2 & 4\end{bmatrix} + 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{10}\left\{\begin{bmatrix}^-1 & ^-3\\ 2 & ^-4\end{bmatrix} + \begin{bmatrix}\cdots & \cdots\\ \cdots & \cdots\end{bmatrix}\right\}$$

$$= \frac{1}{10}\begin{bmatrix}4 & ^-3\\ 2 & 1\end{bmatrix} = \begin{bmatrix}\cdots & \cdots\\ \cdots & \cdots\end{bmatrix}$$

(iii) $A = \begin{bmatrix}5 & ^-1\\ 2 & 1\end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & ^{-}1 \\ 2 & 4 \end{vmatrix} = 5\times4-2\times-1 = 20+2 = ...$$

$$A(adjA) = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times4+-1\times-2 & 5\times1+-1\times5 \\ 2\times4+4\times-2 & 2\times1+4\times5 \end{bmatrix}$$

$$= \begin{bmatrix} ...... & ...... \\ ...... & ...... \end{bmatrix}$$

$$= 22\times\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 22I = |A|\times I$$

$$(adjA)A = \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}\times\begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times5+1\times2 & 4\times-1+1\times4 \\ -2\times5+5\times2 & -2\times-1+5\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 20+2 & -4+4 \\ -10+10 & 2+20 \end{bmatrix} = \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$$

$$= 22\times\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} = 22\times I = |A|\times I$$

Worksheet - 3

Activity - 1

Solve the system of equations $5x+2y=4, 7x+3y=5$ using matrix method

Ans: Writing the equations in matrix from

$$\begin{bmatrix}5 & 2\\7 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$$

$$AX = B, \text{ where } A=\begin{bmatrix}\cdots & \cdots\\\cdots & \cdots\end{bmatrix}, \quad X=\begin{bmatrix}x\\y\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\5\end{bmatrix}$$

$$|A|=\begin{vmatrix}5 & 2\\7 & 3\end{vmatrix}=5\times3-7\times2=15-14=.....$$

Since $|A|\neq 0$, the system of equations is consistent and has unique solution $X=A^{-1}B$

$$adjA=\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{1}\begin{bmatrix}3 & -2\\-7 & 5\end{bmatrix}=\begin{bmatrix}\cdots & \cdots\\\cdots & \cdots\end{bmatrix}$$

$$X=A^{-1}B=\begin{bmatrix}3 & ^{-}2\\^{-}7 & 5\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}=\begin{bmatrix}3\times4+{}^{-}2\times5\\{}^{-}7\times4+5\times5\end{bmatrix}=\begin{bmatrix}12-10\\-28+25\end{bmatrix}=\begin{bmatrix}\cdots\\\cdots\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\^{-}3\end{bmatrix}$$

$$x=2,\ y={}^{-}3$$

Activity -2

Consider the matrix $A=\begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}$

(i) Find A^{-1}

(ii) Use this result to solve the system of equations

$x-y+z=4, 2x+y-3z=0, x+y+z=2$

Ans: (i) $|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(1\times1-1\times-3)+1(2\times1-1\times-3)+1(2\times1-1\times1)$

$=$.............................

Co-factors are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = (-1)^2(1\times1-1\times-3) = 1(1+3) = 4$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = (-1)^3(2\times1-1\times-3) = -1(2+3) = \ldots\ldots\ldots$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^4(2\times1-1\times1) = \ldots\ldots\ldots\ldots$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^3(-1\times1-1\times1) = \ldots\ldots\ldots\ldots$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (-1)^4(1\times1-1\times1) = \ldots\ldots\ldots\ldots$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)^5(1\times1-1\times-1) = \ldots\ldots\ldots$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = (-1)^5(1\times1-1\times-1) = \ldots\ldots\ldots$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = (-1)^4(-1\times-3-1\times1) = \ldots\ldots\ldots$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = (-1)^5(1\times-3-2\times1) = \ldots\ldots\ldots$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (-1)^6(1\times1-2\times1) = \ldots\ldots\ldots$$

$$\text{Co-factors matrix} = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$adjA = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

(ii) Given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{AX} = \text{B where } A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Since $|A| \neq 0$, the system of equations has a unique solution $X = A^{-1}B$

$$X = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 4\times 4 + 2\times 0 + 2\times 2 \\ -5\times 4 + 0\times 0 + 5\times 2 \\ 1\times 4 + -2\times 0 + 3\times 2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix} = 10\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \therefore \; x = \ldots\ldots, \quad y = \ldots\ldots., \quad z = \ldots\ldots$$

Chapter - 5

Continuity and Differentiability

Define the continuity of a function $f(x)$ at a point 'a' of its domain:

A function $f(x)$ is said to be continuos at the point $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

$$ie \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Worksheet-1

Activity - 1

(i) Consider $f(x) = \begin{cases} 3x - 5 & if \quad x \leq 5 \\ 2k + 3 & if \quad x > 5 \end{cases}$. For what value of k , $f(x)$ is continuous at $x = 5$

Ans:

$$\Rightarrow \lim_{x \to 5} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

$$\Rightarrow \lim_{x \to 5} .... = \lim_{x \to 5} 2k + 3 = 3(5) - 5$$

$$\Rightarrow \ .... \ = 2k + 3 = 3(5) - 5$$

$$\Rightarrow 2k + 3 = 10 \Rightarrow 2k = 7 \Rightarrow k = \frac{7}{2}$$

Activity - 2

Consider the function $f(x)=\begin{cases}5, & if\ x\le 5\\ ax+b, & if\ 2<x<10\\ 21, & if\ x\ge 10\end{cases}$

a) Find the relationship between a and b if $f(x)$ is continuous at $x=2$

b) Find the relationship between a and b if $f(x)$ is continuous at $x=10$

c) Find $f(5)$ if $f(x)$ is a continuous function.

Ans: a) $f(x)$ is continuous at $x=2$

$\Rightarrow \lim_{x\to 2} f(x) = f(2)$

$\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow \ldots\ldots. = \lim_{x\to 2}\left(ax+b\right) = \ldots\ldots..$

$\Rightarrow 2a+b=5 \quad \to (1)$

b) $f(x)$ is continuous at $x=2$

$\Rightarrow \lim_{x\to 10} f(x) = f(10)$

$\Rightarrow \lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = 21$

$\Rightarrow \lim_{x\to 10} \ldots\ldots\ldots = \lim_{x\to 10} 21 = \ldots\ldots..$

$\Rightarrow 10a+b=21 \quad \to (2)$

c) $f(x)$ is a continuous function

$\Rightarrow f(x)$ is continuous at $x=2$ and $x=10$

$$10a+b=21 \quad \to (1)$$
$$2a+b=5 \quad \to (2)$$
$$(1)-(2) \Rightarrow 8a = \ldots\ldots..$$
$$\Rightarrow a = \ldots\ldots.$$
$$\therefore Eqn(2) \Rightarrow b = 5-2a$$
$$= 5-2(\ldots\ldots) = \ldots..$$
$$\Rightarrow b=1$$

d)

$$f(x)=\begin{cases}5 & if\ x\le 2\\ 2x+1 & if\ 2<x<10\\ 21 & if\ x\ge 10\end{cases}$$

$\therefore f(5)=2(5)+1 = \ldots\ldots\ldots.$

Activity 3

Consider $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Discuss the continuity of $f(x)$

Ans: case (i) if $x < 0$

Then $f(x) = \frac{-x}{x} = -1,$ a constant function hence continuos

case (ii) if $x > 0$

Then $f(x) = \frac{x}{x} = 1,$ a constant function hence continuous

case (iii) if $x = 0$

Then $x = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0} .... = ...$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0} .... = ...$

$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

Hence $\lim_{x \to 0} f(x)$ does not exist

$\therefore f(x)$ is not continuous at $x = 0$

Activity - 4

Discuss the continuinty of the following functions

a) $f(x) = \sin|x|$

b) $g(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Ans: a) $f(x) = \sin|x|$

Let $h(x) = \sin x$ and $g(x) = |x|$ both are continuous functions

Now $hog(x) = h(g(x)) = h(...........) = ...$

Since composition of two continuous functions is continuous,

$h \circ g(x) = \sin|x|$ is continuous

b) $g(x) = \begin{cases} x\sin\dfrac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$

case (i) If $x \neq 0$, then $g(x) = x\sin\frac{1}{x}$ is continnous funtion

case (ii) if $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = \text{..............}$$

$$f(0) = \text{............}$$

$\therefore f(x)$ is continuous at $x = 0$

A function $f(x)$ is said to be differestiable at the point $x = a$ if

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f^1(a)$$ exists and finite.

Remarks

(i)	Differentiability implies continuity.
(ii)	Continuity need not imply Differentiability

Worksheet -2

Activity -1

Is $f(x)=|x|$ is differentiable at $x=0$? why?

Ans: $f(x)=|x|$ is not differentiable at $x=0$

Since $f'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{|0+h|-|0|}{h}$$

$$=\lim_{h\to 0}\frac{|h|}{h}$$ which does not exist.

$$\left[\begin{array}{lll}\text{Since} & \lim_{x\to o^-}\dfrac{|h|}{h}=\lim_{x\to o}-\dfrac{h}{h}=-1 \\ \text{and} & \lim_{x\to o+}\dfrac{|h|}{h}=\lim_{x\to o}-\dfrac{h}{h}=1\end{array}\right]$$

Remarks $f(x)=|x+a|$ is not differentiable where

$x+a=0$ or $x={}^{-}a$

Derivatives of composite functions :- (When? - How?)
When $y=f(g(x))$ then $\dfrac{dy}{dx}=f'[g(x)].g'(x)$

Activity - 2

Find $\dfrac{dy}{dx}$ for the following functions.

(i) $y=\sin\sqrt{x}$ (ii) $y=\tan(x^2.\cos x)$

(iii) $y=\tan(\sin^2 x)$ (iv) $y=\sec(\tan\sqrt{x})$

(v) $y=2\sqrt{\cot(x^2)}$

Ans: (i) $y=\sin\sqrt{x}$

$$\frac{dy}{dx} = [\cos..........]\left[\frac{1}{2\sqrt{x}}\right]$$

(ii) $y = \tan(x^2 \cos x)$

$$\frac{dy}{dx} = \left[\text{Sec}^2(x^2.\cos x)\right]\left[x^2(-.......) + (\cos x)(.........)\right]$$

(iii) $y = \tan(\sin^2 x)$

$$\frac{dy}{dx} = \left[\text{Sec}^2(..........)\right][2\sin x][..........]$$

(iv) $y = \sec(\tan\sqrt{x})$

$$\frac{dy}{dx} = \left[\text{Sec}(\tan\sqrt{x})\tan(\tan\sqrt{x}\right]\left[\sec^2\sqrt{x}\right][........]$$

(v) $y = 2\sqrt{\cot(x^2)}$

Derivatives of implicit functions (When? How?)

When we are given a relation connecting x and y where it is difficult or impossible to express y a function of x alone.

Then we differentiate term by term with reference to x and solve for $\frac{dy}{dx}$

Activity 3

Find $\frac{dy}{dx}$ for the following

(i) $x^2 + xy + y^2 = 100$

(ii) $\sin^2 x + \cos^2 y = 1$

(iii $x^3 + y^3 = 3a \times y$

Ans: (i) $x^2 + xy + y^2 = 100$

diffentiating w. r. t x

$$2x + x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y - \text{.........}$$

$$\Rightarrow (x + 2y)\frac{dy}{dx} = -(y + \text{..........})$$

\

$$\Rightarrow \frac{dy}{dx} = -\frac{(y + 2x)}{(\text{..........})}$$

(ii) $\text{Sin}^2 x + \cos^2 y = 1$

diffentiating w.r.t x

$$2\,\text{Sin}\,x\text{.......} - 2\cos y \sin y\frac{dy}{dx} = 0$$

$$2\cos y \sin y\frac{dy}{dx} = \text{...............}$$

$$\frac{dy}{dx} = \text{..............}$$

(iii) $x^3 + y^3 = 3axy$

diffentiating wrt x

$$\text{........} + 3y^2.\frac{dy}{dx} = 3a(x\frac{dy}{dx} + \text{........})$$

$$\Rightarrow x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = \text{...........}$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{\text{...........}}$$

(ii) $\sin^2 x + \cos^2 y = 1$

diffentiating w.r.t x

$$2\sin x.......-2\cos y \sin y \frac{dy}{dx} = 0$$

$$2\cos y \sin y \frac{dy}{dx} = ...$$

$$\frac{dy}{dx} = ...$$

(iii) $x^3 + y^3 = 3axy$

diffentiating wrt x

$$........+3y^2.\frac{dy}{dx} = 3a(x\frac{dy}{dx} + ...)$$

$$\Rightarrow x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$\Rightarrow (y^2 - ax)\frac{dy}{dx} = ...$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{..........}$$

Derivatives of Inverse Trgonometric functions

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ (vi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Activity 4

Find $\frac{dy}{dx}$ in the following

(*i*) $y=\tan^{-1}(2x^2)$

(*ii*) $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

(*iii*) $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(*iv*) $y=\sin^{-1}2x\sqrt{1-x^2}$

(*v*) $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Ans:

(i) $y=\tan^{-1}(2x^2)$

$$\frac{dy}{dx}=\left(\frac{1}{1+\left(2x^2\right)^2}\right)[2(2x)]=\frac{4x}{1+\ldots\ldots}$$

(ii) $y=\cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\frac{\pi}{2}-\sin^{-1}\left(\frac{2x}{1+x^2}\right)\quad\left[\text{since }\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$$

Put $x=\tan\theta$

$\Rightarrow\theta=\tan^{-1}x$

$$\Rightarrow y=\frac{\pi}{2}-\sin^{-1}\left[\frac{2\times\ldots\ldots\ldots}{1+\tan^2\theta}\right]$$

$$=\frac{\pi}{2}-\sin^{-1}(\sin 2\theta)$$

$$=\frac{\pi}{2}-2\theta$$

$$=\frac{\pi}{2}-2\ \ldots\ldots\ldots\ldots$$

$$\frac{dy}{dx}=0-2\left(\frac{1}{1+x^2}\right)=\frac{-2}{1+x^2}$$

(iii) $$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$y = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1} x$

$$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-\text{.........}}{1+\tan^2\theta}\right)$$

$$= \frac{\pi}{2} - \cos^{-1}(\text{..........})$$

$$= \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2\text{..........}$$

$$\frac{dy}{dx} = 0 - 2\left(\frac{1}{1+x^2}\right) = \frac{-2}{1+x^2}$$

(iv) $$y = \sin^{-1} 2x\sqrt{1-x^2}$$

Put $x = \sin\theta$

$\Rightarrow \theta = \sin^{-1} x$

$$\Rightarrow y = \sin^{-1}(2\text{..........}\sqrt{1-\sin^2\theta})$$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\text{..........})$$

$$= 2\theta$$

$$= 2\text{.............}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

(v) $y = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$

Put $x = \cos\theta$

$\Rightarrow \theta = \cos^{-1} x$

$$\Rightarrow y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right)$$

$$= \sec^{-1}\left(\frac{1}{\ldots\ldots\ldots\ldots\ldots}\right)$$

$$= \sec^{-1}(\sec \ldots)$$

$$= 2\theta$$

$$= 2\ldots\ldots\ldots\ldots\ldots$$

$$\therefore \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

Rolle's Theorem :

A real function f is such that

(i) f is continuous in [a,b]

(ii) f is differentible in (a, b)

(iii) f (a)= f (b).

Then there exits some $c \in (a,b)$ such that $f^1(c) = 0$

Mean Value Theorem:

Let f be a real function such that

(i) f is continuous in [a,b]

(ii) f is differentaible in (a,b).

Then there exists some $c \in (a,b)$ such that $f^1(c) = \dfrac{f(b)-f(a)}{b-a}$.

Activity 5

(i) Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$ for $x \in [-4, 2]$

(ii) Verify Mean Value Theorem for $f(x) = x^2 - 4x - 3$ in $[1,4]$

(i) $f(x) = x^2 + 2x - 8$

(i) $f(x)$ is .... is [-4,2] being a polynomial

(ii) $f^1(x) = 2x + 2$

(iii) $f(-4) = 16 - 8 - 8 = ...$

$f(2) = 4 + 4 - 8 = ...$

$\therefore f(x)$ statisfies conditions of Rolle's Theorm.

Hence there exise some $c \in (-4, 2)$ such that.

$f^1(c) = 0$

$\Rightarrow 2c + 2 = 0$

$\Rightarrow 2c = ...$

$\Rightarrow c = ...$

Clearly $-4 < -1 < 2$. Hence Rolle's Theroem is verified.

(ii) $f(x) = x^2 - 4x - 3$ in $[1, 4]$

(i) $f(x)$ is continuous in $[1, 4]$, being a polynomial

(ii) $f^1(x) = 2x - 4$ exists on (1,4). Hence $f(x)$ is differntiable in (1,4)

$\therefore f(x)$ satisfies conditions of Mean value theorem. hence there exist some

$c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2c - 4 = \frac{f(4) - f(1)}{4 - 1}$$

$$2c - 4 = \frac{.......... - ....}{3}$$

$$= \frac{-3+6}{3}$$

$$= ....$$

$$2C = 1 + 4$$

$$C = ...$$

Clearly $1 < \frac{5}{2} < 4$. Hence Mean Value Thereom is verifed.

Chapter -6

Applications of Derivaties

Rate of change of Quantities

Consider the function $y = f(x)$. Then $\frac{dy}{dx} = f^1(x)$ is called rate of change of y w.r.t. x Further if both x and y vary w.r.t a third varibale t then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \Rightarrow \frac{dy}{dt} = \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \frac{dy}{dt} = f'(x).\frac{dx}{dt} ----(1)$$

From (1) If the rate of change of one variable is known, the rate of change of other cand be calculated.

Activity - 1

A balloon, which always remains spherical on inflation, is being inflated by puming in 900 cubic centimeteres of gas per second. Then

(i) what is the rate at which the radius of the balloon increases when the radius is 15cm?

(ii) what is the rate at which the surface area of the balloon increase when the radius is 15cm?

Ans: Let V, r, S be volume, radius, surface area of the baloon at any time t.

Given that $\frac{dV}{dt} = 900cc/\sec$.

$$Now \qquad V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi ....\frac{dr}{dt}$$

$$\Rightarrow 900 = ...\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi(15)^2} = \frac{900}{900\pi} = ...\text{cm/sec}$$

Activity -2

A ladder 5m long is leaning against a wall. The bottom of the ladder pulled along the ground, away from the wall at the rate of 2cm/sec. How fast is its height on the wall decreasing when the foot of the ladder 4m away from the wall.

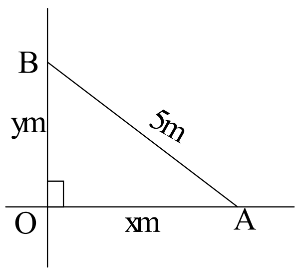
Ans:

Let AB = 5m, length of the ladder

OA= x, distance of the foot of the ladder from the wall

OB= y, height of the ladder on the wall.

Given $\frac{dy}{dx} = 2cm/\sec = \frac{2}{100} m/\sec.$



From ΔOAB,

$x^2 + y^2 = 25$

differentiating wrt. t,

$$.......\frac{dx}{dt} + ...\frac{dy}{dt} = 0$$

$$\Rightarrow 2x\left(\frac{2}{100}\right) + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 2y\frac{dy}{dt} = \frac{-4x}{100}$$

$$\frac{dy}{dt} = \frac{-4x}{2y(100)} = ...$$

When $x = 4\text{m}, y = \sqrt{25-42} = ...\text{m}$

$$\therefore \left(\frac{dy}{dt}\right) = \frac{-2(4)}{100(3)} = ...\text{cm/sec} \qquad (\because x = 4\text{m},\ y = 3\text{m})$$

$\therefore$ Height on the wall is decreasing at the rate of $\frac{8}{3}$ cm/sec.

Activity - 3

A particle is moving along the curve $6y = x^2 + 2$, find the points on the curve at which y-coordinate is changing 8 times as fast as the x-coordinate.

Ans: Given that $6y = x^3 + 2$

differentiating wrt t

$$6\frac{dy}{dt} = \text{.........}\frac{dx}{dt} \quad \left[\text{given } \frac{dy}{dt} = 8\frac{dx}{dt}\right]$$

$$\Rightarrow 6\left(8\frac{dx}{dt}\right) = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \text{..........} = 3x^2$$

$$\Rightarrow x^2 = \text{...........} \Rightarrow x = \pm 4$$

when $x = 4,$ *then* $y = \frac{x^3+2}{6} = \frac{4^3+2}{6} = \text{..............}$

when $x = {}^-4,$ *then* $y = \frac{(-4)^3+2}{6} = \text{.............}$

$\therefore$ Required points are $(......,11)$ and $(-4,.......)$

========

Activity - 4

The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. What is the marginal revenue when x=15

Ans $R(x) = 3x^2 + 36x + 5$

Differentiating wrt x, we have

$$\frac{dR}{dx} = \text{............} + 36$$

Marginal revenue $= \left[\frac{dR}{dx}\right]_{x=15} = 6(15) + 36 = \text{.........}$

Whorksheet -2

Increasing and Decreasing Functions.

A function is increasing on an open interval iff its derivative is positive in the interval.

A function is decreasing on an open interval iff its derivative is negative in the interval.

Activity 1.

Consider the function $f(x)=x^3-3x^2+4x+5$. Is the function f is strictly increasing on R?

Ans:

$$f(x)=x^3-3x^2+4x+5$$
$$f^1(x)=\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$=3x^2-6x+3+1$$
$$=3(\ldots\ldots\ldots\ldots\ldots)+1$$
$$=3(x-1)^2+1>0$$ always being a perfect squre and positive number

$\therefore f(x)$ is strictly increasing on R

Activity 2

Consider $f(x)=\log\sin x, 0<x<\pi$. .Find the intervals in which $f(x)$ is strictly increasing or decreasing

Ans: $f(x)=\log\sin x$

$$f'(x)=\left[\frac{1}{\sin x}\right][\ldots\ldots\ldots\ldots]=\cot x$$

Now $f'(x)=\cot x>0$ in $0<x<\frac{\pi}{2}$, I[st] Quadrant

$\therefore f(x)=\log\sin x$ is strictly increasing in $\left(0,\frac{\pi}{2}\right)$

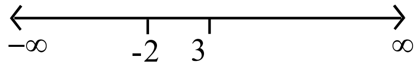
Also $f'(x)=\cot x<0$ in $\frac{\pi}{2}<x<\pi$, II[nd] Quadrant

$\therefore f(x)=\log\sin x$ is strictly decreasing in $\left(\pi/2,\pi\right)$

Activity 3

Find the intervals in which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is strictly incrasing or strictly decreasing

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$
$$f^1(x) = 6x^2 - ....$$
$$= 6(..............)$$
$$= 6(x+2)(x-3)$$
$$f'(x) = 0 \quad \Rightarrow \quad x = -2 \text{ or } x - 3$$



Intervals	Sign of $f'(x) = (6)(x+2)(x-3)$	Nature of $f(x)$
$(-\infty, -2)$		$f(x)$ is increasing
$(-2, 3)$		$f(x)$ is ....
$(3, \infty)$		$f(x)$ is ....

$\therefore f(x)$ is strictly increasing on $(-\infty, -2) \cup (3, \infty)$ and $f(x)$ is strictly decreasing on ....

Worksheet -3

Tangents and Normals:

1. Slope of the tangent to the curve $y = f(x)$ at (x_0, y_0)

$$= \left[\frac{dy}{dx}\right]_{(x_0,y_0)} = f'(x_0)$$

2. Slope of the normal to the curve $y = f(x)$ at (x_0, y_0)

$$= \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_0,y_0)}} = \frac{-1}{f'(x_0)}$$

3. Equations of the tangent to the curve $y = f(x)$ at (x_0, y_0) is

$$y - y_o = f'(x_0)(x - x_0)$$

4. Equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is

$$y - y_o = \frac{-1}{f'(x_0)}(x - x_0)$$

Activity 1.

Consider the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

(i) Find the equation of the tangent to the curve at (1,3)

(ii) Find the equation of the normal to the curve at (1,3)

Ans:

$$y = f(x) = x^4 - 6x^3 + 13x^2 - 10x + 5$$

$$\frac{dy}{dx} = f'(x) = 4x^3 - \text{.........................}$$

$$\left[\frac{dy}{dx}\right]_{(1,3)} = f'(1) = 4 - 18 + 26 - 10 = \text{......}$$

(i) Equation of tangent $y - y_0 = f'(x_0)(x - x_0)$

$$y - 3 = \text{..........}(x - 1)$$

(ii) Equation of normal $y - y_0 = \left[\frac{-1}{f'(x_0)}\right](x - x_0)$

$$y - .... = \frac{-1}{2}(x - ....)$$
$$\Rightarrow x + 2y - 7 = 0$$

Activity 2.

Consider the curve $y = x^3$. Find the points on the curve at which the slope of the tangent is equal to y-coordinate of the point

Ans:

$$y = x^3$$
$$\frac{dy}{dx} = 3.......$$

Slope of the tangent = ... coordinate

$$\Rightarrow \frac{dy}{dx} = y$$
$$\Rightarrow x^3 = 3x^2$$
$$\Rightarrow x^3 - .... = 0$$
$$\Rightarrow x^2(x-3) = 0 \Rightarrow x = ... \quad \text{or} \quad x = ...$$

when $x = 0, \quad y = 0^3 = ...$

when $x = 3, \quad y = 3^3 = ...$

$\therefore$ Required points are (0,...) and (....., 27)

Activity - 3

Find points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which tangents are

(i) parallel to x-axis

(ii) parallel to y-axis

Ans:

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

Differentiating w. r. t. x

$$\frac{1}{9}.......+\frac{1}{16}2y\frac{dy}{dx}=0$$

$$\frac{y}{8}\frac{dy}{dx}=\frac{-2.....}{9}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-16x}{9.....}$$

(i) Tangent is parallel to x-axis

$\Rightarrow$ Slope of the tangent = Slope of x-axis

$\Rightarrow$ $\frac{dy}{dx}=.......$

$\Rightarrow$ $\frac{-16x}{9y}=0\Rightarrow x=.......$

when $x=0,\ \frac{0^2}{9}+\frac{y^2}{16}=1\Rightarrow y^2=......\Rightarrow y=\pm.....$

Therefore tangents at (0,4) and (0,-4) are parallel to x-axis

(ii) Tangent is parallel to y-axis

$\Rightarrow$ Slope of the tangent = Slope of y-axis

$\Rightarrow$ $\frac{dy}{dx}=\frac{-16x}{9y}$ is not defined

$\Rightarrow$ $y=0$

when $y = 0, \frac{x^2}{9} + \frac{0^2}{16} = 1 \Rightarrow x^2 = ... \Rightarrow x = \pm ......$

Therefore tangents at (3,0) and (-3, 0) are parallel to y-axis

Activity -4.

What is the slope of the normal to the curve $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4.}$?

Ans:

$$x = a\cos^3\theta \quad \Rightarrow \frac{dy}{d\theta} = a\left[3\cos^2\theta\right][- ....]$$

$$= -3a\sin\theta\cos^2\theta.$$

$$y = a\sin^3\theta \quad \Rightarrow \frac{dy}{d\theta} = a\left[3\sin^2\theta\right][........]$$

$$= 3a\sin^2\theta\cos\theta.$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{..................}{..................} = -\tan\theta$$

Slope of the normal at $\theta = \frac{\pi}{4}$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{4}}}$$

$$= \frac{-1}{\tan\frac{\pi}{4}} = ...$$

==================

Chapter 7

INTEGRALS

WORKSHEET 1

At a Glance

- If $\frac{dy}{dx} = f(x)$, then $y = \int f(x)dx$
- **Some standard integrals**

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$ In particular $\int dx = x + C$
2. $\int \frac{1}{x} dx = \log|x| + C$
3. $\int \sin x \, dx = -\cos x + C$
4. $\int \cos x \, dx = \sin x + C$
5. $\int \sec^2 x dx = \tan x + C$
6. $\int \operatorname{cosec}^2 x dx = -\cot x + C$
7. $\int \sec x \tan x dx = \sec x + C$
8. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$
9. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
10. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
11. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$
12. $\int e^x dx = e^x + C$
13. $\int a^x dx = \frac{a^x}{\log a} + C$

ACTIVITY 1

Evaluate the following integrals

a) $\int(\cos x-\sin x)dx$

b) $\int\sec x(\sec x+\tan x)dx$

c) $\int\tan^2 xdx$

d) $\int\left(2x^2+e^x\right)dx$

a) $\int(\cos x-\sin x)dx$

$\int\cos x-\sin xdx=\int\cos xdx-\int\sin xdx$

$=\dots\dots\dots\dots\dots -\dots\dots\dots\dots+C$

$\left[\begin{array}{l}\int\cos xdx=\dots\dots\\ \int\sin xdx=\dots\dots\end{array}\right]$

b) $\int\sec x(\sec x+\tan x)dx$

$\int\sec x(\sec x+\tan x)dx=\int\left(\sec^2 x+\sec x\tan x\right)dx$

$=\int\dots\dots dx+\int\dots\dots dx$

$=\dots\dots\dots+\dots\dots\dots+C$

$\left[\begin{array}{l}\int\sec^2 xdx=\dots\dots\dots\dots\\ \int\sec x\tan xdx=\dots\dots\dots\end{array}\right]$

c) $\int\tan^2 xdx$

$\int\tan^2 xdx=\int(\dots\dots-1)dx$

$=\dots\dots\dots-x+C$

$\left\langle\tan^2 x=\sec^2 x-1\right\rangle$

d) $\int\left(2x^2+e^x\right)dx$

$\int\left(2x^2+e^x\right)dx=2\int\dots dx+\int\dots dx$

$=\dots\dots\dots+\dots\dots\dots+C$

$\left[\begin{array}{l}\int x^2dx=\frac{x^3}{3}+C\\ \int e^xdx=e^x+C\end{array}\right]$

ACTIVITY 2

Evaluate the following integrals

e) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$

f) $\int\frac{x^3+5x^2-4}{x^2}dx$

g) $\int\frac{2-3\sin x}{\cos^2 x}dx$

a) $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx=\int\left(x-2+\frac{1}{x}\right)dx$$

$$=\text{........} - \text{........} + \text{.......} + C$$

$\left[\begin{array}{l}\int 2dx = 2x + C \\ \int \frac{1}{x}dx = \text{.....}\end{array}\right]$

b) $\int\frac{x^3+5x^2-4}{x^2}dx$

$$\int\frac{x^3+5x^2-4}{x^2}dx=\int x+5-\frac{4}{x^2}dx$$

$$=\int\text{.....}dx+\int\text{......}dx-\int\text{.....}dx$$

$$=\text{.......}+\text{.......}-\text{........}+C$$

$\left[\int\frac{1}{x^2}dx=-\frac{1}{x}+C\right]$

c) $\int\frac{2-3\sin x}{\cos^2 x}dx$

$$\int\frac{2-3\sin x}{\cos^2 x}dx=2\int\frac{1}{\cos^2 x}dx-3\int\frac{\sin x}{\text{......}}dx$$

$$=2\int\text{..........}dx-3\int\text{.......}dx$$

$$=\text{..................}-3\text{............}+C$$

$\left[\begin{array}{l}\frac{1}{\cos^2 x}=\sec^2 x \\ \frac{\sin x}{\cos^2 x}=\sec x\tan x\end{array}\right]$

WORKSHEET 2

Focus Area 7.3.1 Integration by substitution

At a Glance

Some standard integrals

1. $\int \tan x \, dx = \log|\sec x| + C$

2. $\int \cot x \, dx = \log|\sin x| + C$

3. $\int \sec x dx = \log|\sec x + \tan x| + C$

4. $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

Some useful Trigonometric Identities

- $\sin^2 ax = \frac{1-\cos 2ax}{2}$
- $\cos^2 ax = \frac{1+\cos 2ax}{2}$
- $\sin mx.\cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$
- $\cos mx.\cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x]$
- $\sin mx.\sin nx = \frac{1}{2}[\cos(m+n)x - \cos(m-n)x]$

ACTIVITY 1

Integrate the following functions w.r.t. x:

a) $2x\cos\left(x^2+1\right)$

b) $\dfrac{(1+\log x)^3}{x}$

c) $\dfrac{e^{\tan^{-1}x}}{1+x^2}$

d) $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

e) $\dfrac{\sin x}{\sin(x+a)}$

a) $\int 2x\cos\left(x^2+1\right)dx$

$x^2+1=u \Rightarrow 2xdx=du$ $\left[\int \cos x dx = ...\right]$

$\int 2x\cos\left(x^2+1\right)dx = \int \cos u.du.$

$= ....+C$

$= \sin\left(x^2+1\right)+C$

b) $\int \dfrac{(1+\log x)^3}{x}dx$

$1+\log x = u \Rightarrow \dfrac{1}{x}dx = du$

$\int \dfrac{(1+\log x)^3}{x}dx = \int u^3 du$

$= ....+C$ $\left[\int x^n dx = ...\right]$

$= ....+C$

c) $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

$\tan^{-1}x = u \Rightarrow \frac{1}{1+x^2}dx = du$ $\left[\int e^x dx = ...\right]$

$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^u du.$

$=.............+C$

$=..............+C$

d) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

$\sqrt{x} = u \Rightarrow \frac{1}{2\sqrt{x}}dx = du \Rightarrow \frac{1}{\sqrt{x}}dx = ...du$

$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx = 2\int \cos u\, du.$

$=.............+C$

$=..............+C$

e) $\int \frac{\sin x}{\sin(x+a)}dx$

$(x+a) = u \Rightarrow dx = du$

$x = u - a$

$\int \frac{\sin x}{\sin(x+a)}dx = \int \frac{\sin(u-a)}{\sin u}dx$

$= \int \frac{....................}{\sin u}dx$

$= \int (\cos a - \sin a \cot u)\, du$

$= u\cos a - \sin a \int \cot u\, du$

$\left[\begin{array}{l} \sin(x-y) = \sin x\cos y - \cos x\sin y \\ \int \cot x dx = \log|\sin x| + C \end{array}\right]$

$= .... + C$

ACTIVITY 2

Integrate the following functions w.r.t. x:

a) $\sin^2(2x+1)$

b) $\cos 4x.\cos 6x$

c) $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}$

a) $\sin^2(2x+1) = \dfrac{1-\cos(4x+2)}{2}$

$$\int \sin^2(2x+1)\,dx = \int \frac{\text{..........}}{2}\,dx$$

$$= \frac{1}{2}[\text{.........} - \text{...............}] + C$$

$$\left[\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C\right]$$

b) $\cos 6x \cos 4x = \dfrac{1}{2}(\cos 10x + \cos 2x)$

$$\int \cos 6x \cos 4x\,dx = \frac{1}{2}\int(\cos 10x + \cos 2x)\,dx$$

$$= \frac{1}{2}[\text{.........} - \text{...............}] + C$$

c) $\displaystyle\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx = 2\int \frac{(\text{............}) - (\text{...........})}{\cos x - \cos\alpha}\,dx$$

$$\left[\cos 2x = 2\cos^2 x - 1\right]$$

$$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}\,dx$$

$$= 2\int \frac{(\text{........} - \text{......})(\text{.........} - \text{.........})}{\cos x - \cos\alpha}\,dx$$

$$= 2\int (\cos x + \cos\alpha)\,dx$$

$$= \text{.........................} + C$$

a)

WORKSHEET 3

Focus Area 7.4 Integrals of some particular functions

At a Glance

❖ Integrals of Some Particular Functions

1. $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

2. $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

3. $\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

4. $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

5. $\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

6. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{1}\left(\frac{x}{a}\right) + C$

ACTIVITY 1

Integrate the following functions w.r.t. x:

a) $\frac{1}{\sqrt{9-25x^2}}$

b) $\frac{x^2}{x^6-a^6}$

c) $\frac{\sec^2 x}{\sqrt{9+\tan^2 x}}$

a) $\int \frac{1}{\sqrt{9-25x^2}}dx$

$$\int \frac{1}{\sqrt{9-25x^2}}dx = \int \frac{1}{\sqrt{3^2-(5x)^2}}dx = \frac{1}{5}............+C \quad \left[\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C\right]$$

b) $\int \frac{x^2}{x^6-a^6}dx$

Put $x^3 = u \Rightarrow 3x^2dx = du \Rightarrow x^2dx = ...du$

$$\int \frac{x^2}{x^6-a^6}dx = \int \frac{x^2}{\left(x^3\right)^2-\left(a^3\right)^2}dx$$

$$= \frac{1}{3}\int \frac{...}{.......-\left(a^3\right)^2}du \quad \left[\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C\right]$$

$$= \frac{1}{3}\cdot\frac{1}{2a^3}\log\left|\frac{.........}{.........}\right|+C$$

$$= \frac{1}{6a^3}\log\left|\frac{.........}{.........}\right|+C$$

c) $\int \frac{\sec^2 x}{\sqrt{9+\tan^2 x}}dx$

Put $\tan x = u \Rightarrow \sec^2 x dx = du$

$$\int \frac{\sec^2 x}{\sqrt{9+\tan^2 x}}dx = \int \frac{1}{\sqrt{3^2+u^2}}du$$

$$= \log\left|.....+\sqrt{..........}\right|+c \quad \left[\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|+C\right]$$

$$= \log\left|\tan x+\sqrt{..........}\right|+c$$

ACTIVITY 2

Integrate the following functions w.r.t. x:

a) $\dfrac{1}{9x^2+6x+5}$

b) $\dfrac{1}{\sqrt{x^2+4x+13}}$

c) $\dfrac{1}{\sqrt{7-6x-x^2}}$

a) $\dfrac{1}{9x^2+6x+5}$

$$\int \frac{1}{9x^2+6x+5}dx = \frac{1}{9}\int \frac{1}{\ldots\ldots\ldots\ldots\ldots\ldots}dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\ldots\ldots}dx$$

$$= \frac{1}{9}\times\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{\ldots\ldots}{\ldots\ldots}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{\ldots\ldots}\right)+C$$

$$9x^2+6x+5 = 9\left(x^2+\frac{2}{3}x+\frac{5}{9}\right)$$
$$= 9\left(\left(x+\frac{1}{3}\right)^2+\frac{5}{9}-\frac{1}{9}\right)$$
$$= 9\left(\left(x+\frac{1}{3}\right)^2+\frac{4}{9}\right)$$
$$= 9\left(\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right)$$

b) $\dfrac{1}{\sqrt{x^2+4x+13}}$

$$\int \frac{1}{\sqrt{x^2+4x+13}}dx = \int \frac{1}{\sqrt{(x+\ldots)^2+13-\ldots}}dx$$

$$= \int \frac{1}{\sqrt{\ldots\ldots\ldots+3^2}}dx$$

$$= \log\left|\ldots\ldots\ldots\ldots+\sqrt{x^2+4x+13}\right|+C$$

c) $\frac{1}{\sqrt{7-6x-x^2}}$

$$\int \frac{1}{\sqrt{7-6x-x^2}}dx = \int \frac{1}{\sqrt{-\left(x^2+6x-7\right)}}dx$$

$$= \int \frac{1}{\sqrt{-\left\{\left((x+....\right)^2-7-....\right\}}}dx$$

$$= \int \frac{1}{\sqrt{-\left\{\left((x+....\right)^2-....\right\}}}dx$$

$$= \int \frac{1}{\sqrt{4^2-\left(x+....\right)^2}}dx$$

$$= \sin^{-1}\frac{.....}{.....}+C$$

$$\left[\begin{aligned} x^2+6x-7 &= (x+3)^2-7-9 \\ &= (x+3)^2-4^2 \\ \int \frac{1}{\sqrt{a^2-x^2}}\,dx &= \sin^{-1}\left(\frac{x}{a}\right)+C \end{aligned}\right]$$

ACTIVITY 3

Evaluate $\int \frac{x+3}{x^2-2x-5}dx$

$x+3 = A\frac{d}{dx}\left(x^2-2x-5\right)+B$

$x+3 = A\left(........\right)+B$

Equating coefficients of x

$2A = 1 \Rightarrow A = ...$

Equating constants

-2A+B=.......

.....+B=...... $\Rightarrow$ B=.....

$\therefore$ x+3=.....$(2x-2)$+.....

$$\int \frac{x+3}{x^2-2x-5}dx = \int \frac{....(2x-2)+....}{x^2-2x-5}dx$$

$$= ....\int \frac{(2x-2)}{x^2-2x-5}dx + ...\int \frac{1}{x^2-2x-5}dx$$

$$= ....\log\left|x^2-2x-5\right| + ...\int \frac{1}{x^2-2x-5}dx$$

$$= ....\log\left|x^2-2x-5\right| + ...\int \frac{1}{(x-1)^2 - ...}dx$$

$$= ....\log\left|x^2-2x-5\right| + ...\times\frac{1}{2\times...}\log\left|\frac{x-1-....}{x-1+...}\right| + c$$

$$= ....\log\left|x^2-2x-5\right| + ...\log\left|\frac{x-1-....}{x-1+...}\right| + c$$

Remember

$$\left[\int \frac{f'(x)}{f(x)}dx = \log\left|f(x)\right| + C\right]$$

ACTIVITY 3

Evaluate $\int \frac{x+2}{\sqrt{2x^2+6x+5}}\,dx$

$$x+2 = A\frac{d}{dx}\left(2x^2+6x+5\right)+B$$

$$x+2 = A(..........)+B$$

Equating the coefficients of x and constant terms from both sides, we get

$$1 = ...A \quad \text{and} \quad 2 = ...A + B$$

$$\therefore \quad A = ... \quad \text{and} \quad B = ...$$

$$\therefore \quad x+2 = ...(4x+6) + ...$$

$$\therefore \int \frac{x+2}{\sqrt{2x^2+6x+5}}\,dx$$

$$= \int \frac{\frac{1}{4}(4x+6)+\frac{1}{2}}{\sqrt{2x^2+6x+5}}$$

$$= \frac{1}{4}\int \frac{\cdots\cdots}{\sqrt{2x^2+6x+5}}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{2x^2+6x+5}}\,dx$$

$$= \frac{1}{4}\int \frac{4x+6}{\sqrt{2x^2+6x+5}}\,dx + \frac{1}{2\sqrt{2}}\int \frac{1}{\cdots\cdots\cdots}\,dx$$

$$= \frac{1}{4}\times 2\sqrt{2x^2+6x+5} + \frac{1}{2\sqrt{2}}\log\left|\cdots\cdots + \sqrt{x^2+3x+\frac{5}{2}}\right| + c$$

$$= \cdots\cdots\cdots\cdots + \cdots\cdots\cdots\cdots C$$

$$2x^2+6x+5$$

$$= 2\left(x^2+3x+\frac{5}{2}\right)$$

$$= 2\left(\left(x^2+3x+\frac{9}{4}\right)-\frac{9}{4}+\frac{5}{2}\right)$$

$$= 2\left(\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+c$$

Remember

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|+C$$

WORKSIIEET 4

7.9 Evaluation of Dcfinitc Intcgrals by Substitution

At a glance

- Steps for calculating $\int_a^b f(x)\,dx$
  - Find the indefinite integral $\int f(x)\,dx$. Let this be $F(x)$. There is no need to keep integration constant C
  - Evaluate $F(b)$ and $F(a)$
  - Calculatc $F(b)-F(a)$, which is thc valuc of $\int_a^b f(x)\,dx$
- Steps to evaluate $\int_a^b f(x)\,dx$, by substitution
  - Obtain the definite integral and express it in the form $\int_a^b f(g(x))g'(x)\,dx$
  - Put $g(x)=t$ which givcs $g'(x)=dt$
  - Put $x=a$ in $t=g(x)$ to get $t=g(a)$ as the lower limit for t
    Put $x=b$ in $t=g(x)$ to get $t=g(b)$ as the upper limit for t
  - Substitutc $g'(x)=dt$ and rcplacc old limits of intcgration by thc new limits to get $\int_{g(a)}^{g(b)} f(t)dt$. Evaluate this integral by using standard methods.

Activity 1 :

Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$

Put $x^5+1=t$, then $5x^4\,dx=dt$

when $x=-1, t=.....$,when $x=1, t=.........$

Therefore $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_{.....}^{.....}\sqrt{t}\,dt$

$= \frac{2}{3}[......]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = ...$

Remember

$\int\sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$

Activity 2 :

Evaluate $\int_{0}^{1} \frac{\tan^{-1}x}{1+x^2}\,dx$

Put $\tan^{-1}x=t$, then $\frac{1}{1+x^2}\,dx=dt$

when $x=0, t=.....$,when $x=1, t=.........$

Therefore $\int_{0}^{1} \frac{\tan^{-1}x}{1+x^2}\,dx = \int_{0}^{\frac{\pi}{4}}........\,dt$

$= [......]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = ...$

Remember

$\int x\,dx = \frac{x^2}{2} + c$

Activity 3 :

Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

Put $\cos x = t$, then $-\sin x\,dx = dt \quad \therefore \quad \sin x\,dx = -dt$

when $x = 0,\ t = ...$,when $x = \frac{\pi}{2},\ t = ...$

Therefore $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx = -\int_1^0 ....\,dt$

$= -[......]_1^0 = -\left[\tan^{-1} 0 - \tan^{-1} 1\right] = ...$

Remember
$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

Activity 4 :

Evaluate $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{\left(x^2+2x+1\right)+4} = \int_{-1}^{1} \frac{dx}{(x+1)^2+2^2}$$

Put $x+1 = t$, then $dx = dt$

when $x = -1,\ t = ...$,when $x = 1,\ t = ...$

Therefore $\int_{-1}^{1} \frac{dx}{(x+1)^2+2^2} = \int_{...}^{...} ....\,dt$

$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_{......}^{......} = \frac{1}{2}\left[\tan^{-1} 1 - \tan^{-1} ....\right] = ...$

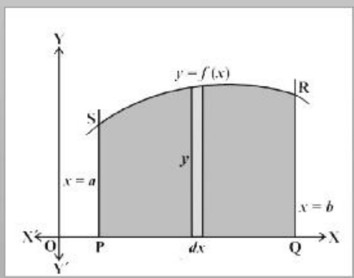
Remember
$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Chapter 8

APPLICATIONS OF INTEGRALS

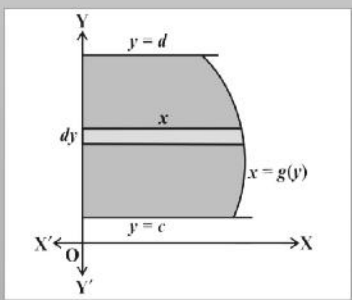
At a glance

8.2 Area under simple curves



The area A bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)dx = \int_a^b ydx$$



The area A of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$, $y = d$ is given by

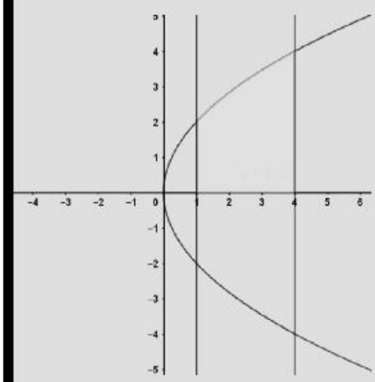
$$A = \int_c^d f(y)dy = \int_c^d xdy$$

$$\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

This integral is useful to find area of circle and ellipse using integration

ACTIVITY 1

Draw a rough sketch of the curve $y^2 = 4x$ and find the area of the region bounded by the curve $y^2 = 4x$ and the lines $x - 1$, $x - 4$ and the x-axis in the first quadrant.



$y^2 = 4x$

$\therefore y = ...$

Area bounded by the curve $y^2 = 4x$ and

the lines x=1 and x=4

In the shaded region x varies from 1 to 4

$\therefore$ Required Area= $\int_a^b y dx$ a=....... And b=......

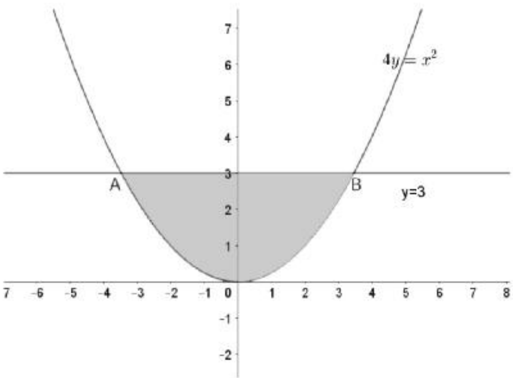
= $\int_{..}^{..} ....dx$

= $[...]_{..}^{..}$

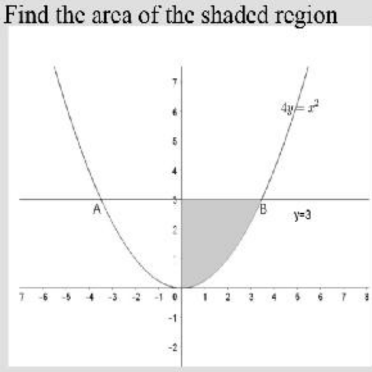
= ....sq.units

ACTIVITY 2

The figure represents the curve $4y = x^2$ and the line $y = 3$



Find the area of the shaded region



This figure shows area in the I st quadrant and it is the area of the region bounded by $4y=x^2$ and the line $y=3$ and y axis

$$4y=x^2 \Rightarrow x=.....$$

Here y varies from 0 to 3

The area in the activity is double of this area

Required Area $=2\int_a^b f(y)dy$

$$=2\int_0^3 x dy$$

$$=2\int_0^3 ....dy$$

$$=2[....]_0^3$$

$$=........sq.units$$

ACTIVITY 3

Find the area of the circle $x^2+y^2=a^2$ using integration

$x^2+y^2=a^2 \Rightarrow y=$

O(0,0) and A(a,0)

Here x varies from ...... to ......

$$Area\ of\ shaded\ region=\int_a^b y dx$$

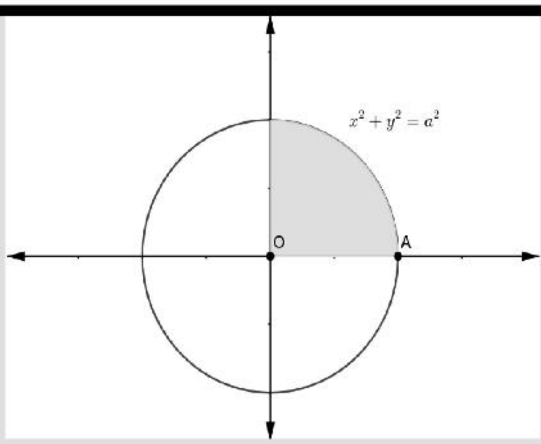
$$=\int_{...}^{...} ....dx$$

$$=[................]$$

$$=.......sq.units$$

Area of circle = 4 x Area of shaded region

=..........sq.units



ACTIVITY 4

Find the area of the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ using integration

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=$

O(0,0) and A(a,0)

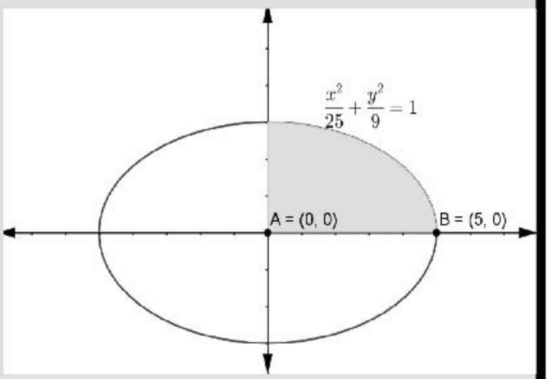
Here x varies from ...... to ......

$$Area\ of\ shaded\ region = \int_a^b y\,dx$$

$$= \int_{...}^{...} ....dx$$

$$= [................]$$

$$= ....sq.units$$



Area of Ellipse = 4 x Area of shaded region

= ....sq.units

CHAPTER 9

DIFFERENTIAL EQUATION

WORKSHEET

At a Glance

9.2 Basic Concepts

- **Order of a Differential Equation**

Order of a differential equation is order of the highest derivative occurring in the equation

- **Degree of a Differential Equation**

Degree of a differential equation is power of the highest derivative occurring in the Equation

- Degree of a polynomial is defined only if it is a polynomial in its derivatives.

- **9.5.1 Variable -Separable Differential Equation**

Differential Equation of the form $f(x)dx + g(y)dy = 0$

Its Solution is $\int f(x)dx + \int g(y)dy = C$

ACTIVITY 1

Find the Order and degree of the following Differential equations

$(a)\left(\frac{dy}{dx}\right)^2 + y = 0$

$(b)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 + xy = 0$

$(c)x^3\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 + \sin y = 0$

$(d)\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 + \sin\left(\frac{dy}{dx}\right) = 0$

$(e)\frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} = 0$

(a) Highest derivative is $\frac{dy}{dx}$

$\therefore$ Order of differential equation is.........

Power of Highest derivative $\frac{dy}{dx}$ is 2

$\therefore$ Degree of differential equation is.......

(b) Highest derivative is ....

$\therefore$ Order of differential equation is.........

Power of Highest derivative is ....

$\therefore$ Degree of differential equation is.......

(c) Highest derivative is ....

$\therefore$ Order of differential equation is.........

Power of Highest derivative is .....

$\therefore$ Degree of differential equation is.......

(d) Highest derivative is $\frac{d^3y}{dx^3}$

$\therefore$ Order of differential equation is.........

Equation is not a polynomial in $\frac{dy}{dx}$

$\therefore$ Degree of differential equation is.......

(e) Highest derivative is ....

$\therefore$ Order of differential equation is.........

Equation is not a polynomial in ....

ACTIVITY 2

Solve $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\frac{dy}{......} = \frac{......}{1+x^2}$$

integrating

$$\int \frac{dy}{......} = \int \frac{......}{1+x^2}$$

$$\left[\text{Remember } \int \frac{1}{1+x^2} dx = \tan^{-1} x + C\right]$$

Solution is

$$..........=..........+C$$

ACTIVITY 3

Solve $sec^2x.tany\ dx + sec^2y.tanx\ dy = 0$

$$sec^2x.tany\ dx + sec^2y.tanx\ dy = 0$$

Dividing through out by tanx.tany

$$\frac{sec^2x}{.......} dx + \frac{......}{\tan y} dy = 0$$

integrating

$$\int \frac{sec^2x}{.......} dx + \int \frac{......}{\tan y} dy = 0$$

$$...............+..................=C$$

$$\left[\begin{array}{l}\text{Remember} \\ \frac{d}{dx}(\tan x) = \sec^2 x \\ \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C\end{array}\right]$$

ACTIVITY 3

Solve $y\log y\, dx - xdy = 0$

$y\log y\, dx - xdy = 0$

Dividing through out by x.ylogy

$\frac{\cdots\cdots}{\cdots\cdots} dx + \frac{\cdots\cdots}{\cdots\cdots} dy = 0$

integrating

$\int \frac{\cdots\cdots}{\cdots\cdots} dx + \int \frac{\cdots\cdots}{\cdots\cdots} dy = 0$

$\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots = C$

Remember

$\frac{d}{dx}(\log x) = \frac{1}{x}$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$

ACTIVITY 4

Solve $\frac{dy}{dx} = y\tan x$; $y = 1$ when $x = 0$

$\frac{dy}{dx} = y\tan x$

$\frac{\cdots\cdots}{\cdots\cdots} dy = \ldots\ldots dx$

integrating

$\int \frac{\cdots\cdots}{\cdots\cdots} dy = \int \ldots\ldots dx$

$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots + C$

$y = 1$ when x=0

we get C=.....

Solution is ....

Remember

$\int \frac{1}{x} dx = \log|x| + C$

$\int \tan x dx = \log|\sec x| + C$

Chapter 10

VECTOR ALGEBRA

WORK SHEET 1 :

10.4 Addition of vectors

10.5 Multiplication of a vector by a scalar

At a glance

- The position vector of a point $P(x,y,z)$ is $\overrightarrow{OP}$ (or $\vec{a}$) $= x\hat{i} + y\hat{j} + z\hat{k}$
- The length or magnitude or modulus of $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$ is $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$
- If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are any two points, then the vector joining P and Q is the vector $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$
- Unit vector in the direction of $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$
- If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$, then its direction ratios are the components x, y, z of the vector and direction cosines are $\frac{x}{|\vec{a}|}, \frac{y}{|\vec{a}|}, \frac{z}{|\vec{a}|}$

  (Scalar components of the unit vector $\hat{a}$)

Activity 1 :

Consider the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

(a) Find the unit vector in the direction of $\vec{a}$

(b) Find a vector in the direction of $\vec{a}$ which has magnitude 8 units

(a) $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$, $|\vec{a}| =$ ....

Unit vector in the direction of $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|} =$ ....

(b) Vector in the direction of $\vec{a}$ which has magnitude 8 units

$= 8\hat{a} =$ ....

Activity 2 :

Let $\vec{a} = 2\hat{i} - j + 2k$ and $\vec{b} = -\hat{i} + j - k$

(a) Find $\vec{a} + \vec{b}$

(b) Find the unit vector in the direction $\vec{a} + \vec{b}$

(a) $\vec{a} = 2\hat{i} - j + 2k$

$\vec{b} = -\hat{i} + j - k$

$\vec{a} + \vec{b} =$ .... , $\left|\vec{a} + \vec{b}\right| =$ ....

(b) Unit vector in the direction of $\vec{a} + \vec{b}$ is $\dfrac{\vec{a} + \vec{b}}{\left|\vec{a} + \vec{b}\right|} =$ ....

Activity 3 :

Let $P(1,2,3)$ and $Q(4,5,6)$

(a) Find the vector $\overrightarrow{PQ}$

(b) Find the unit vector in the direction of $\overrightarrow{PQ}$

(c) Find the direction cosines of the vector $\overrightarrow{PQ}$

(a) $P(x_1, y_1, z_1) = (1,2,3)$ and $Q(x_2, y_2, z_2) = (4,5,6)$. Then

$\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)j + (z_2 - z_1)k$

= ....

(b) $\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + 3^2 + 3^2} =$ ....

Unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{\text{..............}}{\text{..............}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$$

(b) Direction cosines of the vector $\overrightarrow{PQ}$ are $\frac{1}{\sqrt{3}}$, ..., ...

Activity 4:

Show that the points $A\left(2\hat{i}-j+k\right)$, $B\left(\hat{i}-3j-5k\right)$, $C\left(3\hat{i}-4j-4k\right)$ are the vertices of a right angled triangle

(a) $A\left(2\hat{i}-j+k\right)$, $B\left(\hat{i}-3j-5k\right)$, $C\left(3\hat{i}-4j-4k\right)$

$\overrightarrow{AB}=(1-2)\hat{i}+(-3+1)j+(-5-1)k=-\hat{i}-2j-6k$

$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)j+(-4+5)k=$............

$\overrightarrow{CA}=$ .... = ...

$\left|\overrightarrow{AB}\right|=$ ....

$\left|\overrightarrow{BC}\right|=$ ....

$\left|\overrightarrow{CA}\right|=$ ....

$\left|\overrightarrow{AB}\right|^2=$............, $\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{CA}\right|^2=$.........

$\therefore \left|\overrightarrow{AB}\right|^2=\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{CA}\right|^2$

Hence, the triangle is a right angled triangle

WORK SHEET 2

10.6.1 Scalar (or dot) product of two vectors

At a glance

- The scalar product of two non zero vectors $\vec{a}$ and $\vec{b}$ is defined as

  $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

- $\vec{a}\cdot\vec{b}$ is a real number
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$
- $\vec{a}\cdot\vec{b} = 0 \quad \Leftrightarrow \quad \vec{a}\perp\vec{b}$
- $\hat{i}\cdot\hat{i} = j\cdot j = k\cdot k = 1$ and $\hat{i}\cdot j = j\cdot k = k\cdot\hat{i} = 0$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2 = (\vec{a})^2 = a^2$
- $\vec{a} = a_1\hat{i} + a_2 j + a_3 k$ and $\vec{b} = b_1\hat{i} + b_2 j + b_3 k$, then

  $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

- Projection of a vector $\vec{a}$ on $\vec{b}$, is given by $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Activity 1 :

Consider the vectors $\vec{a}=\hat{i}-2j+3k$ and $\vec{b}=3\hat{i}-2j+k$

(a) Find the angle between $\vec{a}$ and $\vec{b}$

(b) Find the projection of $\vec{a}$ on $\vec{b}$

(a) $\vec{a}=\hat{i}-2j+3k$, $\vec{b}=3\hat{i}-2j+k$

$\vec{a}\cdot\vec{b}=............$

$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=.......$, and $|\vec{b}|=.................=.......$

$\text{Cos}\,\theta=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\dfrac{.........}{........}=........$ $\quad\therefore\theta=\cos^{-1}(.........)$

(b) Projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}=\dfrac{.........}{........}=........$

Activity 2 :

If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$, then find $|\vec{x}|$

$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8$

$\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=......$

We have $\vec{x}\cdot\vec{a}=\vec{a}\cdot\vec{x}$ and $\vec{x}\cdot\vec{x}=|\vec{x}|^2$, $\vec{a}\cdot\vec{a}=|\vec{a}|^2$

$|\vec{x}|^2-|\vec{a}|^2=8$

Since $\vec{a}$ is a unit vector , $|\vec{a}|=1$ and $|\vec{a}|^2=1$

$\therefore\ |\vec{x}|^2-\ ......\ =8\ \therefore\ |\vec{x}|^2=9$

Hence $|\vec{x}|=3$

Activity 3 :

$\vec{a} = 5\hat{i} - j - 3k$ and $\vec{b} = \hat{i} + 3j + 5k$, then show that the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular

$\vec{a} = 5\hat{i} - j - 3k$

$\vec{b} = \hat{i} + 3j + 5k$

$\vec{a}+\vec{b} = 6\hat{i} + 2j + 2k$

$\vec{a}-\vec{b} = \ldots\ldots\ldots\ldots\ldots$

$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 6\times 4 + \ldots\ldots + \ldots\ldots = 0$

Since the dot product of $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is ……… ,
$\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are …………………

Activity 4 :

Consider the triangle ABC with vertices $A(1,2,3)$, $B(-1,0,4)$ and $C(0,1,2)$

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) Find $\angle A$

(a) $\overrightarrow{AB} = (-1-1)\hat{i} + (0-2)j + (4-3)k = \ldots\ldots\ldots\ldots$

$\overrightarrow{AC} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

(b) $\overrightarrow{AB}\cdot\overrightarrow{AC} = \ldots\ldots\ldots\ldots\ldots$

$|\overrightarrow{AB}| = \ldots\ldots\ldots\ldots$

$|\overrightarrow{AC}| = \ldots\ldots\ldots\ldots\ldots$

$$\text{Cos}\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{\ldots\ldots}{\ldots\ldots} = \ldots\ldots$$

$\therefore \angle A = \text{Cos}^{-1}(\ldots\ldots)$

WORK SHEET 3

10.6.3 Vector (or cross) product of two vectors

At a glance

- The vector product of two nonzero vectors $\vec{a}$ and $\vec{b}$ is defined as
  $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\, n$
- $\vec{a}\times\vec{b}$ is a vector
- $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$
- $\sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$
- $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}\parallel\vec{b}$
- $\hat{i}\times\hat{i}=j\times j=k\times k=0$ and $\hat{i}\times j=k,\quad j\times k=\hat{i},\quad k\times\hat{i}=j$

  $j\times\hat{i}=-k,\quad k\times j=-\hat{i},\quad \hat{i}\times k=-j$
- $\vec{a}\cdot\vec{a}=|\vec{a}|^2=(\vec{a})^2=a^2$
- $\vec{a}=a_1\hat{i}+a_2 j+a_3 k$ and $\vec{b}=b_1\hat{i}+b_2 j+b_3 k$, then

  $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Activity 1:

Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7j+7k$, $\vec{a}=3\hat{i}-2j+2k$

(a) $\vec{a}=\hat{i}-7j+7k$, $\vec{a}=3\hat{i}-2j+2k$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & j & k\\ 1 & -7 & 7\\ \ldots & \ldots & \ldots\end{vmatrix}=\ldots\ldots\ldots\ldots$$

$$\left|\vec{a}\times\vec{b}\right|=\ldots\ldots\ldots\ldots\ldots\ldots$$

Activity 2 :

Consider the vectors $\vec{a}=3\hat{i}+2j+2k$ and $\vec{b}=\hat{i}+2j-2k$

(a) Find $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$

(b) Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$

(a) $\vec{a}=3\hat{i}+2j+2k$

$\vec{b}=\hat{i}+2j-2k$

$\vec{a}+\vec{b}=4\hat{i}+4j+0k$

$\vec{a}-\vec{b}=\ldots\ldots\ldots\ldots\ldots$

(b) $$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & j & k\\ 4 & 4 & 0\\ \ldots & \ldots & \ldots\end{vmatrix}=\ldots\ldots\ldots\ldots$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\ldots\ldots\ldots\ldots\ldots\ldots$$

Unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is

$$\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}=\frac{\ldots\ldots\ldots}{\ldots\ldots\ldots}=\ldots\ldots\ldots$$

Activity 3 :

(a) $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$ is equal to

(i) $\vec{0}$ (ii) $|\vec{a}|^2-|\vec{b}|^2$ (iii) $\vec{a}\times\vec{b}$ (iv) $2\left(\vec{a}\times\vec{b}\right)$

(b) Find the area of the parallelogram whose adjacent sides are given by the vectors $\vec{a}=3\hat{i}+j+4k$ and $\vec{b}=\hat{i}-j+k$

(a) $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=\vec{a}\times\vec{a}+$ .... $-$ .... $-$ ....

$=0+\vec{a}\times\vec{b}+$ .... $+0$

$=2\left(\vec{a}\times\vec{b}\right)$

(b) $\vec{a}=3\hat{i}+j+4k$ and $\vec{b}=\hat{i}-j+k$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & j & k\\ 3 & 1 & 4\\ \ldots & \ldots & \ldots\end{vmatrix}=\ldots\ldots\ldots\ldots.$$

$\left|\vec{a}\times\vec{b}\right|=$.................

$\therefore$ Area of the parallelogram $=\left|\vec{a}\times\vec{b}\right|=$.......

Activity 4 :

Let $A(1,1,1)$, $B(1,2,3)$ and $C(2,3,1)$ are the vertices of a triangle

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) Find the area of the triangle ABC

(a) $\overrightarrow{AB}=(1-1)\hat{i}+(2-1)j+(3-1)k=$...........

$\overrightarrow{AC}=$... = ...

(b) $$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & j & k\\ \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots\end{vmatrix}=\ldots\ldots\ldots\ldots.$$

Therefore $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=$ ....

Thus required area $=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=$...............

Chapter 11

THREE DIMENSIONAL GEOMETRY

WORKSHEET 1

11.3 Equation of a line in space

At a glance

- Equation of a line through a given point and parallel to a given vector is
  - *Vector form*

    $\vec{r} = \vec{a} + \lambda\vec{b}$

    $\vec{a}$ is the position vector of the given point

    $\vec{b}$ is the parallel vector

  - *Cartesian form*

    $$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

    (x_1, y_1, z_1) is the given point

    a, b, c are the direction ratios of a line

- Equation of a line passing through two given points
  - *Vector form*

    $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$

    $\vec{a}$ and $\vec{b}$ are position vectors of first and second points

  - *Cartesian form*

    $$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

    (x_1, y_1, z_1) and (x_2, y_2, z_2) are first and second points

Activity 1:

Find the vector and cartesian equations of the line which passes through the point $(1,2,3)$ and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$

Given point , $(x_1, y_1, z_1) = (1,2,3)$, its position vector is $\vec{a} = \hat{i} + 2j + 3k$

Parallel vector is $\vec{b} = 3\hat{i}+2\hat{j}-2\hat{k}$, direction ratios of the line are $a = 3,\ b = 2,\ c = -2$

Therefore, the vector equation of the line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\vec{r} = \hat{i} + 2j + 3k + \lambda(\text{..................})$$

Also, the cartesian equation of the line is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\frac{x-1}{3} = \frac{y-\cdots\cdots}{\cdots\cdots} = \frac{z-\cdots\cdots}{\cdots\cdots\cdot}$$

Activity 2:

Find the vector and cartesian equations of the line which passes through the point with position vector $-2\hat{i}+4\hat{j}-5\hat{k}$ and is parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Position vector of the given point is $\vec{a} = -2\hat{i}+4\hat{j}-5\hat{k}$ $\therefore$ $(x_1, y_1, z_1) = (-2,4,5)$

Parallel line is $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$. So parallel vector is $\vec{b} = 3\vec{i}+5\vec{j}+6\vec{k}$ and direction ratios of the line are $a = 3,\ b = 5,\ c = 6$

Therefore, the vector equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$

i.e., $\vec{r} = -2\hat{i} + 4j - 5k + \lambda(\text{..................})$

Also, the cartesian equation of the line is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

i.e., $\frac{x+2}{3} = \frac{y-\cdots\cdots}{\cdots\cdots} = \frac{z-\cdots\cdots}{\cdots\cdots\cdot}$

Activity 3:

The cartesian equation of the line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$, write its vector form

From the given equation we observe that

$$(x_1,y_1,z_1)=(5,-4,6) \text{ and } a=3,\ b=7,\ c=2$$

Therefore $\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$ and $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

Thus the vector equation of the line is $\vec{r}=\vec{a}+\lambda\vec{b}$

$$\vec{r}=(\ldots\ldots\ldots\ldots\ldots\ldots\ldots)+\lambda(\ldots\ldots\ldots\ldots\ldots\ldots)$$

Activity 4:

Find the vector and cartesian equations of the line that passes through the points $(3,-2,-5)$, $(3,-2,6)$

Given points are $(x_1,y_1,z_1)=(3,-2,-5)$, $(x_2,y_2,z_2)=(3,-2,6)$,

Position vectors of the given points are $\vec{a}=3\hat{i}-2j-5k$, $\vec{b}=3\hat{i}-2j+6k$

Therefore, the vector equation of the line is

$$\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$$

$$\vec{r}=3i-2j+5k+\lambda(\ (\ldots\ldots\ldots\ldots\ldots\ldots)-(\ldots\ldots\ldots\ldots\ldots\ldots)\ )$$

$$\vec{r}=3\hat{i}-2j+5k+\lambda(\ldots\ldots\ldots\ldots\ldots\ldots)$$

Also, the cartesian equation of the line is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-3}{3-3}=\frac{y-\cdots\cdots}{\cdots-\cdots}=\frac{z-\cdots\cdots}{\cdots-\cdots}$$

$$\frac{x-3}{0}=\frac{y-\cdots\cdots}{\cdots\cdots}=\frac{z-\cdots\cdots}{\cdots\cdots}$$

WORKSHEET 2

11.5.1 Distance between two skew lines

At a glance

Shortest distance between two skew lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is

$$\left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

Activity 1:

Find the shortest distance between the lines $\vec{r} = \hat{i} + 2j + 3k + \lambda(\hat{i} - 3j + 2k)$ and

$$\vec{r} = 4\hat{i} + 5j + 6k + \mu(2\hat{i} + 3j + k)$$

Given lines are $\vec{r} = \hat{i} + 2j + 3k + \lambda(\hat{i} - 3j + 2k)$ and $\vec{r} = 4\hat{i} + 5j + 6k + \mu(2\hat{i} + 3j + k)$

$\vec{a}_1 = \hat{i} + 2j + 3k,\ \vec{b}_1 = \hat{i} - 3j + 2k$ and $\vec{a}_2 = 4\hat{i} + 5j + 6k,\ \vec{b}_2 = 2\hat{i} + 3j + k$

$\vec{a}_2 - \vec{a}_1 = 3\hat{i} + \dots\dots\dots\dots$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = \hat{i}(-3-6) - \cdots\cdots\cdots + \cdots\cdots = -9\hat{i} + \cdots\cdots\cdots$$

$(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2) = 3 \times -9 + \cdots\cdots\cdots\cdots = \cdots\cdots\cdots$

$|\vec{b}_1 \times \vec{b}_2| = \cdots\cdots\cdots$

Shortest distance between the lines is

$$\left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \frac{\cdots}{\cdots} = \cdots$$

Activity 2:

Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$$

Given lines are $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

$$\vec{a}_1=-\hat{i}-j-k,\ \vec{b}_1=7\hat{i}-6j+k \text{ and } \vec{a}_2=3\hat{i}+5j+7k,\ \vec{b}_2=\hat{i}-2j+k$$

$$\vec{a}_2-\vec{a}_1=4\hat{i}+\ldots\ldots\ldots\ldots$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\end{vmatrix}=\hat{i}(-6+2)-\cdots\cdots\cdots+\cdots\cdots=-4\hat{i}+\cdots\cdots\cdots$$

$$(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=4\times-4+\cdots\cdots\cdots\cdots=\cdots\cdots\cdots$$

$$|\vec{b}_1\times\vec{b}_2|=\cdots\cdots\cdots$$

Shortest distance between the lines is

$$\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|=\frac{\cdots}{\cdots}=\cdots$$

WORKSHEET 3

11.6.2 Equation of a plane perpendicular to a given vector and passing through a given point

11.6.3 Equation of a plane passing through three non collinear points

At a glance

- The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r}-\vec{a})\cdot\vec{N}=0$
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is
  $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$
- Equation of a plane passing through three non collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

- Cartesian equation of a plane passing through three non collinear points having position vectors $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r}-\vec{a})\cdot\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$

Activity 1:

Find the vector and cartesian equations of the plane that passes through the point $(1,0,-2)$ and normal to the plane is $\hat{i}+j-k$

Position vector of the given point $(1,0,-2)$ is $\vec{a}=\hat{i}+0j-2k$ and the normal vector is $\vec{N}=\hat{i}+j-k$

Therefore vector equation of the plane is given by $(\vec{r}-\vec{a})\cdot\vec{N}=0$

$$\text{i.e., } (\vec{r}-\cdots\cdots\cdots\cdots)\cdot(\hat{i}+j-k)=0$$

Also $(x_1,y_1,z_1)=(1,0,-2)$ and direction ratios of $\vec{N}$ are $A=1,\ B=1,\ C=-1$

Cartesian equation of the plane is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$$1(x-1)+\cdots\cdots\cdots+\cdots\cdots=0$$

$$x+y-\cdots=3$$

Activity 2:

Find the vector and cartesian equations of the plane that passes through the point $(5,2,-4)$ and perpendicular to the line with direction ratios $2, 3, -1$

Position vector of the given point $(5,2,-4)$ is $\vec{a}=5\hat{i}+2j-4k$ and the normal vector is $\vec{N}=2\hat{i}+3j-k$

Therefore vector equation of the plane is given by $(\vec{r}-\vec{a})\cdot\vec{N}=0$

$$\text{i.e., } (\vec{r}-\cdots\cdots\cdots\cdots)\cdot(2\hat{i}+3j-k)=0$$

Also $(x_1,y_1,z_1)=(5,2,-4)$ and direction ratios of $\vec{N}$ are $A=2,\ B=3,\ C=-1$

Cartesian equation of the plane is $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$

$$2(x-5)+\cdots\cdots\cdots+\cdots\cdots=0$$

$$2x+3y-\cdots=20$$

Activity 3:

Find the equation of the plane that passes through the points $(1,1,0)$, $(1,2,1)$, $(-2,2,-1)$

Given points are $(x_1,y_1,z_1)=(1,1,0),(x_2,y_2,z_2)=(1,2,1)$ and $(x_3,y_3,z_3)=(-2,2,-1)$

Equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\begin{vmatrix} x-1 & y-1 & z-0 \\ 1-1 & 2-1 & \cdots\cdots \\ -2-1 & \cdots\cdots & \cdots\cdots \end{vmatrix}=0$$

……………………………..

$2x+\cdots\cdots\cdots\cdots\cdots$

Activity 4:

Find the vector equation of the plane that passes through the points
$(2,5,-3)$, $(-2,-3,5)$, $(5,3,-3)$

Let $\vec{a}=2\hat{i}+5j-3k,\ \vec{b}=-2\hat{i}-3j+5k,\ \vec{c}=5\hat{i}+3j-3k$

$\vec{b}-\vec{a}=-4\hat{i}-8j+8k$ and $\vec{c}-\vec{a}=\cdots\cdots\cdots\cdots$

Then the vector equation of the plane is

$$(\vec{r}-\vec{a})\cdot\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$$

$$\left(\vec{r}-\left(2\hat{i}+5j-3k\right)\right)\cdot\left[\left(-4\hat{i}-8j+8k\right)\times\left(\cdots\cdots\cdots\cdots\right)\right]=0$$

CHAPTER 12

LINEAR PROGRAMMING

POINTS AT A GLANCE

1. A linear programming problem (LPP) is one that is concerned with finding the optimal value (maximum or minimum) of a linear functions of several variables (called objective function) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints).
   Variables are sometimes called decision variables and are non-negative.
   Eg: Maximise $Z= 250x+75y$ (called objective functions)
   $5x+y\leq 100$, $x+y\leq 60$ (constraints)
   $x\geq 0$, $y\geq 0$ (non-negative restrictions)
2. Feasible Region: - The common region determined by all the constraints including non-negative restrictions $x, y\geq 0$ of a L.P.P is called the feasible region (or solution region) for the problem.
3. Optimal (feasible) solution :- Any point in the feasible region that gives the optimal value (maximum or minimum) to the objective function is called an optimal solution.
4. The optimal solution to a L.P.P lies at the corners of the feasible region.
5. Any point outside the feasible region is called an infeasible solution.
6. If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of feasible region.
7. If two corner points of the feasible region are both optimal solutions of the same type, i.e; both produce the same maximum or minimum, then any point on the line segment joining these two points is also an optimal solution of the same type.
8. Corner point method:- For solving a L.P.P
   (i) Find the feasible region
   (ii) Determine its corner points
   (iii) Evaluate the objective function $Z=ax+by$ at each corner point . Let M and m respectively be the largest and smallest values at these points.
   (iv) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective functions.

Chapter -12

Linear Progamming

Work Sheet - I

Graphical Method of Solving LPP

Activity -1

Consider the follwing LPP

Maximise Z= x+y

Subject to .

$$2x+y-3\geq 0$$
$$x-2y+1\leq 0$$
$$y\leq 3$$
$$x\leq 0, y\geq 0$$

(a) Draw the feasible region

(b) Find the corner points

(c) Find the corner point at which z attains the maximum

Ans: Given problem is, Maximise $z=x+y$, such that

$$2x+y\geq 3$$

$$x-2y\geq -1$$
$$y\leq 3$$
$$x\leq 0, y\geq 0$$

First, draw the lines $2x+y=3, x-2y=-1$ & $y=3$

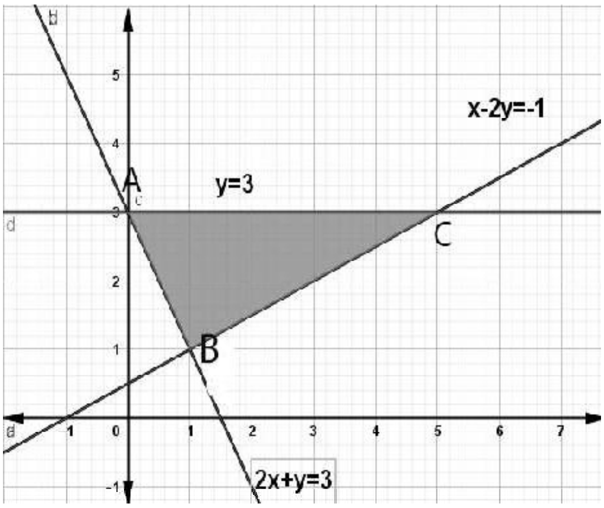
$2x+y=3$

x	0	$3/2$
y	3	0

$x-2y=-1$

x	0	-1
y	$1/2$	0

$y=3$



At(0,0), $2x+y \geq 3 \Rightarrow 0 \geq 3$, which is false

$\therefore$ feasible region lies at right side of the line $2x+y=3$

At (0,0),

$x-2y \leq {}^{-}1 \Rightarrow 0 \leq {}^{-}1$, , which is false

$\therefore$ feasible region lies above the line

$x-2y = {}^{-}1$

for $y \leq 3$, feasible region lies below the line y=3

shaded portion ABC respesents the feasible region.

corner points A(0,3)

<u>To find corner point B</u>

Solve the equations $2x+y=3$ & $x-2y=-1$

(Solve here)

$x=1, \quad y=1$

$\therefore B(1,1)$

<u>To find corner point C</u>

Solve me equations $x-2y=-1$ & $y=3$

(Solve here)

$x = 5, \quad y = 3,$

$\therefore C(5,3)$

$\therefore$ corner points are A(0,3), B(1,1) & C (5,3)

Corner points	Value of $z=x+y$
$A(0,3)$	$z=0+3=3$
B(1,1)	$z=1+1=2$
$C(5,3)$	$z=5+3=8$ — Maximum value

$\therefore$ corner points at which z attains its maximum is C (5,3)

Solution is x=5, y=3

$\therefore$ Maximum value of z=5+3=8

Activity 2

Solve the following LPP

Minimise $z = 200x + 500y$

Subject to the constraints $x + 2y \geq 10$

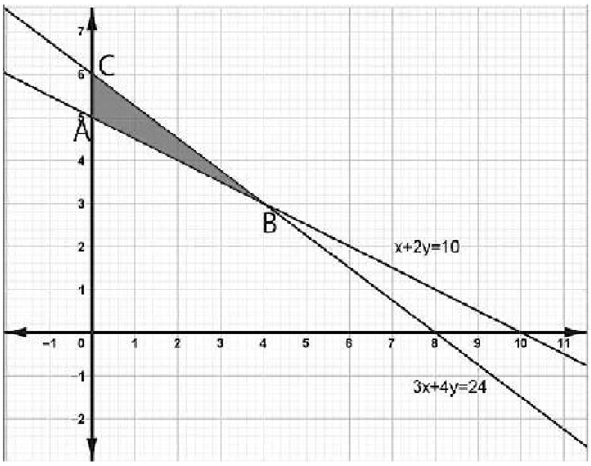
$3x + 4y \leq 24$

$x \geq 0, y \geq 0$

Ans: Draw the lines $x + 2y = 10$ and $3x + 4y = 24$

x	0	10
y	5	0

x	0	8
y	6	0



Shaded portion ABC represents the feasible region.

Find the corner points A, B and C
A(0,5), C(0,6)

To find the co-ordinates of B, Solve $x + 2y = 10$ & $3x + 4y = 24$
(Solve here)

B (4,3)

Corner points	Value of $z = 200x + 500y$
$A(0,5)$	--------
B$(4,3)$	--------
$C(0,6)$	---------

Hence, minium value of z is 2300 attained at........................

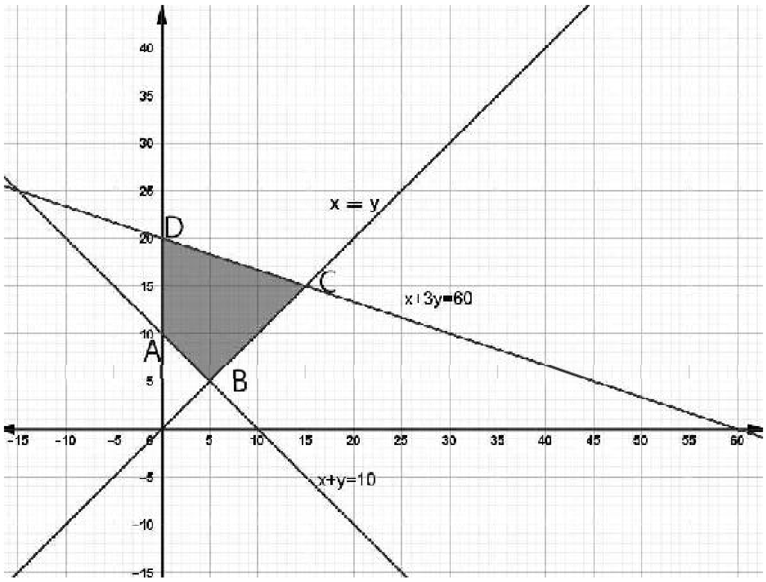
Activity -3

Solve the following problem graphically
Minimise and Maximise $z = 3x + 9y$
Subject to the constaints

$$x + 3y \leq 60$$
$$x + y \geq 10$$
$$x \leq y$$
$$x \geq 0, y \geq 0$$

Ans: Draw the lines

$$x + 3y = 60$$
$$x + y = 10 \;\&\; x = y$$



$x + 3y = 60$

x	0	60
y	20	0

$x + y = 10$

x	0	10
y	10	0

Identify the region ABCD

The corner points are A(..,..), B(...,...), C(...,..) and D(...,...)

Corner points	Value of $z = 3x + 9y$
A(...,...) B(....,....) C(...,...) D(....,...)	90 60 $\rightarrow$ Minimum 180 } Maximum 180 }

The maximum value attains at C(...,..) and D(...,...)and it is 180 in each case. $\therefore$ Every point on the line segment CD gives maximum value.
The minium value of z is 60 and occurs at B(5,5).

Activity :4

Solve : Minimise $z = 3x + 2y$

Subject to $x + y \geq 8$

$3x + 5y \leq 15$

$x \geq 0, y \geq 0$

Ans:

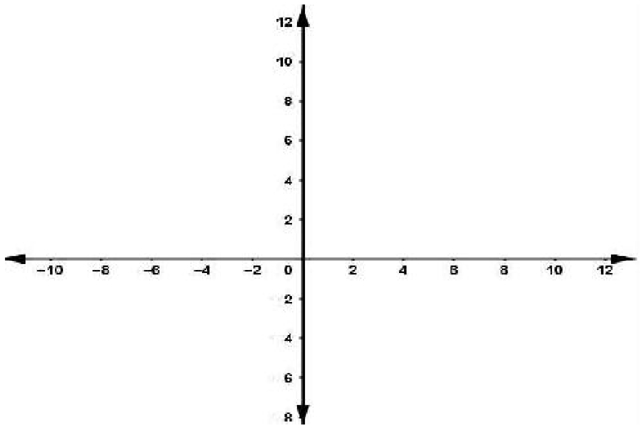
Complete the graph

Draw the lines $x + y = 8$

x	0	8
y	8	0

and $3x + 5y = 15$

x	0	3
y	3	0



There is no point satisfying all the constraints simultaneously. Thus the problem is has no feasible region and hence no feasible solution.

CHAPTER 13

PROBABILITY

Worksheet-1

13.2. Conditional Probability

Let E and F be two events associated with the same sample space of a random experiment. Then the probability of occurrence of the event E given that event F has already occurred is called the conditional propability of E given F

It is denoted of by P(E|F)

P(E/F)= $\frac{P(E \cap F)}{P(F)}$, provided, $P(F) \neq 0$

Properties of Conditional Probability

(i) Let E and F be events of a sample space S of an experiment

Then $P(S \mid F) = P(F \mid F) = 1$

(ii) If A and B are two events of a sample space S and F is an event of S such that $P(F) \neq 0$

Then $P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F) - P((A \cap B) \mid F)$

In particular, if A and B are disjoint events,

Then $P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F)$

(iii) $P(E' \mid F) = 1 - P(E \mid F)$

Activity -1

A family has 2 children. What is the probability that both the children are boys given that atleast one of them is a boy.

Ans: Let 'b' stands for boy and 'g' for girl.

Sample space, $S = \{(b,b),(b,g),.........,..........\}$

Let E and F denote the following events.

E : both the children are boys

F : atleast one of the children is boy.

Then $E = \{(b,b)\}$

$F = \{(b,b), ...,........\}$

$E \cap F = ...$

$P(F) = 3/4$ and $P(E \cap F) = 1/4$

$\therefore$ P (both the children are boys given that atleast one of them is a boy)

$$= P(E \mid F)$$
$$= \frac{P(E \cap F)}{P(F)}$$
$$= ....$$
$$= ....$$

Activity -2

Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$

Ans: Given that $2P(A) = P(B) = \frac{1}{13}$

Then $2P(A) = \frac{5}{13}$ and $P(B) = \frac{5}{13}$

i.e. $P(A) = ...$

We have, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$...............(1)

$P(B) = ...$

$P(A \mid B) = \frac{2}{5}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore P(A \cap B) = P(B).P(A \mid B)$$

$$= ....$$

(1) becomes

$$P(A \cup B) = ... + \frac{5}{13} - .....$$

$$= ....$$

Activity :3

An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questins, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question.

Ans: Let E and F be the events,
E: getting an easy question
F: getting a multiple choice question.

Total no. of question = ...+........................+...................+...................

=1400

$$N(F) = 500 + 400 = 900$$

$$N(E \cap F) = 500 \text{ (easy \& multiple choice questions)}$$

$$P(E \cap F) = \frac{500}{---} = \frac{5}{14}$$

$$P(F) = \frac{-}{1400} = ------$$

Required probability,

$$P(E/F) = \frac{P(E \cap F)}{P(F)}$$

$$=$$

$$=$$

Activity-4

A black and a red dice are rolled.

(i) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

(ii) Find the conditinal probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Ans: (i) Here, the sample space 'S' has---- elements
E and F be events,
E: getting a sum greater than 9
F: black die resulted in 5

We have to find $P(E \mid F)$

$$E = \{(4,6),(5,5),(5,6),(6,4),(6,5),(6,6)\}$$

$$F = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$$

$$E \cap F = \{.....,.......\}$$

$$P(E \cap F) = \frac{...}{36} = ...$$

$$P(F) = ...$$

Required probabiltiy,

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$$= ....$$

(ii) E: getting sum 8
F: red die resulted in a number less than 4

$$E = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

$$F = \{(1,1),(1,2),(1,3),(2,1),(2,2),........(6,3)\}$$ (write all outcomes)

$$E \cap F = \{5,3),(6,2)\}$$

$$P(E \cap F) = ...$$

$$P(F) = \frac{18}{36}$$

$$P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)}$$

$$= .... = \frac{1}{9}$$

Activity 5

If $P(A) = \frac{6}{11}, P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$ .Find

(i) $P(A \cap B)$

(ii) $P(A \mid B)$

(iii) $P(B \mid A)$

Ans: (i) Given $P(A)=\frac{6}{11}$

$$P(B)=\frac{5}{11}$$

$$P(A\cup B)=\frac{7}{11}$$

$$\frac{6}{11}+\frac{5}{11}-P(A\cap B)=\frac{7}{11}$$

$$P(A\cap B)=\frac{6}{11}+\frac{5}{11}-\ldots\ldots\ldots\ldots$$

$$\therefore P(A\cap B)=\frac{4}{11}$$

(ii) $$P(B|A)=\frac{P(A\cap B)}{P(B)}$$

$$=\ldots\ldots\ldots$$

$$=\ldots\ldots\ldots$$

(iii) $$P(B|A)=\frac{P(A\cap B)}{P(A)}$$

$$=\ldots\ldots\ldots$$

$$=\ldots\ldots\ldots$$

Worksheet II

13.4 Independent Events

Two events E and F are said to be independent if the probability of occurrence of any one of them is not affected by the occurrence of the other.

In this case, $P\left(E \mid F\right) = P(E), here\ P(F) \neq 0$

$$P\left(F \mid E\right) = P(F), \text{here } P(E) \neq 0$$

Thus, if E and F are independent events,

$$P(E \cap F) = P(E).P(F).$$

* The term 'independent' is defined in terms of 'probability of events' where as 'mutually exclusive' is defined in tems of events (subsets of sample space).
* Independent events may have common outcome
* Mutually exclusive ents never have a common outcome
* Two mutually exclusive events having non-zero probabilities of occurrence cannot be independent
* Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A).P(B)$$
$$P(A \cap C) = P(A).P(C)$$
$$P(B \cap C) = P(B).P(C)$$
$$P(A \cap B \cap C) = P(A).P(B).P(C)$$

* If the events E & F are independent, then

(a) E`and F are independent

(b) E and F` are independent

(c) E` and F` are independent

Activity -1

A die is thrown E and F are events such that
E: the number appearing is a multiple of 3
F: the number appearing is even.
Find whether E and F are independent.

Ans: Here sample space $S = \{1,2,3,4,5,6\}$

Now $E = \{3,6\}$

$F = \{\_\_\_\_, \_\_\_\_, \_\_\_\_\}$

$E \cap F = \{6\}$

$P(F) = \frac{3}{6} = \frac{1}{2}$

$P(E \cap F) = \_\_\_\_\_\_$ ....(1)

$P(E).P(F) = \frac{1}{3}.\frac{1}{2} = \frac{1}{6}$ ....(2)

from (1) & (2)

$\therefore E$ *and* F are independent

Activity -2

Three coins are tossed simultaneously.
E, F & G are events such that
E: three heads or three tails
F: alterast 2 heads
G: atmost 2 heads.
Of the pairts (E,F), (E,G) and (F,G), which are independent, which are dependent?

Ans: Here Sample space $S = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}$

$E = \{HHH, TTT\}$

$F = \{HHH, HHT, HTH, THH\}$

$G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$E \cap F = \_\_\_\_\_\_\_\_\_\_$

$E \cap G = \_\_\_\_\_\_\_\_\_\_$

$F \cap G = \_\_\_\_\_\_\_\_\_\_$

$$P(E)=\frac{2}{8}=\frac{1}{4}, P(F)=\frac{4}{8}=\frac{1}{2}, P(G)=\frac{7}{8}$$

$$P(E\cap F)=\frac{1}{8}$$

$$P(E\cap G)= \_\_\_\_\_\_\_$$

$$P(F\cap G)= \_\_\_\_\_\_\_\_\_\_$$

$$P(F).P(G)= \_\_\_\_\_\_\_\_\_\_$$

$$P(E).P(F)=\frac{1}{4}.\frac{1}{2}=\frac{1}{8}=P(F\cap E)$$

Hence, the events E and F are independent events.

The pairs of events (E,G) and (F,G) are dependent.

Activity -3

Events A and B are such that $P(A)=\frac{1}{2}, P(B)=\frac{7}{12}$ and $P(\text{not A or not B})=\frac{1}{4}$ state whether A and B are independent.

Ans:

$$P(\text{not A or not B})=\frac{1}{4}$$

$$P(A)=\frac{1}{2}, P(B)=\frac{7}{12}$$

$$P(A'\cup B')=\frac{1}{4}$$

$$\Rightarrow P(A\cap B)'=\frac{1}{4}$$

$$\Rightarrow 1-\_\_\_\_\_\_\_\_=\frac{1}{4}$$

$$\Rightarrow P(A\cap B)=1-\frac{1}{4}=\frac{3}{4}$$

Now, P(A). P(B) =____________ × ____________

= ....

$\therefore$ A and B are not indepedent

Activity -4

(1) The probability of solving a problem independently by A and B are $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that exactly one of them solves the problem

(2) A and B try to solve a problem independently.

The probability that A solves a problem is $\frac{1}{2}$ and that B solves the problem is $\frac{1}{3}$.

Find the probability that

(a) Both of them solves the problem.

(b) Problem is solved.

Ans:

(1) $P(A)=\frac{1}{3}$ $P(B)=\frac{1}{4}$

$$P(A^1)=1-P(A)=\_\_\_\_\_\_\_\_\_$$
$$P(B^1)=1-P(B)=\_\_\_\_\_\_\_\_\_\_$$

Probability of .... one of them solves the problem

$$=P(A).P(B^1)+P(B).P(A^1)$$
$$=\_\_\_\_\_\_\_\_\_\_\_\_=\frac{5}{12}$$

(2) Let A: problem is solved by A

B: Problem is solved by B

$$P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$$

(a) P (both solve the problem) $=P(A\cap B)$

$$=P(A).P(B)=\_\_\_\_\_\_\_\_=\frac{1}{6}$$

(b) P(problem is solved) $= 1 - P(\text{Problem not solved})$

$$= 1 - P(A' \cap B')$$
$$= 1 - P(A^1)(B^1)$$
$$= 1 - ....$$
$$= \frac{2}{3}$$

$P(A^1) =$ ______

$P(B^1) =$ ________

Activity -6

Rani and Joy appear in an interview for 2 vacancies in the same post. The probability of Rani's selection is $\frac{1}{7}$ and that of Joy's selection is $\frac{1}{5}$. What is the probability that

(a) Rani will not be selected

(b) Both of them will be selected

(c) None of them will be selected

Ans:

Let Rani's selection be the event A & Joy's selection be the event B.

$$P(A) = \frac{1}{7}, P(B) = \frac{1}{5}$$

(a) P(Rani will not be selected) $= P(A^1)$

$$= 1 - P(A)$$
$$= \_\_\_\_\_\_\_$$
$$= \frac{6}{7}$$

(b) P(Both of them will be selected)

$$= P(A \cap B) = P(A).P(B)$$
$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$
$$= \frac{1}{35}$$

(iii) P(None of them will be selected)

$$= P(A' \cap B')$$
$$= P(A').P(B')$$
$$= \ldots\ldots \times \ldots\ldots\ldots\ldots$$
$$= \ldots\ldots\ldots\ldots\ldots$$

GENERAL EDUCATION DEPARTMENT

SAMAGRA SHIKSHA, KERALA