SOLUTIONS TO NCERT TEXT BOOK EXERCISE 3.5

For any ΔABC, prove that

3. $$\frac{a+b}{c}=\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$$

Solution

Using the sine rule $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k,$

we get $a = k\sin A$, $b = k\sin B$, $c = k\sin C$

$$\frac{a+b}{c}=\frac{k\sin A+k\sin B}{k\sin C}=\frac{k(\sin A+\sin B)}{k\sin C}=\frac{\sin A+\sin B}{\sin C}$$

$$=\frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)}=\frac{2\cos\left(\frac{C}{2}\right).\cos\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{C}{2}\right).\cos\left(\frac{C}{2}\right)}$$

$$A+B=180°-C$$

$$\frac{A+B}{2}=90°-\frac{C}{2}$$

i.e., $\frac{a+b}{c}=\frac{\cos\left(\frac{A-B}{2}\right)}{\sin\left(\frac{C}{2}\right)}$

$$\sin\left(\frac{A+B}{2}\right)=\sin\left(90°-\frac{C}{2}\right)=\cos\left(\frac{C}{2}\right)$$

5. $\sin\left(\frac{B-C}{2}\right)=\frac{b-c}{a}\cos\frac{A}{2}$ **(NCERT)**

Solution

Using sine formula, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k,$ we get

$a = k\sin A$, $b = k\sin B$, $c = k\sin C$

Consider $\frac{b-c}{a}=\frac{k\sin B-k\sin C}{k\sin A}=\frac{k(\sin B-\sin C)}{k\sin A}$

i.e., $\frac{b-c}{a}=\frac{\sin B-\sin C}{\sin A}=\frac{2\cos\left(\frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$

$$= \frac{\sin\left(\frac{A}{2}\right).\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)}$$

$$\frac{b-c}{a} = \frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

$$\therefore \sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right)\cos\left(\frac{A}{2}\right)$$

$$B + C = 180° - A$$

$$\frac{B+C}{2} = 90° - \frac{A}{2}$$

$$\cos\left(\frac{B+C}{2}\right) = \cos\left(90° - \frac{A}{2}\right) = \sin\frac{A}{2}$$

7. $a(\cos C - \cos B) = 2(b - c)\cos^2\left(\frac{A}{2}\right)$

Solution

Using cosine formula,

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} \text{ and } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore a(\cos C - \cos B) = a\left[\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - \left(\frac{c^2 + a^2 - b^2}{2ca}\right)\right]$$

$$= \left(\frac{a^2 + b^2 - c^2}{2b}\right) - \left(\frac{c^2 + a^2 - b^2}{2c}\right)$$

$$= \frac{a^2c + b^2c - c^3 - bc^2 - a^2b + b^3}{2bc}$$

$$= \frac{b^3 - c^3 + a^2c - a^2b + b^2c - bc^2}{2bc}$$

$$= \frac{(b-c)(b^2 + bc + c^2) - a^2(b-c) + bc(b-c)}{2bc}$$

$$= \frac{(b-c)[b^2 + bc + c^2 - a^2 + bc]}{2bc}$$

$$=(b-c)\left[\frac{b^2+c^2-a^2}{2bc}+\frac{2bc}{2bc}\right]$$

$$=(b-c)[\cos A+1]$$

$$=(b-c)(1+\cos A)$$

$$=(b-c)\,2\cos^2\left(\frac{A}{2}\right)$$

$$\therefore a(\cos C-\cos B)=2(b-c)\cos^2\left(\frac{A}{2}\right)$$

9. $(b+c)\cos\left(\frac{B+C}{2}\right)=a\cos\left(\frac{B-C}{2}\right)$

Solution

Using sine rule, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k,$

we get $a=k\sin A,\ b=k\sin B,\ c=k\sin C$

$$(b+c)\cos\left(\frac{B+C}{2}\right)=(k\sin B+k\sin C)\cos\left(\frac{B+C}{2}\right)$$

$$=k(\sin B+\sin C)\cos\left(\frac{B+C}{2}\right)$$

$$=k.2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right).\cos\left(\frac{B+C}{2}\right)$$

$$=k.2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B+C}{2}\right).\cos\left(\frac{B-C}{2}\right)$$

$$=k\sin(B+C).\cos\left(\frac{B-C}{2}\right)$$

$$=k\sin A.\cos\left(\frac{B-C}{2}\right)$$

| $B+C=\pi-A$, $\sin(B+C)=\sin(\pi-A)=\sin A$

$$=a\cos\left(\frac{B-C}{2}\right),\qquad \text{since } a=k\sin A$$

10. $a\cos A + b\cos B + c\cos C = 2a\sin B.\sin C$

Solution

Using the sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$,

we get $a = k\sin A$, $b = k\sin B$, $c = k\sin C$

$\therefore a\cos A + b\cos B + c\cos C = k\sin A\cos A + k\sin B\cos B + k\sin C\cos C$

$$= \frac{k}{2}[2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C]$$

$$= \frac{k}{2}[\sin 2A + \sin 2B + \sin 2C]$$

$$= \frac{k}{2}[2\sin(A+B)\cos(A-B) + 2\sin C\cos C]$$

$$= \frac{k}{2}[2\sin C.\cos(A-B) + 2\sin C\cos C]$$

$$= k\sin C[\cos(A-B) + \cos C]$$

$A + B = \pi - C$
$\sin(A+B) = \sin(\pi - C)$
$\sin(A+B) = \sin C$

$$= k\sin C[\cos(A-B) - \cos(A+B)]$$

$$= k\sin C(2\sin A\sin B)$$

$$= 2.k\sin A.\sin B.\sin C$$

$C = \pi - (A+B)$
$\cos C = \cos(\pi - (A+B))$
$\cos C = -\cos(A+B)$

$$= 2a\sin B.\sin C, \text{ since } a = k\sin A$$

12. $(b^2 - c^2)\cot A + (c^2 - a^2)\cot B + (a^2 - b^2)\cot C = 0$

Solution

Using the sine rule $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$

we have $\sin A = ka$, $\sin B = kb$, $\sin C = kc$

By cosine rule, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Consider $(b^2 - c^2)\cot A = (b^2 - c^2)\frac{\cos A}{\sin A}$

$$= \frac{(b^2-c^2)\left(\frac{b^2+c^2-a^2}{2bc}\right)}{ka} = \frac{(b^2-c^2)(b^2+c^2-a^2)}{2kabc}$$

$$= \frac{(b^2-c^2)(b^2+c^2)-a^2(b^2-c^2)}{2kabc}$$

$$(b^2-c^2)\cot A = \frac{(b^4-c^4)-a^2(b^2-c^2)}{2kabc}$$

$\therefore$ LHS $= (b^2-c^2)\cot A + (c^2-a^2)\cot B + (a^2-b^2)\cot C$

$$= \frac{b^4-c^4-a^2(b^2-c^2)}{2kabc} + \frac{c^4-a^4-b^2(c^2-a^2)}{2kabc} + \frac{a^4-b^4-c^2(a^2-b^2)}{2kabc}$$

$$= \frac{b^4-c^4+c^4-a^4+a^4-b^4-a^2b^2+a^2c^2-b^2c^2+b^2a^2-c^2a^2+c^2b^2}{2kabc}$$

$$= \frac{0}{2kabc} = 0 = \text{RHS}$$

13. $\frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C = 0$

Solution

Using sine rule $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$, we get

$\sin A = ka$, $\sin B = kb$, $\sin C = kc$

Consider $\frac{b^2-c^2}{a^2}\sin 2A = \frac{b^2-c^2}{a^2} . 2\sin A\cos A$

$$= \frac{(b^2-c^2)}{a^2} 2ka . \frac{(b^2+c^2-a^2)}{2bc} = \frac{k}{abc}(b^2-c^2)(b^2+c^2-a^2)$$

$$\therefore \frac{b^2-c^2}{a^2}\sin 2A + \frac{c^2-a^2}{b^2}\sin 2B + \frac{a^2-b^2}{c^2}\sin 2C$$

$$= \frac{k}{abc}[(b^2-c^2)(b^2+c^2-a^2)] + \frac{k}{abc}[(c^2-a^2)(c^2+a^2-b^2)] + \frac{k}{abc}[(a^2-b^2)(a^2+b^2-c^2)]$$

$$= \frac{k}{abc}[(b^2-c^2)(b^2+c^2)-a^2(b^2-c^2)+(c^2-a^2)(c^2+a^2)-b^2(c^2-a^2)+ (a^2-b^2)(a^2+b^2)-c^2(a^2-b^2)]$$

$$= \frac{k}{abc}[b^4-c^4+c^4-a^4+a^4-b^4-a^2b^2+a^2c^2-b^2c^2+a^2b^2-a^2c^2+b^2c^2]$$

$$= \frac{k}{abc}(0) = 0$$

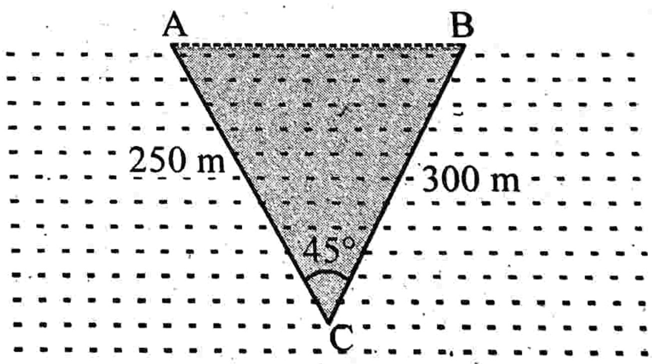
15. Two ships leave a port at the same time. One goes 24 km per hour in the direction N45°E and the other travels 32 km per hour in the direction S75°E. Find the distance between the ships at the end of 3 hours.

Solution

Refer Example 81

16. Two trees A and B are on the same side of a river. From a point C in the river, the distance of the trees A and B is 250 m and 300 m respectively. If the angle C is 45°, find the distance between the trees $\left(\text{use } \sqrt{2} = 1.414\right)$.

Solution



Join AB, BC and AC, we get ΔABC

$\angle C = 45°$, AC = 250 m, BC = 300 m

i.e., $C = 45°$, $b = 250$, $a = 300$

Using cosine formula, $c^2 = a^2 + b^2 - 2ab\cos C$

$\therefore c^2 = 300^2 + 250^2 - 2 \times 300 \times 250 \times \cos 45°$

$= 90000 + 62500 - 150000 \times \dfrac{1}{\sqrt{2}}$

$= 152500 - 75000\sqrt{2}$

$= 152500 - 106050 = 46450$

$\therefore c = \sqrt{46450} = 215.5$ m

$\therefore$ Distance between the trees = 215.5 m

SOLUTIONS TO NCERT MISCELLANEOUS EXERCISE

Prove that

1. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Solution

We have $2\cos x\cos y=\cos(x+y)+\cos(x-y)$

$$\text{LHS}=\cos\left(\frac{\pi}{13}+\frac{9\pi}{13}\right)+\cos\left(\frac{\pi}{13}-\frac{9\pi}{13}\right)+\cos\left(\frac{3\pi}{13}\right)+\cos\left(\frac{5\pi}{13}\right)$$

$$=\cos\left(\frac{10\pi}{13}\right)+\cos\left(\frac{8\pi}{13}\right)+\cos\left(\frac{3\pi}{13}\right)+\cos\left(\frac{5\pi}{13}\right)$$

$$=\cos\left(\pi-\frac{3\pi}{13}\right)+\cos\left(\pi-\frac{5\pi}{13}\right)+\cos\left(\frac{3\pi}{13}\right)+\cos\left(\frac{5\pi}{13}\right)$$

$$=-\cos\frac{3\pi}{13}-\cos\frac{5\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0=\text{RHS}$$

2. $(\sin 3x+\sin x)\sin x+(\cos 3x-\cos x)\cos x=0$

Solution

$\text{LHS}=(2\sin 2x\cos x)\sin x+(-2\sin 2x\sin x)\cos x$

$=2\sin 2x\cos x\sin x-2\sin 2x\cos x\sin x=0=\text{RHS}$

Another method

$\text{LHS}=\sin 3x\sin x+\sin^2 x+\cos 3x\cos x-\cos^2 x$

$=\cos 3x\cos x+\sin 3x\sin x-(\cos^2 x-\sin^2 x)=\cos(3x-x)-\cos 2x$

$=\cos 2x-\cos 2x=0=\text{RHS}$

3. $(\cos x+\cos y)^2+(\sin x-\sin y)^2=4\cos^2\frac{x+y}{2}$

Solution

$$\text{LHS} = \left[2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]^2 + \left[2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right]^2$$

$$= 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right)\sin^2\left(\frac{x-y}{2}\right)$$

$$= 4\cos^2\left(\frac{x+y}{2}\right)\left[\cos^2\left(\frac{x-y}{2}\right) + \sin^2\left(\frac{x-y}{2}\right)\right]$$

$$= 4\cos^2\left(\frac{x+y}{2}\right) = \text{RHS}$$

5. $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$

Solution

$$\text{LHS} = 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right) + 2\sin\left(\frac{5x+7x}{2}\right)\cos\left(\frac{5x-7x}{2}\right)$$

$$= 2\sin 2x \cos x + 2\sin 6x \cos x = 2\cos x\,(\sin 2x + \sin 6x)$$

$$= 2\cos x\left(2\sin\frac{2x+6x}{2}\cos\frac{2x-6x}{2}\right) = 4\cos x \cos 2x \sin 4x = \text{RHS}$$

6. $\dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

Solution

$$\text{LHS} = \frac{2\sin\left(\frac{7x+5x}{2}\right).\cos\left(\frac{7x-5x}{2}\right) + 2\sin\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}{2\cos\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right) + 2\cos\left(\frac{9x+3x}{2}\right)\cos\left(\frac{9x-3x}{2}\right)}$$

$$= \frac{\sin 6x\cos x + \sin 6x\cos 3x}{\cos 6x\cos x + \cos 6x\cos 3x} = \frac{\sin 6x}{\cos 6x}\left(\frac{\cos x + \cos 3x}{\cos x + \cos 3x}\right) = \tan 6x = \text{RHS}$$

7. $\sin 3x + \sin 2x - \sin x = 4\sin x \cos\frac{x}{2}\cos\frac{3x}{2}$

Solution

$$\text{LHS} = \sin 3x - \sin x + \sin 2x = 2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right) + \sin 2x$$

$= 2\cos 2x \sin x + 2\sin x \cos x$

$= 2\sin x[\cos 2x + \cos x] = 2\sin x\left(2\cos\frac{3x}{2}.\cos\frac{x}{2}\right) = 4\sin x.\cos\frac{x}{2}.\cos\frac{3x}{2} = \text{RHS}$

Find $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ in each of the following: (Questions 8 to 10)

8. $\tan x = \frac{-4}{3}$, x in quadrant II

Solution

Since x is in second quadrant, $\frac{\pi}{2} < x < \pi$. $\quad \therefore \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

Therefore $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are positive

Now $\sec^2 x = 1 + \tan^2 x = 1 + \left(\frac{-4}{3}\right)^2 = \frac{9+16}{9} = \frac{25}{9}$

$\therefore \sec x = \frac{-5}{3}$ or $\cos x = \frac{-3}{5}$ *(since x is in second quadrant)*

Now $2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{3}{5} = \frac{8}{5}$

$\therefore \sin^2\frac{x}{2} = \frac{4}{5} \Rightarrow \sin\frac{x}{2} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$, $\quad$ Again $2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore \cos^2\frac{x}{2} = \frac{1}{5} \Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$ $\quad$ Hence $\tan\frac{x}{2} = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \frac{\frac{2\sqrt{5}}{5}}{\frac{\sqrt{5}}{5}} = 2$

9. $\cos x = \frac{-1}{3}$, x in quadrant III

Solution

Since x is in third quadrant, $\pi < x < \frac{3\pi}{2}$. $\quad \therefore \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$.

Therefore $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative and $\tan\frac{x}{2}$ is negative.

Now $2\cos^2\frac{x}{2} = 1 + \cos x = 1 + \frac{-1}{3} = \frac{2}{3}$ $\quad \therefore \cos^2\frac{x}{2} = \frac{1}{3} \Rightarrow \cos\frac{x}{2} = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

Now $2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{1}{3} = \frac{4}{3}$ $\quad \therefore \sin^2\frac{x}{2} = \frac{2}{3} \Rightarrow \sin\frac{x}{2} = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$

Hence $\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\frac{\sqrt{6}}{3}}{\frac{-\sqrt{3}}{3}} = -\sqrt{2}$

10. $\sin x = \frac{1}{4}$, x is in quadrant II

Solution

Since x is in second quadrant, $\frac{\pi}{2} < x < \pi$ $\quad \therefore \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$

$\therefore \sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are positive

$\sin x = \frac{1}{4}$ $\quad \therefore \cos^2 x = 1 - \sin^2 x = 1 - \frac{1}{16} = \frac{15}{16}$

$\therefore \cos x = -\sqrt{\frac{15}{16}} = \frac{-\sqrt{15}}{4}$ since x is in second quadrant

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2} = \frac{1+\frac{\sqrt{15}}{4}}{2} = \frac{4+\sqrt{15}}{8} = \frac{8+2\sqrt{15}}{16} \qquad \therefore \sin\frac{x}{2} = \frac{\sqrt{8+2\sqrt{15}}}{4}$$

$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} = \frac{1+\frac{-\sqrt{15}}{4}}{2} = \frac{4-\sqrt{15}}{8} = \frac{8-2\sqrt{15}}{16}$$

$$\cos\frac{x}{2} = \frac{\sqrt{8-2\sqrt{15}}}{4}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\frac{\sqrt{8+2\sqrt{15}}}{4}}{\frac{\sqrt{8-2\sqrt{15}}}{4}} = \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}}} = \sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}}$$

$$= \sqrt{\frac{(4+\sqrt{15})}{(4-\sqrt{15})} \cdot \frac{(4+\sqrt{15})}{(4+\sqrt{15})}} = \frac{4+\sqrt{15}}{16-15} = 4+\sqrt{15}$$