7.8.4 Torque and Angular Momentum For a System of Particles

Consider a system of 'n' particles. Total angular momentum of the system is the sum of angular momentum of the individual particles.

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \ldots\ldots\ldots\ldots + \mathbf{l}_n = \sum_{i=1}^{n} l_i$$

The angular momentum of the i^{th} particle is $\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i$

Hence, for the system, $\mathbf{L} = \Sigma l_i = \sum_i r_i \times p_i$

But from the equation for torque, it is clear that for a system of particles,

$$\frac{dL}{dt} = \frac{d}{dt}\sum_i l_i = \sum_i \frac{dl_i}{dt} = \sum_i \tau_i$$

But $\sum_i \tau_i = \tau_{ext} + \tau_{int} = \tau_{ext}$

Since the total internal torques are contributed by the internal forces, as for a pair of particles it is equal and opposite.

$$\therefore \frac{dL}{dt} = \tau_{ext}$$

7.8.5 Conservation of Angular Momentum

We have, $\dfrac{dL}{dt} = \tau$

If $\tau = 0$, then $\frac{dL}{dt} = 0$ or $\mathbf{L}$ = a constant

Hence, ***the law of conservation of angular momentum*** *states that, if no external torque is acting on a particle, its angular momentum remains constant.*

Solved Examples

5. A force $5\hat{i} - 2\hat{j} + 3\hat{k}$ acts on a particle whose position vector is $2\hat{i} + 3\hat{j} + 5\hat{k}$. Find the torque about the origin.

Sol.

Here $\mathbf{F} = 5\hat{i} - 2\hat{j} + 3\hat{k}$ and

$\mathbf{r} = 2\hat{i} + 3\hat{j} + 5\hat{k}$

Torque $\tau = \mathbf{r} \times \mathbf{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -2 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= \hat{i}(-10-9) - \hat{j}(25-6) + \hat{k}(15+4)$$

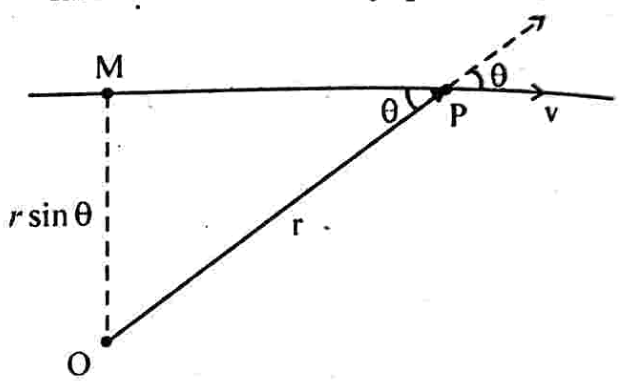
$$= -19\hat{i} - 19\hat{j} + 19\hat{k}$$

$$= 19(-\hat{i} - \hat{j} + \hat{k})$$

6. Show that the angular momentum about any point of a single particle moving with constant velocity remains constant throughout the motion.

Sol.

Let the particle with velocity $\mathbf{v}$ be at point P at some instant t. We want to calculate the angular momentum of the particle about an arbitrary point O.



The angular momentum is $\boldsymbol{l} = \mathbf{r} \times m\mathbf{v}$. Its magnitude is $mvr \sin\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{v}$ as shown in Fig. Although the particle changes position with time, the line of direction of $\mathbf{v}$ remains the same and hence OM = $r \sin \theta$ is a constant.

Further, the direction of $\boldsymbol{l}$ is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{v}$. It is into the page of the figure. This direction does not change with time. Thus, $\boldsymbol{l}$ remains the same in magnitude and direction and is therefore conserved.

7.9 Equilibrium of a Rigid Body

A rigid body is said to be in equilibrium, if the net external force acting on it does not change the translational and rotational states of the body.

7.9.1 Conditions for Translational Equilibrium

In translational equilibrium the rigid body will be at rest or it moves with a constant velocity.

Consider a rigid body of mass m'. Let $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$..... be the external forces acting on it.

$\therefore$ Net external force $\mathbf{F}_{ext} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots\ldots\ldots$

Now, $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots\ldots = \frac{d\mathbf{P}}{dt}$ .... (1) (according to 2[nd] law)

$$\mathbf{F}_{ext} = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}(m\mathbf{v})$$

$$\mathbf{F}_{ext} = m\frac{d\mathbf{v}}{dt} \quad \ldots\ldots\ldots\ (2)$$

The body is in translational equilibrium, if $\mathbf{v}$ = constant or

$$\frac{dv}{dt} = 0 \quad \ldots\ldots\ (3)$$

$$\mathbf{F}_{ext} = 0 \quad \ldots\ldots\ (4)$$

Thus a rigid body is said to be at ***translational equilibrium*** if the net *external force acting on it is zero.*

7.9.2 Condition for Rotational Equilibrium

A rigid body is said to be in rotational equilibrium if it does not rotate or rotate with a constant angular velocity.

Let $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3$ .... be the external torques acting on a rigid body of mass 'm'.

The net external torque on the body

$\vec{\tau}_{ext} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \ldots\ldots\ldots$ (1)

But for a rotating body, its angular momentum is related to torque as,

$\vec{\tau}_{ext} = \frac{d\vec{l}}{dt}$ .... (2) But $\vec{l} = I\omega$

where 'ω' is the angular velocity of the rigid body, $\therefore \vec{\tau}_{ext} = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt}$ ... (3)

The body is in rotational equilibrium, if

$$\omega = \text{a constant . Hence } \frac{d\omega}{dt} = 0$$

$\therefore$ eqn. (3) becomes $\tau_{ext} = 0$ .... (4)

Thus, *a rigid body is said to be in* ***rotational equilibrium****, if the net external torque acting on it is zero.*

7.9.3 Principle of Moments

It is a usual scene in the sawmill, the timber workers are trying hard to move large wooden pieces with a strong lever (iron rod). Did you think, why they are using large rods to lift wooden pieces? What is the use of the small wooden piece (normally using) as a support to lift the large mass? All these questions point to the principle of moments.

The point about which the rod is supported, is called the fulcrum O. The force (F_2) applied to lift the mass is called the effort and the weight to be lifted is called the load (F_1).

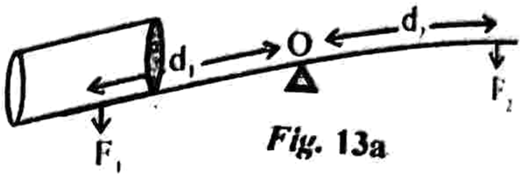


Fig. 13a

As in fig. 13b, through O, a reaction R of the forces F_1 and F_2 is acting. The lever is a system in mechanical equilibrium. For its translational equilibrium,

$R - F_1 - F_2 = 0$ .... (1)

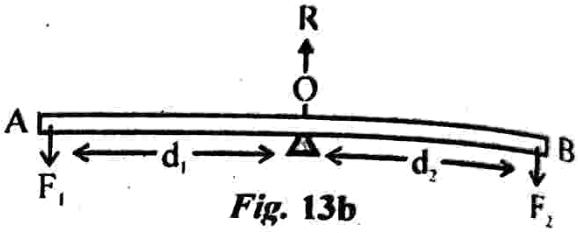


Fig. 13b

For rotational equilirbium, the sum of moments of all forces about the fulcrum must be zero

i.e., $d_1F_1 - d_2F_2 = 0$ .... (2)

Since the reaction R is acting along the fulcrum, moment about O is zero and clockwise moments taken to be negative while anticlockwise moments positive.

The distance of the load 'd_1' from O is called 'load arm' and that of effort (d_2) is called 'effort arm'.

Hence $d_1F_1 = d_2F_2$ .... (3)

or load arm × load = effort arm × effort

This equation represents the principle of moments for a lever.

The ratio of load to effort i.e., $\left(\frac{F_1}{F_2}\right)$ is called mechanical advantage (MA).

i.e., $MA = \frac{F_1}{F_2} = \frac{d_2}{d_1}$

Hence by increasing the effort arm we can decrease the effort.

7.9.4 Centre of Gravity

We can balance a notebook or a scale on our finger tip. The point where the body balances is called centre of gravity.

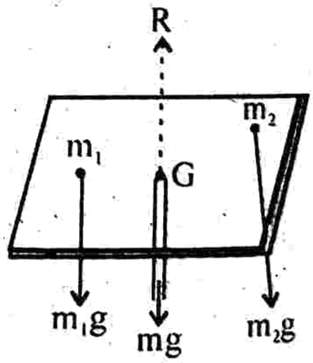


Fig. 14

When we balance a body, its weight mg is acting vertically downward and a reaction R is acting upward through G. These two forces are equal but opposite such that the body is in translational equilibrium.

The body is also in rotational equilibrium about G, such that the total gravitational torque must be zero. For the i^{th} particle, torque is given by $\tau_i = r_i \times m_i g$, where r_i is the position vector of the i^{th} particle w.r.to G.

$\therefore$ Total gravitational torque $\tau_g = \Sigma\tau_i$

$$= \Sigma r_i \times m_i g$$

$$= g\Sigma m_i r_i = 0$$

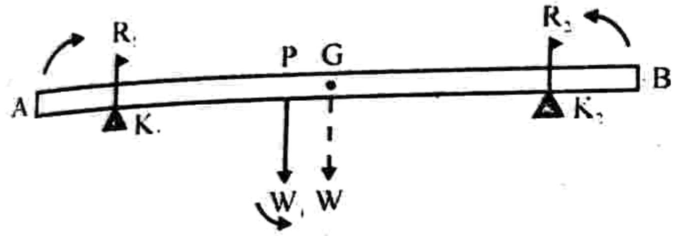
From the above equation it is clear that $\Sigma m_i r_i = 0$, since $g \neq 0$.

Hence in a region where gravity is constant, then centre of gravity coincides with centre of mass of the body.

Solved Examples

7. A metal bar 70 cm long and 4.00 kg in mass is supported on two knife-edges placed 10 cm from each end. A 6.00 kg weight is suspended at 30 cm from one end. Find the reactions at the knife-edges. (Assume the bar to be of uniform cross section and homogeneous.)

Sol.



The figure shows the rod AB, the positions of the knife edges K_1 and K_2, the centre of gravity of the rod at G and the suspended weight at P.

Note the weight of the rod W acts at its centre of gravity G. The rod is uniform in cross section and homogeneous; hence G is at the centre of the rod; AB = 70 cm. AG = 35 cm, AP = 30 cm, PG = 5 cm, $AK_1 = BK_2$ = 10 cm and $K_1G = K_2G$ = 25 cm. Also, W= weight of the rod = 4.00 kg and W_1= suspended weight = 6.00 kg; R_1 and R_2 are the normal reactions of the support at the knife edges.

For translational equilibrium of the rod, $R_1+R_2-W_1-W=0$ ...(i)

Note W_1 and W act vertically down and R_1 and R_2 act vertically up.

For considering rotational equilibrium, we take moments of the forces. A convenient point to take moments about is G. The moments of R_2 and W_1 are anticlockwise (+ve), whereas the moment of R_1 is clockwise (–ve).

For rotational equilibrium,

$-R_1(K_1G) + W_1(PG) + R_2(K_2G) = 0$ ...... (ii)

It is given that W = 4.00g N and W_1 = 6.00g N, where g = acceleration due to gravity. We take g = 9.8 m/s².

With numerical values inserted, from (i)

$R_1 + R_2 - 4.00g - 6.00g = 0$

or $R_1 + R_2 = 10.00g$ N .... (iii)

= 98.00 N

From (ii) $-0.25R_1 + 0.05W_1 + 0.25R_2 = 0$

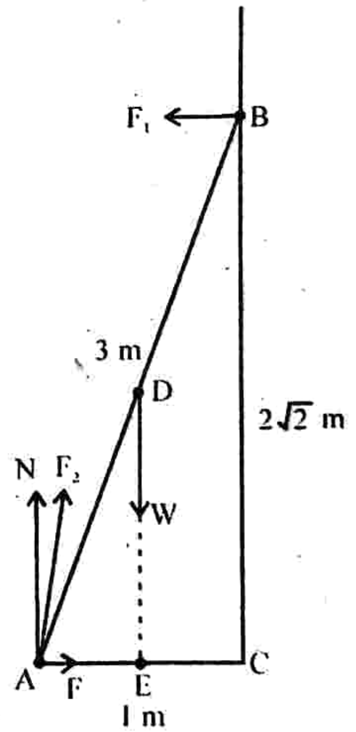
or $R_2 - R_1 = 1.2g$ N = 11.76 N ... (iv)

From (iii) and (iv), R_1 = 54.88 N, R_2 = 43.12 N

Thus the reactions of the support are about 55 N at K_1 and 43 N at K_2.

8. A 3m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in Fig. Find the reaction forces of the wall and the floor.

Sol.



The ladder AB is 3 m long, its foot A is at distance AC = 1 m from the wall. From Pythagoras theorem, BC = $2\sqrt{2}$ m. The forces on the ladder are its weight W acting at its centre of gravity D, reaction forces F_1 and F_2 of the wall and the floor respectively. Force F_1 is perpendicular to the wall, since the wall is frictionless. Force F_2 is resolved into two components, the normal reaction N and the force of friction F. Note that F prevents the ladder from sliding away from the wall and is therefore directed towards the wall.

For translational equilibrium, taking the forces in the vertical direction,

$N - W = 0$ .... (i)

Taking the forces in the horizontal direction,

$F - F_1 = 0$ .... (ii)

For rotational equilibrium, taking the moments of the forces about A,

$$2\sqrt{2}F_1 - \frac{1}{2}W = 0 \quad \text{.................. (iii)}$$

Now W = 20 g = 20 × 9.8 N = 196.0 N

From (i) N = 196.0

From (iii) $F_1 = \frac{W}{4\sqrt{2}} = \frac{196.0}{4\sqrt{2}} = 34.6N$

From (ii) $F = F_1 = 34.6$ N

$$F_2 = \sqrt{F^2 + N^2} = 199.0N$$

The force F_2 makes an angle á with the horizontal,

$$\tan\alpha = \frac{N}{F} = 4\sqrt{2}, \quad \alpha = \tan^{-1}(4\sqrt{2}) \approx 80^\circ$$

7.10 Moment of Inertia

Inertia is the inability of a body to change by itself its state of rest or of uniform motion. This is the case with translational motion.

In the case of a rotating body, it cannot change its state of rest or of uniform rotation about an axis by itself. This inability is called ***rotational inertia or moment of inertia***.

Moment of inertia of a particle is measured as the product of mass of the particle and square of its distance from the axis of rotation.

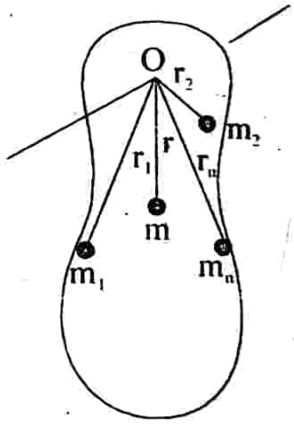


Fig. 15

Consider a particle of mass 'm' rotating about an axis O at a distance of r from it, its moment of inertia is given by, $I = mr^2$

In the case of a rigid body, which is constituted of a number of particles of masses m_1, m_2, ..., m_n at distances r_1, r_2, ..., r_n respectively from the axis.

Moment of inertia is given by, $I = m_1r_1^2 + m_2r_2^2 + ... + m_nr_n^2$ or $I = \sum_{i=1}^{n} m_ir_i^2$

Inertia depends only on the mass of the body while moment of inertia depends on mass and distribution of the mass with respect to the axis of rotation. It is a scalar quantity.

Unit

In SI - kg m² CGS - g cm²

Dimensional formula : $[ML^2T^0]$