7.10.1 Physical Significance of Moment of Inertia

According to Newton's first law, an external force is necessary to change the state of rest or of uniform motion of a body. Hence a body cannot change its state by itself. This inability of a body in linear motion is called inertia. A massive body requires more force to produce linear acceleration. Hence mass of a body is a measure of its inertia.

In the case of a rotating body, an external torque in required to produce an acceleration on it. The inability of a rotating body to produce any change in its state is called rotational inertia or moment of inertia. Moment of inertia depends not only the mass but also its distribution about the axis of rotation. Hence Moment of inertia in rotation plays the same role as mass does in linear motion.

7.10.2 Radius of Gyration

Consider a rigid body rotating about an axis with masses m_1, m_2, ..., m_n at distances r_1, r_2, ..., r_n respectively. Now, the moment of inertia of the body is given by,

$$I = m_1r_1^2 + m_2r_2^2 + ... + m_nr_n^2 \quad \text{or} \quad I = \Sigma mr^2 = MK^2 \quad .... (1)$$

where K is called the radius of gyration.

***Radius of gyration** is the distance of the equivalent point where the whole mass of the rotating body is assigned to a particle such that the moment of inertia of this particle about any axis is same as the moment of inertia of the body about the same axis.*

Radius of gyration can also be defined as the distance, whose square when multiplied by the whole mass gives the moment of inertia of the body.

7.11 Theorems of Perpendicular and Parallel Axes

In order to calculate the moment of inertia of some regular shaped bodies, two theorems are used.

i. Theorem of perpendicular axis

and

ii. Theorem of parallel axes

7.11.1 Theorem of Perpendicular Axis

It states that, *'the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of moments of inertia about two mutually perpendicular axes in its plane and intersecting each other at the point where the perpendicular axis meets the lamina'.*

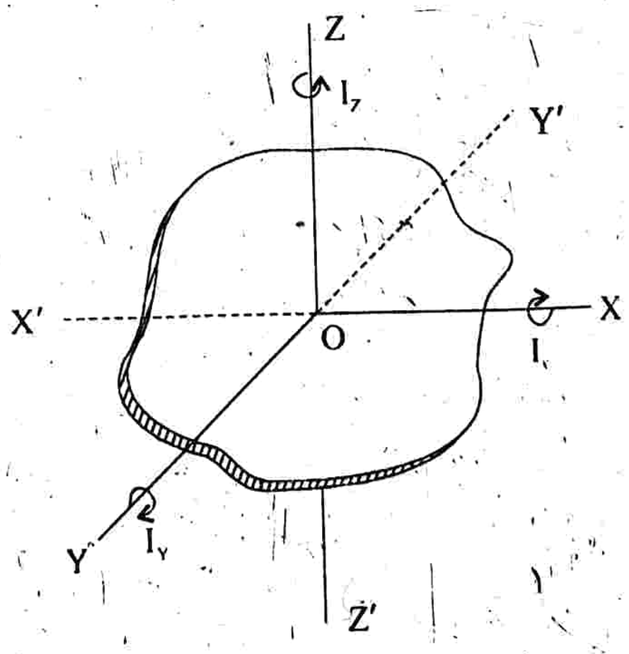


Fig. 16
Theorem of perpendicular axes applicable to a planar body; X and Y axes are two perpendicular axes in the plane and the Z-axis is $\perp^r$ to the plane

Let I_x and I_y are moments of inertia of a plane lamina about two mutually perpendicular axes XX' and YY' on its surface.

Let ZZ' be an axis perpendicular to XX' and YY'. O is the common point of intersection of the three axes.

The moment of inertia about ZZ' is $I_z = I_x + I_y$

7.11.2 Theorem of Parallel Axes

This theorem states that *'the moment of inertia of a body about any axis is equal to the sum of the moment of inertia about a parallel axis passing through the centre of mass of the body and the product of the mass of the body and the square of the distance between the two axes'.*

Consider a body of mass M. Let G be its centre of mass. The MI of the body about any axis XY be I. Consider an axis $X'Y'$ parallel to XY and passing through G. Let I_{cm} be the MI of the body about $X'Y'$. Let 'a' be the separation between the axes. Now, according to the parallel axes theorem $I = I_{cm} + Ma^2$

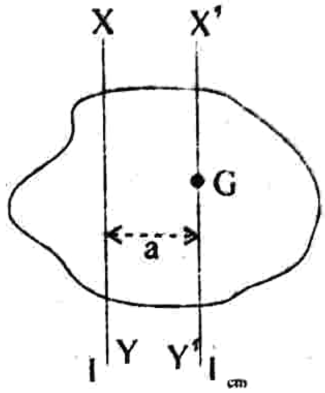


Fig. 17

The theorem of parallel axes. The XY and X'Y' axes are two parallel axes separated by a distance 'a'. G is the centre of mass of the body

7.12 Moment of Inertia of Regular Shaped Bodies

By applying parallel and perpendicular axes theorems, we can reach the equations for moment of inertia of certain regular shaped bodies.

7.12.1 Moment of inertia of a circular ring

i. About an axis passing through the centre and perpendicular to its plane

Consider a circular ring of mass M and radius R. AB be an axis passing through its centre O and perpendicular to its plane as shown in figure.

Consider a particle of mass 'm' on the ring. Since this particle is at a distance of R from the axis of rotation, its moment of inertia is mR^2

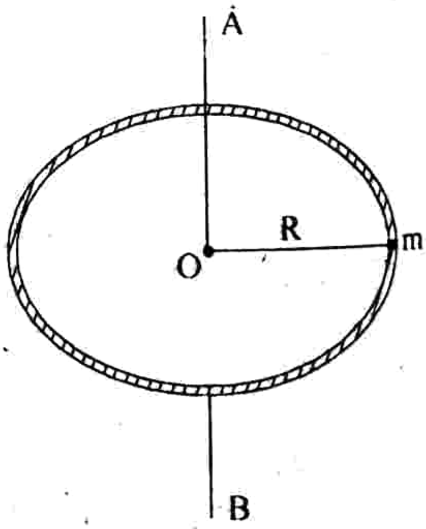


Fig. 18

$\therefore$ Moment of inertia of the ring about AB, $I = \Sigma mR^2$

ie., $I = MR^2$, where $\Sigma m = M$, is the total mass of the ring.

ii. About any diameter

The moment of inertia of the ring about any diameter is the same due to its symmetry. Consider two mutually perpendicular diameters XX' and YY' on its plane. Let I_d be the moment of inertia about any diameter. ZZ' be an axis passing through its centre and perpendicular to its plane. Let I be the moment of inertia about ZZ'.

According to the perpendicular axes theorem

MI about ZZ′ = MI about XX′ + MI about YY′

ie., $I = I_d + I_d$

ie., $2I_d = I = MR^2$

$$I_d = \frac{MR^2}{2}$$

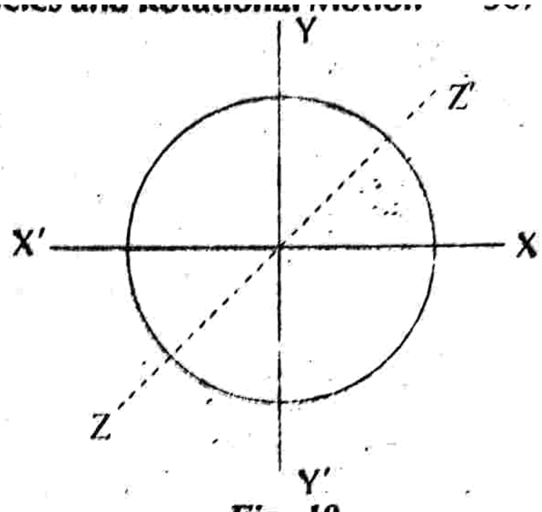


Fig. 19

iii. About a tangent parallel to the diameter of the ring

Let AB be a tangent to the ring, parallel to the diameter XX′ of the ring.

MI about XX′ is $I_d = \frac{MR^2}{2}$. According to parallel axes theorem,

MI about AB = MI about XX′ + MR²

$$I_{AB} = \frac{MR^2}{2} + MR^2$$

$$I_{AB} = \frac{3}{2}MR^2$$

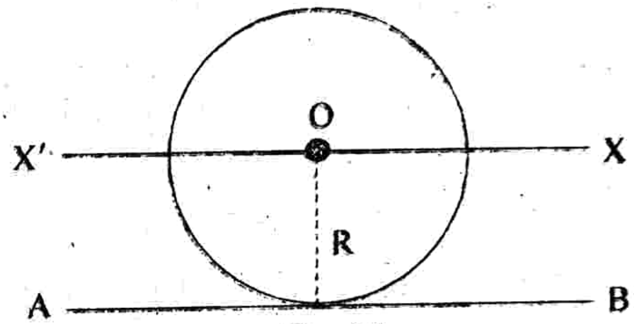


Fig. 20

7.12.2 Moment of Inertia of a Circular Disc

i. About an axis passing through its centre and perpendicular to its plane

Consider a circular disc of mass M and radius R. AB be an axis passing through its centre O and perpendicular to its plane.

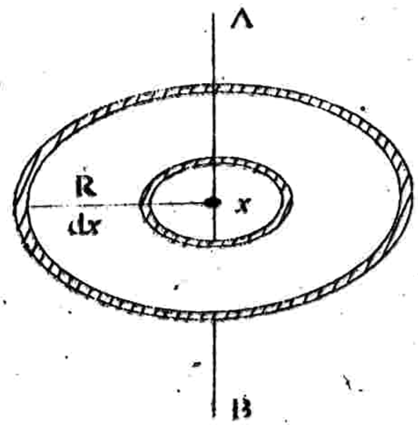


Fig. 21

Mass per unit area of the disc $= \frac{M}{\pi R^2}$

The disc is assumed to be made up of a number of concentric rings whose radii varies from 0 to R.

Consider one such ring of radius x with thickness dx.

Area of the ring $= 2\pi x.dx$

$$\therefore \text{Mass of the ring} = \frac{M}{\pi R^2} \times 2\pi x dx = \frac{2M}{R^2}.x dx$$

∴ MI of the ring about AB = mass × (radius)²

$$= \frac{2M}{R^2} x\,dx \,.\, x^2 = \frac{2M}{R^2} x^3\,dx$$

$\therefore$ MI of the disc about AB, $I = \int_0^R \frac{2M}{R^2} . x^3 dx = \frac{2M}{R^2} \int_0^R x^3 . dx$

$$= \frac{2M}{R^2} \left[\frac{x^4}{4} \right]_0^R = \frac{2M}{R^2} \times \frac{R^4}{4}$$

ie., $I = \frac{MR^2}{2}$

ii. About any diameter

The moment of inertia of the disc about any diameter is the same due to its symmetry. XX' and YY' are two mutually perpendicular diameters of the disc in its plane. Let I_d be the moment of inertia about any diameter.

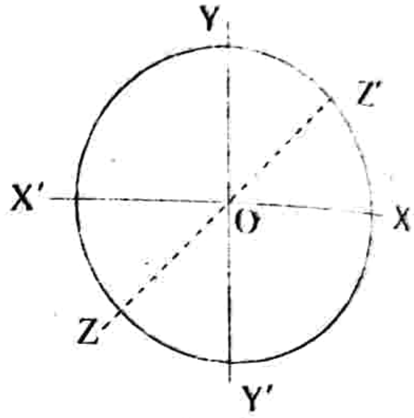


Fig. 22

Moment of inertia of the disc about an axis passing through its centre and perpendicular to its plane

$I = \frac{MR^2}{2}$

According to the perpendicular axes theorem, $I = I_d + I_d$

ie., $2 I_d = \frac{MR^2}{2}$ $\therefore$ $I_d = \frac{MR^2}{4}$

iii. About a tangent parallel to its diameter

Let AB be a tangent to the disc parallel to the diameter XX'.

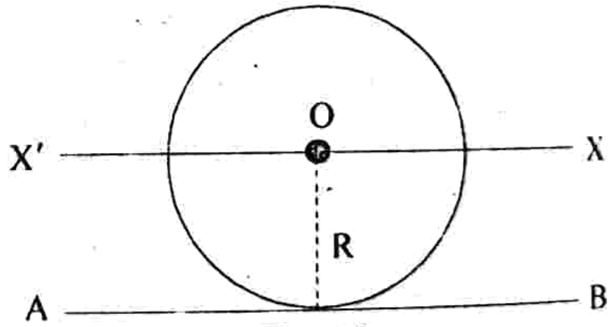


Fig. 23

Moment of inertia of the disc about any diameter

$I_d = \frac{MR^2}{4}$

According to parallel axes theorem,

$I_{AB} = I_d + MR^2 = \frac{MR^2}{4} + MR^2$

$I_{AB} = \frac{5}{4} MR^2$

7.12.3 Moment of Inertia of a Uniform Rod

i. About an axis passing through its centre and perpendicular to its length

Consider a uniform rod AB of length l and mass M. Let O be its centre. YY' is an axis passing through its centre and perpendicular to its length.

Since mass is uniformly distributed, mass/unit length of the rod $= \frac{M}{l}$

Now consider a small element of the rod of length dx at a distance of x

from the axis.

Mass of this element $= \frac{M}{l} dx$

MI of this element about YY' $= \frac{M}{l} dx . x^2$

$\therefore$ MI of the rod about YY', I $= \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx$

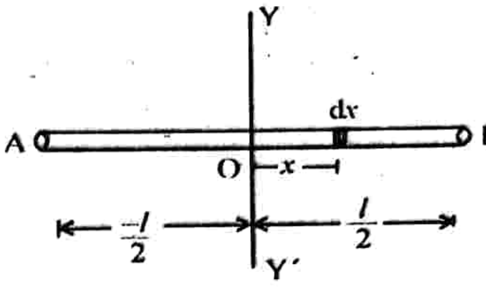


Fig. 24

$$= \frac{M}{l} \times 2 \int_0^{l/2} x^2 dx$$

$$= \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2} = \frac{2M}{l} \times \frac{1}{3} \times \frac{l^3}{8}$$

$$I = \frac{Ml^2}{12}$$

ii. About an axis passing through its one end and perpendicular to its length

Using parallel axes theorem, the moment of inertia about PQ,

$$I_{PQ} = I + M\left(\frac{l}{2}\right)^2$$

$$I_{PQ} = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$$

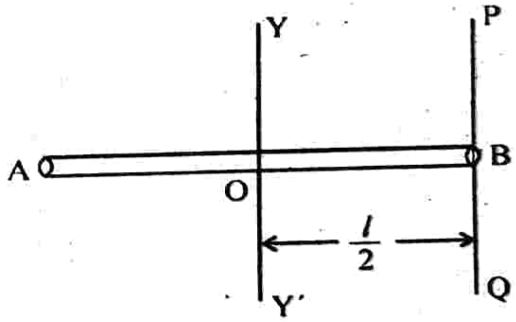


Fig. 25

7.12.4 Moment of Inertia of a Solid Cylinder

i. About its own axis

Consider a solid cylinder of mass M and radius R. Let the cylinder be rotating about an axis AB passing through its centre and along its length.

The cylinder can be considered as the combination of a number of discs.

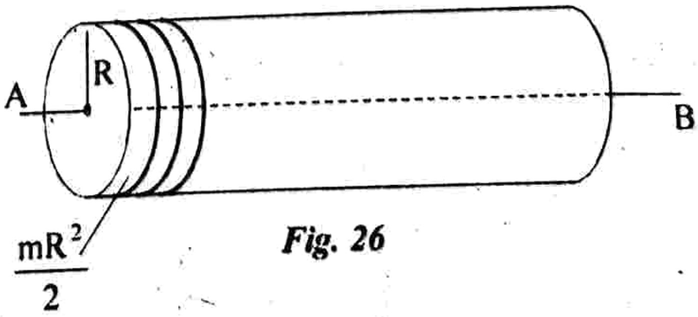


Fig. 26

MI of one disc $= \frac{mR^2}{2}$. Hence MI of the cylinder, about AB is $= I = \sum \frac{mR^2}{2} = \frac{MR^2}{2}$, since $\sum m = M$

ii. About an axis passing through its centre and perpendicular to its axis

Consider a cylinder of mass M and radius R. Let l be its length. PQ be an axis passing through its centre O and perpendicular to its own axis AB.

Mass per unit length of the cylinder

$$= \frac{M}{l} \quad .... (1)$$

Suppose the solid cylinder is made up of a number of circular discs of radius R. Consider one such disc of thickness dx at a distance of x from PQ.

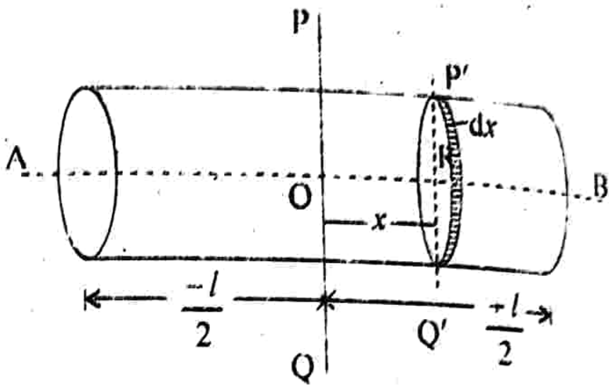


Fig. 27

$$\text{Mass of the disc} = \frac{M}{l} \cdot dx \quad .... (2)$$

The axis P'Q' is passing through the diameter of this disc.

$\therefore$ Moment of inertia about P'Q', $I_{P'Q'} = \frac{1}{4} \times \text{mass} \times (\text{Radius})^2$

$$\text{ie.,} \quad I_{P'Q'} = \frac{1}{4} \times \frac{M}{l} dx \times R^2 = \frac{MR^2}{4l} . dx \quad ..... (3)$$

Now, Moment of inertia about PQ can be found out by using parallel axes theorem.

$$\text{ie.,} \quad I_{PQ} = I_{P'Q'} + \text{mass of the disc} \times x^2 = \frac{MR^2}{4l} . dx + \frac{M}{l} dx \times x^2$$

$$= \frac{MR^2}{4l} dx + \frac{M}{l} x^2 dx \quad ...... (4)$$

$\therefore$ Moment of inertia of the cylinder about PQ,

$$I = \int_{-l/2}^{+l/2} \left[\frac{MR^2}{4l} dx + \frac{M}{l} x^2 dx\right] = \frac{MR^2}{4l} \int_{-l/2}^{l/2} dx + \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx$$

$$= \frac{MR^2}{4l} [x]_{-l/2}^{l/2} + \frac{M}{l} \left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{MR^2}{4l} \left[\frac{l}{2} - \left(\frac{-l}{2}\right)\right] + \frac{M}{3l} \left[\frac{l^3}{8} - \left(\frac{-l^3}{8}\right)\right]$$

$$= \frac{MR^2}{4l} \times l + \frac{M}{3l} \times \frac{l^3}{4} = \frac{MR^2}{4} + \frac{Ml^2}{12}$$

$$I = \frac{M}{4}\left[R^2 + \frac{l^2}{3}\right] \quad ...... (5)$$

Solved Examples

9. A uniform ring and disc have the same radii 0.5 m and mass 10 kg. Calculate the ratio of their moments of inertia about an axis passing through their centres and perpendicular to their planes.

Sol.

Given, Mass of ring/disc, $M = 10$ kg, radius of ring /disc, $R = 0.5$ m

MI of ring about AB, $I_1 = MR^2$

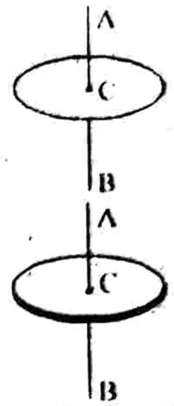
$= 10 \times (0.5)^2 = 2.5 \text{ kg m}^2$

MI of disc about AB, $I_2 = \frac{MR^2}{2}$

$= 1.25 \text{ kgm}^2$

$\therefore$ Ratio of their MI,

$$\frac{I_1}{I_2} = \frac{2.5}{1.25} = 2$$



7.13 Kinematics of Rotational Motion About a Fixed Axis

A rigid body is one in which the distance between any two pair of particles remains constant.

Consider a rigid body capable to rotate about an axis AB as shown in fig. 28. When a force is applied on it, each particle of the body revolves on a circular path of radius equal to its distance from the axis. Now the rigid body is in rotational motion and the physical quantities needed for explaining its motion are angular displacement θ, angular velocity ω and angular acceleration α.

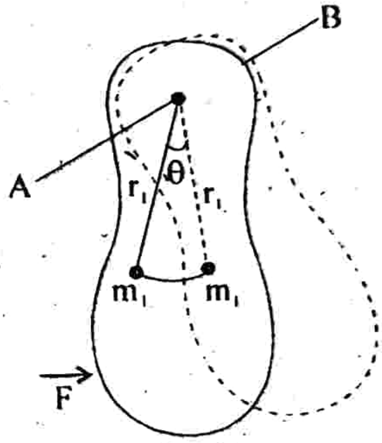


Fig. 28

These quantities were related with the translational motion as,

$x = r\theta$; $v = r\omega$; $a = r\alpha$

Equations of Rotational Motion

Just as the equations of translatory (linear) motion, we can derive equations of rotational motion. Here the angular acceleration of the rotating body is taken as a constant.

a. Angular velocity after any time

Consider a rigid body of mass m rotating about an axis with uniform angular acceleration α. Let ω_0 be its initial angular velocity. After any time t, let ω_t be its angular velocity. Now its angular acceleration, by definition

$$\alpha = \frac{\omega_t - \omega_0}{t}$$

$\therefore \omega_t - \omega_0 = \alpha t$; $\quad \omega_t = \omega_0 + \alpha t$ .... (1)

b. Angular displacement after any time

Let a rigid body capable of rotation about an axis, revolves with a uniform angular acceleration α. Let ω_0 be its initial angular velocity. After any time t, let it has an angular displacement of θ, and its angular velocity becomes ω_t.

Angular displacement = Average angular velocity × time i.e., $\theta = \left(\frac{\omega_t + \omega_0}{2}\right)t$

But $\omega_t = \omega_0 + \alpha t$

$$\therefore \theta = \left(\frac{\omega_0 + \omega_0 + \alpha t}{2}\right)t$$

i.e., $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ .... (2)

c. Angular velocity after some angular displacement

Let a rigid body rotate about an axis with uniform angular acceleration α. Let ω_0 be its initial angular velocity. After an angular displacement θ, let its angular velocity becomes ω_t.

Now from (1) equation, $(\omega_t - \omega_0) = \alpha t$ ...... (1)

Also, Average angular velocity × time = angular displacement

i.e., $\left(\frac{\omega_t + \omega_0}{2}\right)t = \theta$ or $(\omega_t + \omega_0) = \frac{2\theta}{t}$ .... (2)

Multiplying (1) and (2), we get

$$(\omega_t + \omega_0)(\omega_t - \omega_0) = \frac{2\theta}{t} \times \alpha t$$

i.e., $\omega_t^2 - \omega_0^2 = 2\alpha\theta$

$\omega_t^2 = \omega_0^2 + 2\alpha\theta$ .... (3)

d. Kinetic energy of rotation

Consider a rigid body, consisting of n particles, executing rotational motion (See fig 28). The first particle of mass m_1 is at a distance r_1 from AB, second particle of mass m_2 is at a distance r_2 from AB etc. Also the velocity of first particle is v_1 that of second particle is v_2 etc.

Kinetic energy of first particle = $\frac{1}{2}m_1 v_1^2$

Kinetic energy of 2nd particle = $\frac{1}{2}m_2 v_2^2$

Kinetic energy of nth particle = $\frac{1}{2}m_n v_n^2$

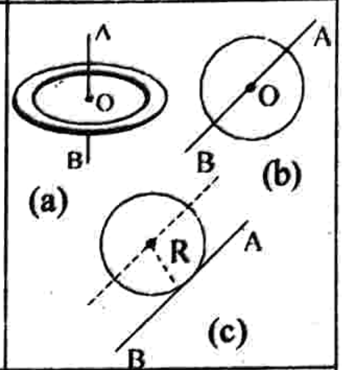
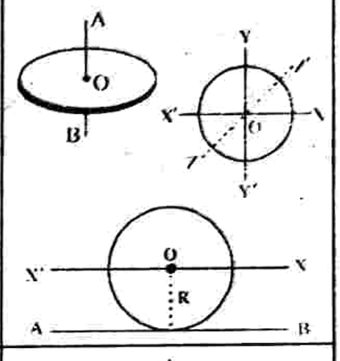
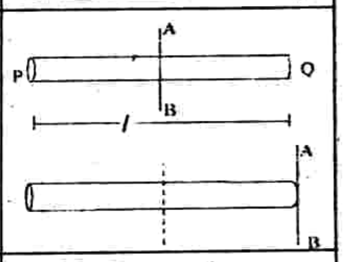
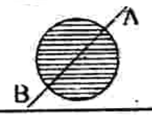
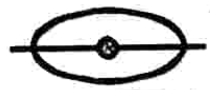
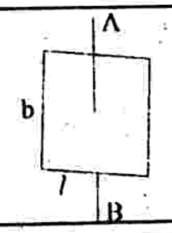
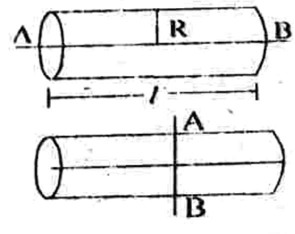
Total KE = $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 .... + \frac{1}{2}m_n v_n^2$

But $v_1 = r_1\omega$, $v_2 = r_2\omega$ etc. Substituting

$$\text{Total KE of rotation} = \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + ... + ... + \frac{1}{2}m_n r_n^2\omega^2$$

$$= \frac{1}{2}\omega^2[m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2] = \frac{1}{2}\omega^2 I$$

TABLE 2

Sl. No.	Body	Axis	MI	Figure
1.	Circular Ring	a. Through its centre and perpendicular to its plane	MR^2	
		b. About any diameter	$\frac{MR^2}{2}$	
		c. About any tangent parallel to the diameter	$\frac{3}{2}MR^2$	
2.	Circular Disc	a. Through its centre and perpendicular to its plane	$\frac{MR^2}{2}$	
		b. About any diameter	$\frac{MR^2}{4}$	
		c. About any tangent parallel to the diameter	$\frac{5}{4}MR^2$	
3.	Thin Rod	a. Through its centre and perpendicular to its length	$\frac{Ml^2}{12}$	
		b. Through one end of the rod and ⊥r to its length	$\frac{Ml^2}{3}$	
4.	Solid Sphere	a. About any diameter	$\frac{2}{5}MR^2$	
5.	Hollow Sphere (Shell)	a. About any diameter	$\frac{2}{3}MR^2$	
6.	Rectangular lamina	a. Through its centre and perpendicular to its plane	$\frac{M}{12}(l^2+b^2)$	
		b. Through one side	$\frac{M}{3}(l^2+b^2)$	
7.	Solid cylinder	a. About the axis	$\frac{1}{2}MR^2$	
		b. About the centre and perpendicular to its own axis	$\frac{M}{4}\left(R^2+\frac{l^2}{3}\right)$	

Solved Examples

10. The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform? (ii) How many revolutions does the engine make during this time?

Sol.

i. We shall use $\omega = \omega_0 + \alpha t$

ω_0 = initial angular speed in rad/s

= $2\pi \times$ angular speed in rev/s

$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$

$= \frac{2\pi \times 1200}{60} = 40\pi$ rad/s

Similarly ω = final angular speed in rad/s

$= \frac{2\pi \times 3120}{60} = 2\pi \times 52$

$= 104\pi$ rad/s

$\therefore$ Angular acceleration

$\alpha = \frac{\omega - \omega_0}{t} = 4\pi$ rad/s^2

The angular acceleration of the engine = 4π rad/s^2

ii. The angular displacement in time t is given by

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$= \left(40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2\right)$.

$= (640\pi + 512\pi) = 1152\pi$ rad

Number of revolutions = $\frac{1152\pi}{2\pi}$

= 576

11. The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds.
   i. What is its angular acceleration, assuming the acceleration to be uniform?
   ii. How many revolutions does the engine make during this time?

Sol. We shall use $\omega = \omega_0 + \alpha t$

ω_0 = initial angular speed in rad/s = $2\pi \times$ angular speed in rev/s

$= \frac{2\pi \times \text{angular speed in rev/min}}{60 \text{ s/min}}$

$= \frac{2\pi \times 1200}{60}$ rad/s $= 40\pi$ rad/s

Similarly ω = final angular speed in rad/s

$= \frac{2\pi \times 3120}{60}$ rad/s $= 2\pi \times 52$ rad/s

$= 104\pi$ rad/s

$\therefore$ Angular acceleration,

$\alpha = \frac{\omega - \omega_0}{t} = 4\pi$ rad/s^2

The angular acceleration of the engine = 4π rad/s^2

ii. The angular displacement in time t is given by

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$= (40\pi \times 16 + \frac{1}{2} \times 4\pi \times 16^2)$ rad

$= (640\pi + 512\pi)$ rad $= 1152\pi$ rad

Number of revolutions = $\frac{1152\pi}{2\pi}$

= 576

12. An electron of mass 9×10^{-31} kg revolves in a circle of radius 0.53 A° around the nucleus of hydrogen with a velocity of 2.2×10^6 ms^{-1}. Show that its angular momentum is equal to $\frac{h}{2\pi}$, where h is Planck's constant.

Sol.

Given, m = 9×10^{-31} kg,

r = 0.53 A° = 0.53×10^{-10}m

v = 2.2×10^6 ms^{-1},

h = 6.6×10^{-34} Js

L = mvr

$= 9 \times 10^{-31} \times 2.2 \times 10^6 \times 0.53 \times 10^{-10}$

$= 1.0494 \times 10^{-34}$ Js —— (1)

We have

$\frac{h}{2\pi} = \frac{6.6 \times 10^{-34}}{2 \times 3.14} = 1.0504 \times 10^{-34}$ Js —— (2)

From eqns. (1) and (2), we get,

$L \cong \frac{h}{2\pi}$