7.13 Kinematics of Rotational Motion About a Fixed Axis

A rigid body is one in which the distance between any two pair of particles remains constant.

Consider a rigid body capable to rotate about an axis AB as shown in fig. 28. When a force is applied on it, each particle of the body revolves on a circular path of radius equal to its distance from the axis. Now the rigid body is in rotational motion and the physical quantities needed for explaining its motion are angular displacement θ, angular velocity ω and angular acceleration α.

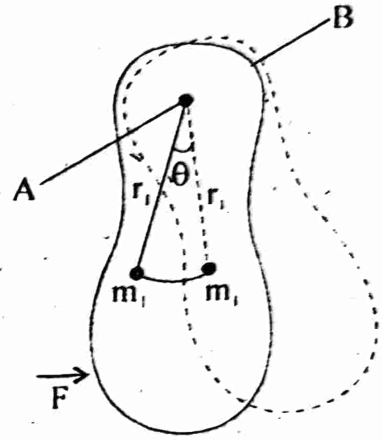


Fig. 28

These quantities were related with the translational motion as,

$x = r\theta$; $v = r\omega$; $a = r\alpha$

Equations of Rotational Motion

Just as the equations of translatory (linear) motion, we can derive equations of rotational motion. Here the angular acceleration of the rotating body is taken as a constant.

a. Angular velocity after any time

Consider a rigid body of mass m rotating about an axis with uniform angular acceleration α. Let ω_0 be its initial angular velocity. After any time t, let ω_t be its angular velocity. Now its angular acceleration, by definition

$$\alpha = \frac{\omega_t - \omega_0}{t}$$

$$\therefore \omega_t - \omega_0 = \alpha t, \qquad \omega_t = \omega_0 + \alpha t \quad \ldots\ldots\ (1)$$

b. Angular displacement after any time

Let a rigid body capable of rotation about an axis, revolves with a uniform angular acceleration α. Let ω_0 be its initial angular velocity. After any time t, let it has an angular displacement of θ, and its angular velocity becomes ω_t.

Angular displacement = Average angular velocity × time i.e., $\theta = \left(\frac{\omega_t + \omega_0}{2}\right)t$

But $\omega_t = \omega_0 + \alpha t$

$$\therefore \theta = \left(\frac{\omega_0 + \omega_0 + \alpha t}{2}\right)t$$

i.e., $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ .... (2)

c. Angular velocity after some angular displacement

Let a rigid body rotate about an axis with uniform angular acceleration α. Let ω_0 be its initial angular velocity. After an angular displacement θ, let its angular velocity becomes ω_t.

Now from (1) equation, $(\omega_t - \omega_0) = \alpha t$ ...... (1)

Also, Average angular velocity × time = angular displacement

i.e., $\left(\frac{\omega_t + \omega_0}{2}\right)t = \theta$ or $(\omega_t + \omega_0) = \frac{2\theta}{t}$ .... (2)

Multiplying (1) and (2), we get

$$(\omega_t + \omega_0)(\omega_t - \omega_0) = \frac{2\theta}{t} \times \alpha t$$

i.e., $\omega_t^2 - \omega_0^2 = 2\alpha\theta$

$\omega_t^2 = \omega_0^2 + 2\alpha\theta$ .... (3)

d. Kinetic energy of rotation

Consider a rigid body, consisting of n particles, executing rotational motion (See fig 28). The first particle of mass m_1 is at a distance r_1 from AB, second particle of mass m_2 is at a distance r_2 from AB etc. Also the velocity of first particle is v_1 that of second particle is v_2 etc.

Kinetic energy of first particle = $\frac{1}{2}m_1 v_1^2$

Kinetic energy of 2nd particle = $\frac{1}{2}m_2 v_2^2$

Kinetic energy of nth particle = $\frac{1}{2}m_n v_n^2$

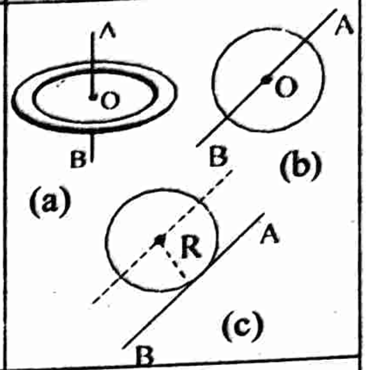
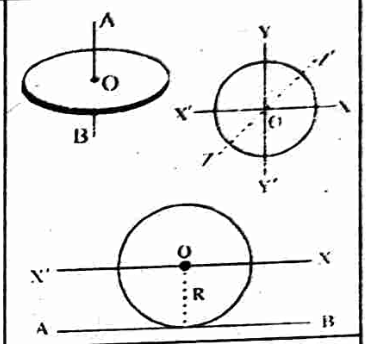
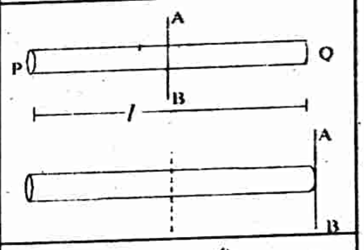
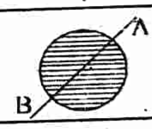
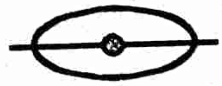
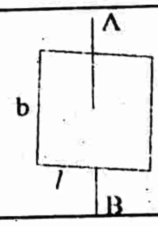
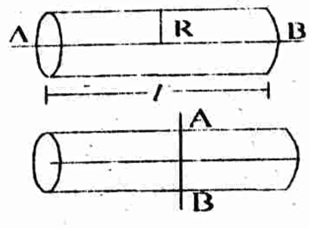
Total KE = $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 .... + \frac{1}{2}m_n v_n^2$

But $v_1 = r_1\omega$, $v_2 = r_2\omega$ etc. Substituting

Total KE of rotation = $\frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + ... + ... + \frac{1}{2}m_n r_n^2\omega^2$

$$= \frac{1}{2}\omega^2[m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2] = \frac{1}{2}\omega^2 I$$

TABLE 2

Sl. No.	Body	Axis	MI	Figure
1.	Circular Ring	a. Through its centre and perpendicular to its plane b. About any diameter c. About any tangent parallel to the diameter	MR^2 $\frac{MR^2}{2}$ $\frac{3}{2}MR^2$	
2.	Circular Disc	a. Through its centre and perpendicular to its plane b. About any diameter c. About any tangent parallel to the diameter	$\frac{MR^2}{2}$ $\frac{MR^2}{4}$ $\frac{5}{4}MR^2$	
3.	Thin Rod	a. Through its centre and perpendicular to its length b. Through one end of the rod and $\perp^r$ to its length	$\frac{Ml^2}{12}$ $\frac{Ml^2}{3}$	
4.	Solid Sphere	a. About any diameter	$\frac{2}{5}MR^2$	
5.	Hollow Sphere (Shell)	a. About any diameter	$\frac{2}{3}MR^2$	
6.	Rectangular lamina	a. Through its centre and perpendicular to its plane b. Through one side	$\frac{M}{12}(l^2+b^2)$ $\frac{M}{3}(l^2+b^2)$	
7.	Solid cylinder	a. About the axis b. About the centre and perpendicular to its own axis	$\frac{1}{2}MR^2$ $\frac{M}{4}\left(R^2+\frac{l^2}{3}\right)$	

Solved Examples

10. The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds. (i) What is its angular acceleration, assuming the acceleration to be uniform? (ii) How many revolutions does the engine make during this time?

Sol.

i. We shall use $\omega = \omega_0 + \alpha t$

ω_0 = initial angular speed in rad/s

$= 2\pi \times$ angular speed in rev/s

$= \dfrac{2\pi \times \text{angular speed in rev/min}}{60 \text{s/min}}$

$= \dfrac{2\pi \times 1200}{60} = 40\pi$ rad/s

Similarly ω = final angular speed in rad/s

$= \dfrac{2\pi \times 3120}{60} = 2\pi \times 52$

$= 104\pi$ rad/s

$\therefore$ Angular acceleration

$\alpha = \dfrac{\omega - \omega_0}{t} = 4\pi \text{ rad/s}^2$

The angular acceleration of the engine = 4π rad/s²

ii. The angular displacement in time t is given by

$\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$

$= \left(40\pi \times 16 + \dfrac{1}{2} \times 4\pi \times 16^2\right)$

$= (640\pi + 512\pi) = 1152\pi$ rad

Number of revolutions $= \dfrac{1152\pi}{2\pi}$

$= 576$

11. The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16 seconds.

i. What is its angular acceleration, assuming the acceleration to be uniform?

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12. An electron of mass 9×10^{-31} kg revolves in a circle of radius 0.53 A° around the nucleus of hydrogen with a velocity of 2.2×10^6 ms⁻¹. Show that its angular momentum is equal to $\dfrac{h}{2\pi}$, where h is Planck's constant.

Sol.

Given, m = 9×10^{-31} kg,

r = 0.53 A° $= 0.53 \times 10^{-10}$ m

v $= 2.2 \times 10^6$ ms⁻¹,

h $= 6.6 \times 10^{-34}$ Js

L = mvr

$= 9 \times 10^{-31} \times 2.2 \times 10^6 \times 0.53 \times 10^{-10}$

$= 1.0494 \times 10^{-34}$ Js —— (1)

We have

$\dfrac{h}{2\pi} = \dfrac{6.6 \times 10^{-34}}{2 \times 3.14} = 1.0504 \times 10^{-34}$ Js —— (2)

From eqns. (1) and (2), we get,

$L \cong \dfrac{h}{2\pi}$