Example 21

How many words, with or without meaning, can be formed using all letters of the word FRIDAY, using each letter exactly once? How many of them have first letter is a vowel?

(September 2013)

Solution

There are 6 letters in the word FRIDAY and all 6 letters are used exactly once.

$\therefore$ Number of words = ${}^6P_6 = 6! = 720$

There are 2 vowels.

Since, the first letter is a vowel, it can be arranged in 2 different ways.

The remaining 5 places can be arranged by the four consonants and the one vowel remaining in 5! ways.

$\therefore$ Total number of words in which the first letter is a vowel = $2 \times 5! = 240$

Example 22

The letters of the word TUESDAY are arranged in a line, each arrangement ending with letter S. How many different arrangements are possible? How many of them start with letter D?

(NCERT)

Solution

There are 7 letters in the word TUESDAY. Since the last letter is 'S', the number of arrangements is equal to the permutation of 6 letters taking all, which is ${}^6P_6 = 6! = 720$

If the words start with 'D' and end in 'S', the number of arrangements is equal to the permutation of 5 letters taking all, which is ${}^5P_5 = 5! = 120$

Example 23

In how many ways can the letters of the word ENGLISH be arranged if

i. all the letters are used at a time

ii. words start with E and end in H

iii. four letters are used at a time

(March 2010)

Solution

There are 7 different letters in the word ENGLISH.

i. If all the letters are used at a time, then

the number of words = ${}^7P_7 = 7!$

ii. If the words start with E and end in H, then two positions are fixed and we have to arrange the remaining 5 positions with the 5 letters.

$\therefore$ The number of words starting with E and ending in H = ${}^5P_5 = 5!$

iii. If 4 letters are used at a time, then the number of words formed

= permutation of 7 letters taking 4 at a time

= ${}^7P_4 = 7 \times 6 \times 5 \times 4 = 840$

Example 24

There are 10 examination papers for a course.

i. Find out in how many ways these examination papers can be arranged.

ii. Find out in how many ways these papers can be arranged so that two papers must be consecutive.

iii. Find out in how many ways these papers can be arranged so that two particular papers should not be consecutive. **(March 2011)**

Solution

i. **There are 10 examination papers**

The number of arrangements of 10 examination papers = ${}^{10}P_{10} = 10!$

ii. **Two papers are consecutive**

Consider the two particular papers as a single unit. Then there are 9 papers and they can be arranged in ${}^{9}P_{9}$ ways = 9!.

The two particular papers can be arranged in 2! ways. Hence the total number of arrangements of the examination papers so that two papers are consecutive = $9! \times 2!$

iii. **Two papers are not consecutive**

The number of arrangements of the examination papers so that two papers are not consecutive = $10! - (9! \times 2!)$

$= 9!\,(10 - 2) = 8 \times 9!$

Example 25

i. How many 4 digit numbers can be formed using the digits 3, 5, 8, 9 if repetition is not allowed?

ii. How many of these numbers end in 3?

iii. How many of the above numbers mentioned in (i) ends in 3 or 5? **(March 2012)**

Solution

i. The number of 4 digit numbers formed by using the digits 3, 5, 8, 9 when repetition is not allowed = ${}^{4}P_{4} = 4! = 24$

ii. If the number ends in 3, the number of 4 digit numbers = ${}^{3}P_{3} \times 1 = 3! = 6$

iii. If the number ends in 5, the number of 4 digit numbers = ${}^{3}P_{3} \times 1 = 3! = 6$

$\therefore$ Total number of numbers ends in 3 or 5 = 6 + 6 = 12

Example 26

Find the value of n such that ${}^{n}P_{3} = 4 \times {}^{n}P_{2}$, $n>3$ **(March 2015)**

Solution

${}^{n}P_{3} = 4 \times {}^{n}P_{2}$

$\Rightarrow n(n-1)(n-2) = 4n(n-1)$

$\Rightarrow n - 2 = 4$

$\Rightarrow n = 6$

Example 27

Prove that i. $^nP_n = 2\,(^nP_{n-2})$ ii. $^{10}P_3 = {}^9P_3 + 3\,{}^9P_2$ iii. $^nP_r = {}^{n-1}P_r + r\,.{}^{n-1}P_{r-1}$

Solution

i. We have, $^nP_n = n!$ and $2(^nP_{n-2}) = 2\dfrac{n!}{2!} = n!$ Hence $^nP_n = 2(^nP_{n-2})$

ii. $^{10}P_3 = 10.9.8 = 720$

$^9P_3 + 3\,{}^9P_2 = 9.8.7 + 3.9.8 = 9.8(7+3) = 720$. Hence $^{10}P_3 = {}^9P_3 + 3\,{}^9P_2$

iii. $^{n-1}P_r + r\,.{}^{n-1}P_{r-1} = \dfrac{(n-1)!}{(n-r-1)!} + r\,\dfrac{(n-1)!}{(n-r)!} = \dfrac{(n-r)(n-1)!}{(n-r)(n-r-1)!} + r\,\dfrac{(n-1)!}{(n-r)!}$

$$= \frac{(n-r)(n-1)!}{(n-r)!} + r\,\frac{(n-1)!}{(n-r)!} = \frac{(n-r+r)(n-1)!}{(n-r)!} = \frac{n(n-1)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!} = {}^nP_r$$

Example 28

Find the value of n such that $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}, n > 4$ **(NCERT)**

Solution

$$\frac{^nP_4}{^{n-1}P_4} = \frac{3}{5}$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{(n-1)(n-2)(n-3)(n-4)} = \frac{5}{3}$$

$$\Rightarrow \frac{n}{n-4} = \frac{5}{3}$$

$\Rightarrow 5(n-4) = 3n$

$\Rightarrow 5n - 3n = 20$

$\Rightarrow 2n = 20$

$\Rightarrow n = 10$

Example 29

Find n if $3 \times {}^nP_4 = 5 \times {}^{n-1}P_4$, $n>3$

Solution **(March 2013)**

$3 \times {}^nP_4 = 5 \times {}^{n-1}P_4$

$\Rightarrow 3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)\,(n-3)(n-4)$

$\Rightarrow 3n = 5(n-4) \Rightarrow 3n = 5n - 20$

$\Rightarrow n = 10$

Example 30

Find n if ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$. **(NCERT, March 2009)**

Solution

${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9 \quad \Rightarrow (n-1)(n-2)(n-3) : n(n-1)(n-2)(n-3) = 1 : 9$

$\Rightarrow 1 : n = 1 : 9 \Rightarrow n = 9$

Example 31

Find r if: i. ${}^{5}P_r = 2\ {}^{6}P_{r-1}$ ii. $5 \,.\, {}^{4}P_r = 6 \,.\, {}^{5}P_{r-1}$ **(NCERT, September 2010)**

iii. ${}^{5}P_r = {}^{6}P_{r-1}$ **(March 2009, August 2009, March 2012)**

Solution

i. $${}^{5}P_r = 2.{}^{6}P_{r-1} \Rightarrow \frac{5!}{(5-r)!} = 2\frac{6!}{(7-r)!} \Rightarrow \frac{1}{(5-r)!} = 2\frac{6}{(7-r)(6-r)(5-r)!}$$

$\Rightarrow (7-r)(6-r) = 12 \Rightarrow r^2 - 13r + 30 = 0 \Rightarrow (r-3)(r-10) = 0$

$\Rightarrow r = 10$ or $r = 3$

Hence $r = 3$, since ${}^{5}P_{10}$ is not defined

ii. $$5.\,{}^{4}P_r = 6.\,{}^{5}P_{r-1} \Rightarrow 5\frac{4!}{(4-r)!} = 6\frac{5!}{(6-r)!} \Rightarrow \frac{5!}{(4-r)!} = \frac{6(5)!}{(6-r)(5-r)(4-r)!}$$

$\Rightarrow (6-r)(5-r) = 6 \Rightarrow r^2 - 11r + 24 = 0 \Rightarrow (r-8)(r-3) = 0$

$\Rightarrow r = 8$ or $r = 3$

Hence $r = 3$, since ${}^{4}P_8$ is not defined

iii. $${}^{5}P_r = {}^{6}P_{r-1} \Rightarrow \frac{5!}{(5-r)!} = \frac{6!}{(7-r)!} \Rightarrow \frac{5!}{(5-r)!} = \frac{6(5)!}{(7-r)(6-r)(5-r)!}$$

$\Rightarrow (7-r)(6-r) = 6 \Rightarrow r^2 - 13r + 36 = 0 \quad \Rightarrow (r-9)(r-4) = 0$

$\Rightarrow r = 9$ or $r = 4$. Hence $r = 4$, since ${}^{5}P_9$ is not defined

Example 32

Find the value of n such that ${}^{n}P_5 = 42.{}^{n}P_3$, $n > 4$. **(March 2010)**

Solution

Given that ${}^{n}P_5 = 42.{}^{n}P_3 \Rightarrow n(n-1)(n-2)(n-3)(n-4) = 42\, n(n-1)(n-2)$

Therefore dividing both sides by $n(n-1)(n-2)$, we obtain $(n-3)(n-4) = 42$

$n^2 - 7n - 30 = 0$ or $(n-10)(n+3) = 0$ Hence $n = 10$ or -3.

Obviously, n cannot be negative. Thus, $n = 10$ is the required value.

Example 33

Find the number of different 6 letter arrangements that can be made from the letters of the word MOTHER assuming that no letter is repeated so that

i. all letters are used at a time ii. all vowels occur together

(March 2014, March 2015)

Solution

The word MOTHER has 6 letters in which there are two vowels and four consonants.

i. Total number of 6 letter arrangements = ${}^6P_6 = 720$

ii. Since the vowels occur together, consider the group of two vowels as a single object. Then the remaining 4 letters with this single group of vowels form 5 objects. These 5 objects can be arranged in ${}^5P_5 = 5! = 120$ ways.

In each arrangement, the single group of 2 vowels can be arranged among themselves in 2! ways.

Hence by FPC, the number of arrangements = ${}^5P_5 \times 2! = 120 \times 2 = 240$ ways

Example 34

If the letters of word "EQUATION" are arranged, find the number of arrangements in which no two consonants are adjacent. **(October 2011)**

Solution

There are 5 vowels and 3 consonants.

The 5 vowels can be arranged in ${}^5P_5$ ways = 5! = 120

Let * denote the position of vowels.

– * – * – * – * – * –

There are 6 positions for the consonants (– marked positions).

3 consonants can be arranged in ${}^6P_3$ ways = 120

∴ By the fundamental principle of counting, the total number of arrangements

$$= 120 \times 120 = 14400$$

Example 35

In the different 8 letter arrangement of the word 'DAUGHTER'

i. Find the number of words in which all vowels occur together.

ii. In the above arrangement, find the number of words in which all vowels do not occur together. **(September 2010)**

Solution

i. There are different 8 letters in the word in which 3 are vowels. Since the vowels occur together, consider the group of 3 vowels as a single object. Then the remaining 5 letters with this single object form 6 objects. These 6 objects can be permuted in ${}^6P_6$ ways = 6!.
In each arrangement, the single group of 3 vowels can be permuted in 3! ways.
Hence by FPC, the total number of arrangements = $6! \times 3! = 720 \times 6 = 4320$

ii. The total number of arrangements of the letters of the word = 8!
The number of arrangements in which the vowels are together = $6! \times 3!$ From (i)
∴ The number of arrangements of the word in which the vowels do not occur together

$$= 8! - (6! \times 3!) = 6!(8 \times 7 - 6) = 6! \times 50$$
$$= 720 \times 50 = 36000$$