Example 43

$^{10}C_4 + {}^{10}C_5 = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution

$$^{10}C_4 + {}^{10}C_5 = \frac{10!}{(10-4)!(4)!} + \frac{10!}{(10-5)!(5)!} = \frac{10!}{6!4!} + \frac{10!}{5!5!}$$

$$= \frac{10!}{6(5!)(4!)} + \frac{10!}{5!(5)4!} \quad \text{since } n! = n(n-1)!$$

$$= \frac{10!(5)+10!(6)}{6(5!)5(4!)} = \frac{10!(11)}{6!\,5!} = \frac{11!}{6!\,5!} = {}^{11}C_5 .$$

Another method

We have $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$\therefore {}^{10}C_4 + {}^{10}C_5 = {}^{10}C_5 + {}^{10}C_4 = {}^{10}C_5 + {}^{10}C_{(5-1)} = {}^{(10+1)}C_5 = {}^{11}C_5$

Example 44

Evaluate: i. $^{13}C_6 + {}^{13}C_5$ ii. $^{19}C_{17} + {}^{19}C_{18}$

Solution

i. $$^{13}C_6 + {}^{13}C_5 = \frac{13!}{(13-6)!.6!} + \frac{13!}{(13-5)!.5!} = \frac{13!}{7!.6!} + \frac{13!}{8!.5!} = 1716 + 1287 = 3003$$

ii. $$^{19}C_{17} + {}^{19}C_{18} = \frac{19!}{2!.17!} + \frac{19!}{1!.18!} = 171 + 19 = 190$$

Example 45

If $^nC_6 = {}^nC_5$, find n and $^nC_2$. **(March 2010)**

Solution

$$^nC_6 = {}^nC_5 \Rightarrow \frac{n!}{6!(n-6)!} = \frac{n!}{5!(n-5)!} \Rightarrow \frac{1}{6(5!)(n-6)!} = \frac{1}{5!(n-5)(n-6)!}$$

$$\Rightarrow \frac{1}{6} = \frac{1}{n-5} \qquad \Rightarrow n-5=6 \qquad \Rightarrow n = 5+6 = 11$$

Hence $^nC_2 = {}^{11}C_2 = \dfrac{11\times 10}{1\times 2} = 55$

Another method

We have $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

Hence $^nC_6 = {}^nC_5 \Rightarrow n = 5 + 6 = 11$

$\therefore {}^nC_2 = {}^{11}C_2 = \dfrac{11\times 10}{1\times 2} = 55$

Example 46

How many values of r will satisfy ${}^{22}C_{r+2} = {}^{22}C_{2r-1}$? **(October 2011)**

Solution

${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $x + y = n$

$\therefore {}^{22}C_{r+2} = {}^{22}C_{2r-1} \Rightarrow r + 2 = 2r - 1$ or $r + 2 + 2r - 1 = 22$

$\Rightarrow r = 3$ or $3r = 21$

$\Rightarrow r = 3$ or $r = 7$

$\therefore$ There are two values for r.

Example 47

Given that ${}^{n}C_{11} = {}^{n}C_{9}$, where n is a natural number. Find the value of r making ${}^{n}C_{r}$ the largest. **(October 2011)**

Solution

${}^{n}C_{x} = {}^{n}C_{y} \Rightarrow x = y$ or $n = x + y$

$\therefore {}^{n}C_{11} = {}^{n}C_{9} \Rightarrow n = 11 + 9 = 20$

${}^{n}C_{r}$ is maximum when $r = \frac{n}{2}$,

$\therefore {}^{n}C_{r}$ is maximum when $r = \frac{20}{2} = 10$

Example 48

Determine n if ${}^{2n}C_{3} = 11 \times {}^{n}C_{3}$

(March 2013)

Solution

${}^{2n}C_{3} = 11 \times {}^{n}C_{3}$

$$\Rightarrow \frac{2n(2n-1)(2n-2)}{1\times2\times3} = \frac{11\times n(n-1)(n-2)}{1\times2\times3}$$

$\Rightarrow 2(2n - 1)2(n - 1) = 11(n - 1)(n - 2)$

$\Rightarrow 4(2n - 1) = 11(n - 2)$

$\Rightarrow 8n - 4 = 11n - 22$

$\Rightarrow 3n = 18$

$\Rightarrow n = 6$

Example 49

i. If ${}^{n}C_{10} = {}^{n}C_{12}$, determine n and hence ${}^{n}C_{5}$.

ii. Determine n if ${}^{2n}C_{3} : {}^{n}C_{2} = 12 : 1$ **(NCERT)**

Solution

i. ${}^nC_{10} = {}^nC_{12} \Rightarrow n = 10 + 12 = 22$

$$\therefore {}^nC_5 = {}^{22}C_5 = \frac{22\times21\times20\times19\times18}{1\times2\times3\times4\times5} = 26334$$

ii. ${}^{2n}C_3 : {}^nC_2 = 12:1 \Rightarrow \frac{{}^{2n}C_3}{{}^nC_2} = \frac{112}{1}$

$$\Rightarrow \frac{\frac{2n(2n-1)(2n-2)}{1\times2\times3}}{\frac{n(n-1)}{1\times2}} = \frac{12}{1}$$

$$\Rightarrow \frac{4(2n-1)}{3} = 12$$

$\Rightarrow 8n - 4 = 36$

$\Rightarrow 8n = 40 \Rightarrow n = 5$

Example 50

i. If ${}^nC_2 : {}^{2n}C_1 = 3 : 2$, find n. **(March 2010)**

ii. Twenty eight matches were played in a volleyball tournament. Each team playing one against each of others. How many teams were there?

Solution

i. ${}^nC_2 : {}^{2n}C_1 = 3 : 2 \Rightarrow \frac{n(n-1)}{1\times2} : 2n = 3 : 2$

$$\Rightarrow \frac{\frac{n(n-1)}{2}}{2n} = \frac{3}{2} \Rightarrow \frac{n-1}{4} = \frac{3}{2} \Rightarrow 2n - 2 = 12 \Rightarrow 2n = 14,\ n = 7$$

ii. Let the number of teams be n.

$\therefore$ Number of matches $= {}^nC_2$

${}^nC_2 = 28 \Rightarrow \frac{n(n-1)}{2} = 28$

$\Rightarrow n^2 - n - 56 = 0$

$\Rightarrow (n - 8)(n + 7) = 0$

$\Rightarrow n = 8$ or $n = -7$

Since n cannot be negative, $n = 8$

$\therefore$ Number of teams played = 8

Example 51

In how many ways can one choose 5 books from 10 different books? **(March 2012)**

Solution

5 books can be selected from 10 books in $^{10}C_5$ ways $= \dfrac{10\times9\times8\times7\times6}{1\times2\times3\times4\times5} = 252$

Example 52

If there are 12 persons in a party and each of them shake hands with all others, what is the total number of handshakes? **(March 2012)**

Solution

If two persons shake hands, it is counted as a hand shake. So this is a problem of combinations.

So the number of hand shakes $= {}^{12}C_2 = \dfrac{12!}{10!.2!} = \dfrac{12\times11}{1\times2} = 66$

Example 53

Determine the number of 5 cards combinations out of a deck of 52 cards if there is exactly one ace in each combination. **(March 2009, NCERT)**

Solution

One ace card will be selected from 4 ace cards in $^4C_1$ ways. The remaining 4 cards will be selected from the remaining 48 cards in $^{48}C_4$ ways. Therefore by the Fundamental Principle of Counting, the total number of ways $= {}^4C_1 \times {}^{48}C_4$ ways $= 4\times194580 = 778320$ ways.

Example 54

What is the number of ways of choosing 4 cards from a pack of 52 playing cards? **(September 2012, March 2014, 2015)**

In how many of these

i. four cards of the same suit **(March 2015, September 2012)**

ii. four cards belong to four different suits **(March 2014, 2015)**

iii. are face cards

iv. two are red cards and two are black cards **(NCERT)**

v. cards are of same colour

Solution

There are 52 cards and four cards can be chosen in $^{52}C_4$ ways $= \dfrac{52!}{48!\times4!} = 270725$

i. There are four suits, namely spade, club, heart and diamond. There are 13 cards in each suit. There are $^{13}C_4$ ways of choosing 4 spades, $^{13}C_4$ ways of choosing 4 clubs, $^{13}C_4$ ways of choosing 4 hearts and $^{13}C_4$ ways of choosing 4 diamonds.

$\therefore$ Total number of ways of choosing 4 cards of same suit $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$$= 4\times{}^{13}C_4 = 4\times\frac{13\times12\times11\times10}{1\times2\times3\times4} = 2860$$

ii. Since the four cards belong to four different suits, one card should be drawn from each suit.
One spade card can be chosen in $^{13}C_1$ ways.
One club card can be chosen in $^{13}C_1$ ways
One heart card can be chosen in $^{13}C_1$ ways and one diamond card can be chosen in $^{13}C_1$ ways.
Hence by the multiplication principle, required number of ways

$$= {}^{13}C_1 . {}^{13}C_1 . {}^{13}C_1 . {}^{13}C_1$$

$$= ({}^{13}C_1)^4 = (13)^4 = 28561$$

iii. There are 3 face cards in each suit. Hence there are 12 face cards in the deck of cards and 4 cards are to be selected from these 12 cards. This can be done in $^{12}C_4$ ways.

$\therefore$ The number of ways of selecting four face cards $= {}^{12}C_4 = \dfrac{12\times11\times10\times9}{1\times2\times3\times4} = 495$

iv. There are 26 black cards and 26 red cards.

$\therefore$ Two red cards and two black cards can be selected in $^{26}C_2 \times {}^{26}C_2$ ways $= ({}^{26}C_2)^2$

$$= \left(\frac{26\times25}{1\times2}\right)^2 = 105625$$

v. Four red cards or four black cards can be selected from 26 cards in $^{26}C_4$ ways each.

$\therefore$ Required number of ways $= {}^{26}C_4 + {}^{26}C_4$

$$= 2 \times {}^{26}C_4 = \frac{2\times26\times25\times24\times23}{1\times2\times3\times4} = 29900$$

Example 55

A student is instructed to answer 8 out of 12 questions.

i. How many different ways can he choose the questions?

ii. How many different ways can he choose the questions so that question number 1 will be included? **(October 2011)**

iii. How many different ways can he choose the questions so that question number 1 will be included and question number 10 will be excluded?

Solution

i. From 12 questions, 8 questions can be selected in $^{12}C_8$ ways $= {}^{12}C_4$

$$= \frac{12\times11\times10\times9}{1\times2\times3\times4} = 495$$

ii. When question number 1 is included, then we have to select 7 questions from 11 questions.
Number of ways of selection $= {}^{11}C_7$

$$= {}^{11}C_4 = \frac{11\times10\times9\times8}{1\times2\times3\times4} = 330$$

iii. When question number 1 is included and question number 10 is excluded, we have to select 7 questions from 10 questions.

$\therefore$ Number of ways of selection $= {}^{10}C_7 = {}^{10}C_3 = \dfrac{10\times 9\times 8}{1\times 2\times 3} = 120$

Example 56

In how many ways can a committee of 3 men and 2 women be selected out of 7 men and 5 women? **(March 2012)**

Solution

3 men can be selected from 7 men in ${}^7C_3$ ways.

2 women can be selected from 5 women in ${}^5C_2$ ways.

$\therefore$ Total number of ways of selection $= {}^7C_3 \times {}^5C_2 = 35 \times 10 = 350$

Example 57

In how many ways can a committee of 3 men and 2 women be formed from a group of 5 men and 4 women if Mr.A is always included and Mrs.B is never included? **(October 2011)**

Solution

Since Mr.A is always included, we can select 2 men from 4 men in ${}^4C_2$ ways.

Since Mrs.B is never included, we can select 2 women from 3 women in ${}^3C_2$ ways.

$\therefore$ Total number of ways of selection $= {}^4C_2 \times {}^3C_2 = 6 \times 3 = 18$

Example 58

A committee of 3 persons is to be constituted from a group of 2 men and 3 women.

a. In how many ways can this be done?

b. How many of these committees would consist of 1 man and 2 women? **(March 2010)**

Solution

a. Number of men = 2

Number of women = 3

Total number of persons = 2 + 3 = 5

Since 3 persons are selected from a group of 5 persons, the order of persons do not matter. Hence the number of committee = combination of 5 different persons taken 3 at a time

$$= {}^5C_3 = \frac{5\times 4\times 3}{1\times 2\times 3} = 10$$

b. The committee should contain 1 man and 2 women.

1 man can be selected from 2 men in ${}^2C_1$ ways

2 women can be selected from 3 women in ${}^3C_2$ ways

$\therefore$ By fundamental principle of counting, the number of committees $= {}^2C_1 . {}^3C_2 = 2 \times 3 = 6$