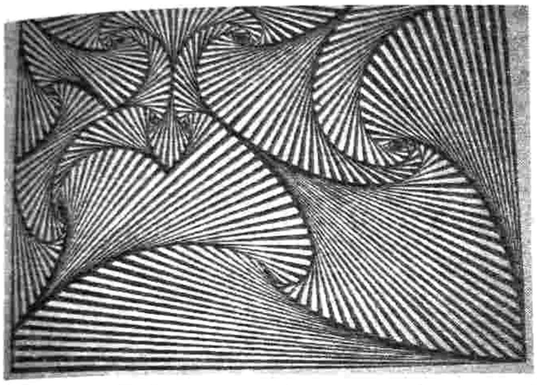
STRAIGHT LINES

10



"What you should learn"

GASPARD MONGE (9 May 1746 – 28 July 1818) was a French mathematician and the inventor of descriptive geometry. He gave the modern *point-slope* form of the equation of a line.

10.1 INTRODUCTION

The geometry that we studied in earlier classes was based on the axioms laid by Euclid. The tools of algebra were not used in it. In 1637, Rene Descartes provided a new tool for unifying the two branches of mathematics such as algebra and geometry. This unification led to a new branch of mathematics called 'Analytic Geometry' (Coordinate geometry). In this branch of mathematics, methods of algebra are used to solve geometrical problems. This is done by representing the points in a plane by ordered pairs of real numbers, lines and curves by algebraic equations.

Coordinate of a point

Let P be any point. From P, draw PL and PM perpendicular to the x and y axes. The length OL (or PM) is called the **x coordinate** or **abscissa** of the point P. The length OM (or PL) is called the **y coordinate** or the **ordinate** of the point P. Any point whose abscissa is x and ordinate y is named as the point (x, y)

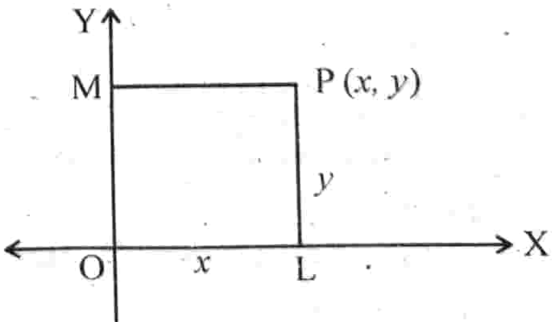


Fig. 10. 1

Distance Formula

To find the distance between two points (x_1, y_1) and (x_2, y_2).

Let P (x_1, y_1) and Q (x_2, y_2) be the two points in xy - plane. With the two points, a right triangle PRQ is formed as shown in Fig 10.2. The length of the vertical side (QR) of the triangle is $y_2 - y_1$ and the length of the horizontal side (PR) is $x_2 - x_1$

$\therefore$ By Pythagoras theorem, $PQ^2 = PR^2 + RQ^2$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

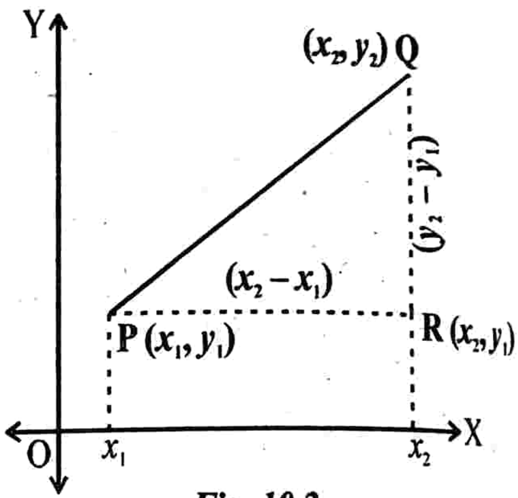


Fig. 10.2

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

or $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

The distance of a point P (x, y) from the origin O (0, 0) is given by $OP = \sqrt{x^2 + y^2}$

If PQ is parallel to x - axis, then $y_1 = y_2$ $\therefore$ PQ $= \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$

If PQ is parallel to y - axis, then $x_1 = x_2$ $\therefore$ $PQ = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$

Collinear points are points lying on the same line.

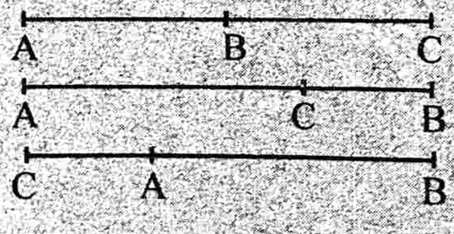
STUDY TIPS

Three points A, B and C are ***collinear*** or lie on a line if one of the following is true

i. AB + BC = AC

ii. AC + CB = AB

iii. CA + AB = CB



Example 1

1. Consider the points A (6, 2), B(3, −1) and C (−2, 4)
   i. Find AB, BC and AC
   ii. Show that ΔABC is right angled.

Solution

i. $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3-6)^2 + (-1-2)^2} = \sqrt{9+9} = \sqrt{18}$

$BC = \sqrt{(-2-3)^2 + (4-{}^{-}1)^2} = \sqrt{25+25} = \sqrt{50}$

$AC = \sqrt{(-2-6)^2 + (4-2)^2} = \sqrt{64+4} = \sqrt{68}$

ii. Consider $AB^2 + BC^2 = 18 + 50 = 68$

$= AC^2$

Thus in ΔABC, $AB^2 + BC^2 = AC^2$ $\therefore$ ΔABC is right angled.

Section Formula

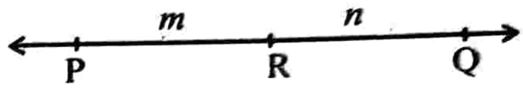


Fig. 10.3

Let R be a point on the line PQ.

i. If R lies between the points P and Q, we say R divides the line segment PQ internally in the ratio PR : RQ = $m : n$ (fig. 10.3)

ii. If R is a point on the line PQ produced or QP produced, then R divides the line segment PQ externally in the ratio PR : RQ = $m : n$ (fig. 10.4)

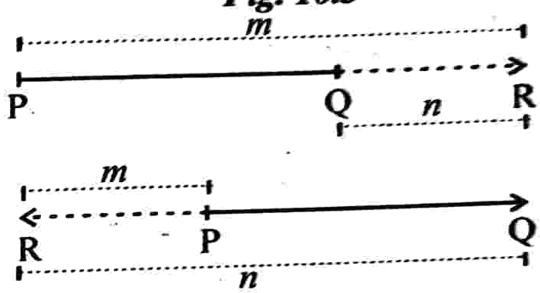


Fig. 10.4

Internal division

Let P (x_1, y_1) and Q (x_2, y_2) be the given points. Let R (x, y) divide PQ internally in the ratio $m : n$. Draw PL, RN, QM perpendicular to the x - axis. Also draw $PS \perp RN$ and $RT \perp QM$ as shown in fig. 10.5.

Here $\frac{PR}{RQ} = \frac{m}{n}$

Triangles PSR and RTQ are similar

$\therefore \frac{PR}{RQ} = \frac{PS}{RT} = \frac{SR}{TQ} \Rightarrow \frac{PS}{RT} = \frac{m}{n} = \frac{SR}{TQ}$

From the figure, $PS = x - x_1$,

$RT = x_2 - x$, $SR = y - y_1$ and

$TQ = y_2 - y$

$\therefore \frac{x - x_1}{x_2 - x} = \frac{m}{n} = \frac{y - y_1}{y_2 - y}$

$\Rightarrow \frac{x - x_1}{x_2 - x} = \frac{m}{n} \Rightarrow m(x_2 - x) = n(x - x_1)$

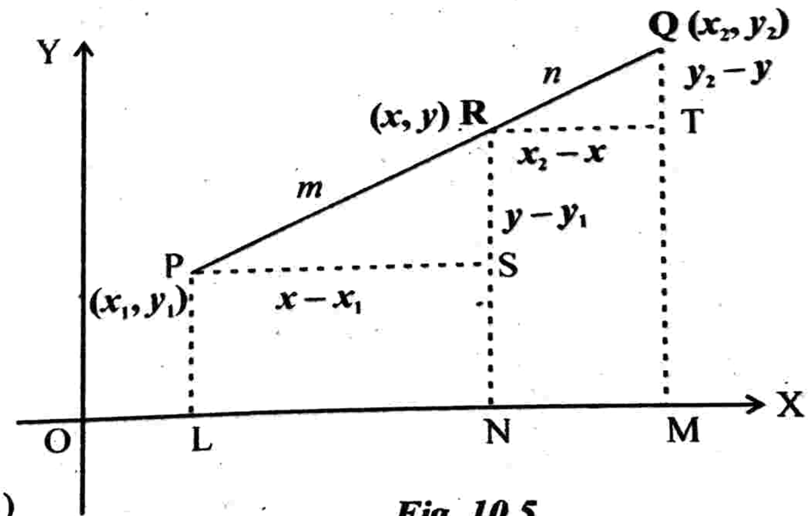


Fig. 10.5

$\Rightarrow mx_2 - mx = nx - nx_1 \quad \Rightarrow (m+n)x = mx_2 + nx_1 \qquad \therefore x = \dfrac{mx_2 + nx_1}{m+n}$

Similarly $y = \dfrac{my_2 + ny_1}{m+n}$

$\therefore$ R is the point $\left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n}\right)$

Fig. 10.6 shows an easy way to remember the section formula (internal division).

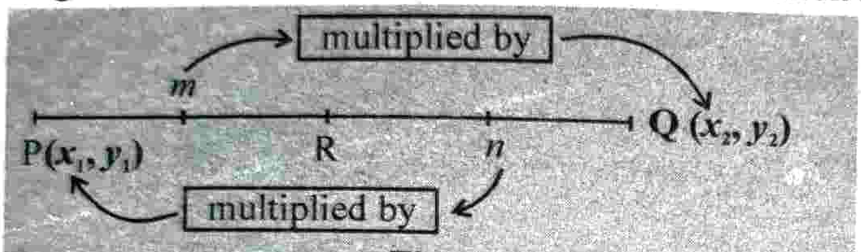


Fig. 10.6

External division

To write the coordinates of the point which divides the join of two given points externally in the ratio $m : n$, replace n by $-n$ in the formula for internal division.

$\therefore$ R is the point $\left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}\right)$

STUDY TIPS

- x - axis divides the line segment joining (x_1, y_1) and (x_2, y_2) in the ratio $-y_1 : y_2 \quad (y_1 \neq y_2)$
- y - axis divides the line segment in the ratio $-x_1 : x_2 \quad (x_1 \neq x_2)$

Midpoint of a line segment

The *midpoint* R of the line segment joining P (x_1, y_1) and Q (x_2, y_2) divides the line segment in the ratio 1 : 1

$\therefore$ ***The midpoint of PQ is R*** $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Example 2

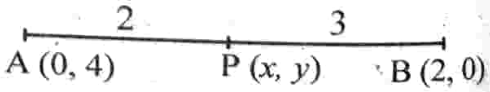
Find a point on the line joining the points (0, 4) and (2, 0) dividing the line segment

i. Internally in the ratio 2 : 3   ii. Externally in the ratio 3 : 2

Solution

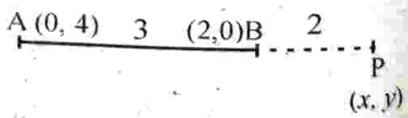
Let A (0, 4) and B (2, 0) be the points.

i. **Internal division**: Ratio 2:3



Coordinates of P are $\left(\dfrac{2(2)+3(0)}{2+3}, \dfrac{2(0)+3(4)}{2+3}\right) = \left(\dfrac{4}{5}, \dfrac{12}{5}\right)$

ii. **External division** : Ratio 3:2



Coordinates of P are $\left(\dfrac{3(2)-2(0)}{3-2}, \dfrac{3(0)-2(4)}{3-2}\right) = (6, -8)$

Area of a Triangle

Let A (x_1, y_1), B(x_2, y_2) and C (x_3, y_3) be the vertices of ΔABC.
Draw AL, BM and CN perpendicular to x - axis.
Area of ΔABC = Area of trapezium ALNC + Area of trapezium (CNMB) – Area of trapezium ALMB

$$= \frac{1}{2}(LA + NC) LN + \frac{1}{2}(NC + MB) NM - \frac{1}{2}(LA + MB) LM$$

$$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}[x_3 y_1 - x_1 y_1 + x_3 y_3 - x_1 y_3 + x_2 y_3 - x_3 y_3 + x_2 y_2 - x_3 y_2 - x_2 y_1 + x_1 y_1 - x_2 y_2 - x_1 y_2]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Area of ΔABC

$$= \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

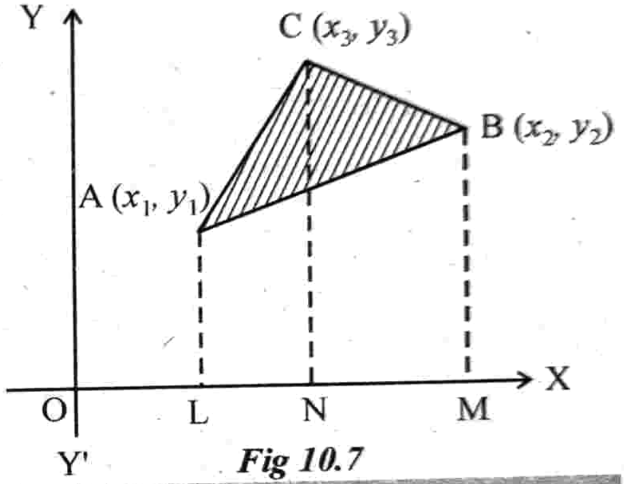


Fig 10.7

NOTE

- Area is always non-negative (positive).
- If area of triangle ABC is zero, then the points A, B, C are collinear.

Example 3

Find the area of the triangle whose vertices are (4, 4), (3, – 2) and (– 3, 16).

Solution

$$\text{Area} = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

$$= \frac{1}{2}|4(-2-16) + 3(16-4) + (-3)(4-(-2))| = \frac{1}{2}|-72 + 36 - 18| = |-27|$$

$\therefore$ Required area = 27 sq. units

Centroid of a triangle

Let D, E and F be the midpoints of the sides BC, CA and AB of ΔABC. Then AD, BE and CF are three medians of ΔABC. The medians of a triangle are intersecting at a point G, called the ***centroid*** of the triangle. The centroid divides each median in the ratio 2 : 1

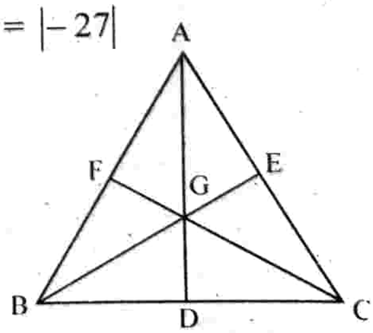


Fig 10.8

Centroid of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3)

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the given vertices. Let D be the midpoint of BC.

$\therefore$ D is the point $\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right)$

The centroid G divides median AD in the ratio 2 : 1

By using section formula, the coordinate of G is

$$\left(\frac{2\left(\frac{x_2+x_3}{2}\right)+1(x_1)}{2+1}, \frac{2\left(\frac{y_2+y_3}{2}\right)+1(y_1)}{2+1}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

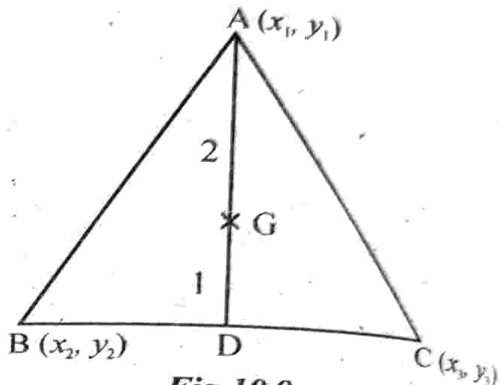


Fig 10.9

10.2 SLOPE OF A LINE

Inclination of a line

The angle made by the line with the positive direction of x - axis measured in the anti clockwise direction is called the ***inclination of the line.*** (See fig.10.10)

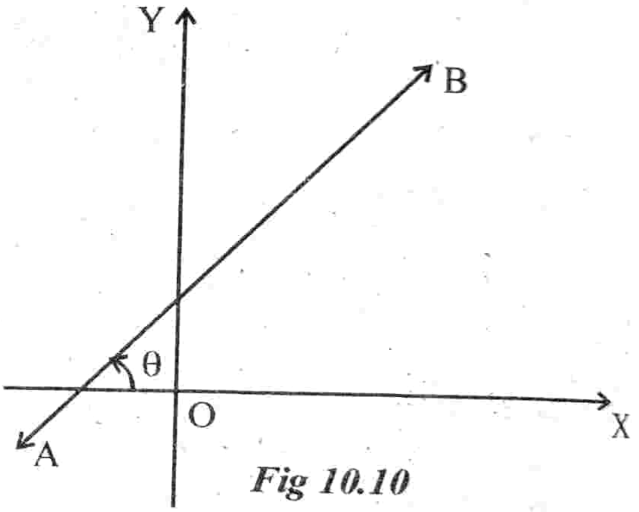


Fig 10.10

For example, in fig. 10.11 children are climbing at an angle of inclination α and descending at an angle of inclination β.

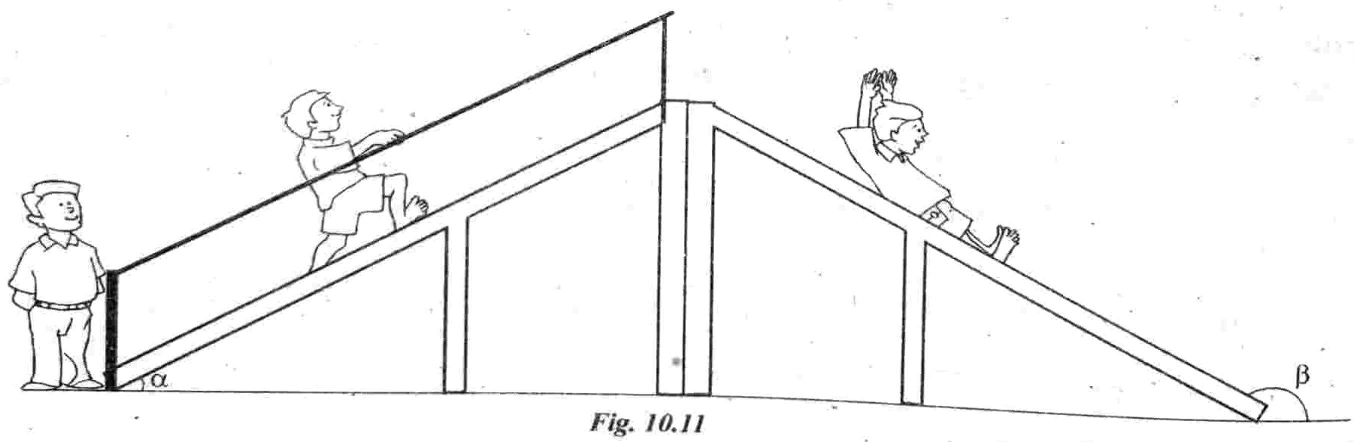
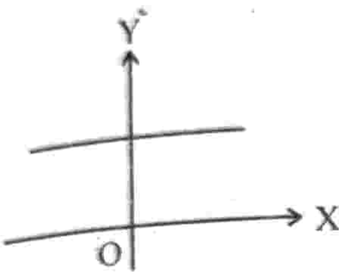


Fig. 10.11

- The inclination of a line always lies between 0° and 180°.
- The inclination of a line parallel to x - axis is 0°.
- The inclination of a line parallel to y - axis is 90°.

The *slope* of a non-vertical line is defined as the tangent of its inclination i.e., if θ is the inclination of a line, then slope of the line is $\tan\theta$. The slope of a non-vertical line is generally denoted by m, hence $m = \tan\theta$. The slope of a line may be positive, zero or negative.



when $\theta = 0°$, $m = 0$

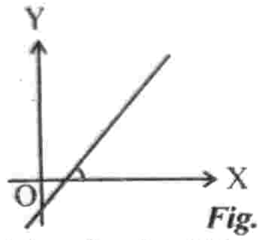
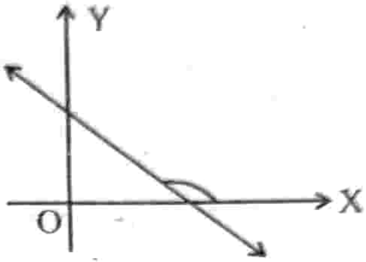


Fig. 10.12

When $0 < \theta < 90°$, $m > 0$, positive



when $90° < \theta < 180°$, $m < 0$, negative

- Slope of any line parallel to the x - axis is zero.
- Slope of any line parallel to the y - axis is not defined as tan90° is not defined.
- Three points A, B and C are collinear if and only if slope of AB = slope of BC = slope of AC

Example 4

Find the slope of a line whose angle of inclination with respect to x - axis is

i. 60° **(NCERT)** ii. 45° iii. 150° **(March 2012)**

Solution

i. Slope, $m = \tan 60° = \sqrt{3}$

ii. Slope, $m = \tan 45° = 1$

iii. Slope, $m = \tan 150° = \tan(180° - 30°) = -\tan 30° = \frac{-1}{\sqrt{3}}$

Example 5

What can be said regarding inclination of a line if its slope is

i. positive ii. zero iii. negative iv. not defined

Solution

i. The inclination is acute

ii. Line is parallel to x - axis or inclination is zero.

iii. The inclination is obtuse

iv. The inclination is right angle or line is perpendicular to x - axis

10.2.1 Slope of a line when coordinates of any two points on the line are given

Slope of a line joining two points

Let A (x_1, y_1) and B (x_2, y_2) be the two given points. Let θ be the inclination of the line AB with x - axis $\therefore$ Slope, $m = \tan\theta$

Case (i) : When angle θ is acute

Draw AM and BN perpendicular to x-axis. Draw AC perpendicular to BN

Since AC is parallel to x-axis and BA is the transversal $\angle BAC = \theta$ (corresponding angles are equal)

$AC = MN = ON - OM = x_2 - x_1$,

$BC = BN - CN = y_2 - y_1$

From the right triangle ACB we get,

$$\tan\theta = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } m = \frac{y_1 - y_2}{x_1 - x_2}, \; x_1 \neq x_2$$

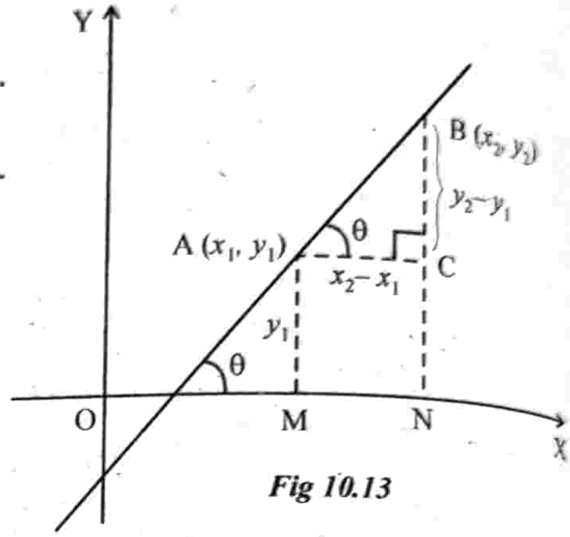


Fig 10.13

The slope m of the non vertical line through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$

Case (ii) When angle θ is obtuse

- Draw AM and BN perpendicular to x - axis, Draw AC perpendicular to BN since AC is parallel to x axis and BA is the transversal $\angle BAC = 180° - \theta$ (corresponding angles are equal)

$AC = MN = OM - ON = x_1 - x_2$

$BC = BN - CN = y_2 - y_1$

From the right triangle ACB we get

$$\tan(180° - \theta) = \frac{BC}{AC} = \frac{y_2 - y_1}{x_1 - x_2}$$

$$\Rightarrow -\tan\theta = \frac{y_2 - y_1}{x_1 - x_2}$$

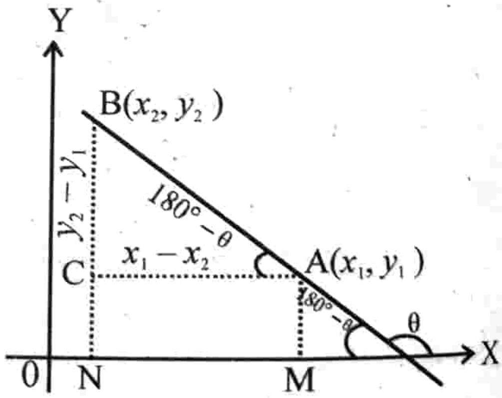


Fig 10.14

$$\therefore \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}, \; m = \frac{y_2 - y_1}{x_2 - x_1}, \; x_1 \neq x_2$$

The slope 'm' of the non vertical line through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$

STUDY TIP

Slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Caution !

If $x_2 = x_1$, slope is not defined (But we can say that the line is parallel to the y - axis)

Example 6

Find the slope of the lines

i. passing through the points (3, –2) and (–1, 4) **(March 2014, 2015)**

ii. passing through the points (3, –2) and (7, –2)

iii. passing through (3, –2) and (3, 4)

Solution

Let m be the slope $\therefore m = \dfrac{y_2 - y_1}{x_2 - x_1}$

i. slope $m = \dfrac{4 - {}^{-}2}{-1-3} = \dfrac{6}{-4} = \dfrac{-3}{2}$

ii. slope $m = \dfrac{-2 - {}^{-}2}{7-3} = \dfrac{0}{4} = 0$

iii. slope $m = \dfrac{4 - {}^{-}2}{3-3} = \dfrac{6}{0}$, which is not defined.

Another Method

ii. Since the y coordinates are equal, the line is parallel to x axis. $\therefore$ Slope = 0

iii. Since the x coordinates are equal, the line is parallel to y axis. $\therefore$ Slope is not defined.

10.2.2 Conditions for parallelism and perpendicularity of lines in terms of their slopes

Condition for parallelism of two non - vertical lines

Let l_1 and l_2 be two non-vertical lines having inclinations α and β. Let their slopes be m_1 and m_2 respectively. Then $m_1 = \tan\alpha$ and $m_2 = \tan\beta$.

Suppose l_1 and l_2 are parallel, then $\alpha = \beta \Rightarrow \tan\alpha = \tan\beta \Rightarrow m_1 = m_2$ i.e., slopes are equal.

Conversely, suppose $m_1 = m_2 \Rightarrow \tan\alpha = \tan\beta$ $\Rightarrow \alpha = \beta$ $\therefore$ lines are parallel

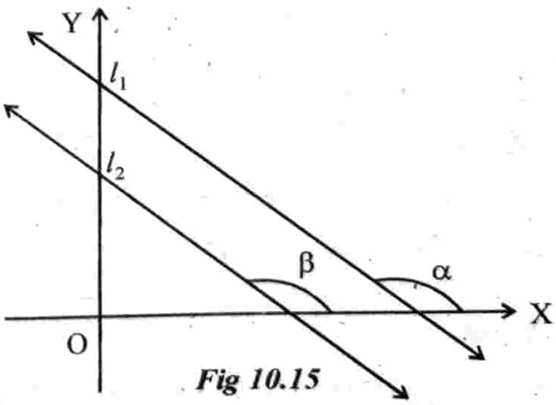


Fig 10.15

- ***Two non-vertical lines are parallel if and only if their slopes are equal.***

Condition for perpendicularity of two non vertical lines

Let l_1 and l_2 be two non-vertical lines having inclinations α and β.

Let their slopes be m_1 and m_2. Then $m_1 = \tan\alpha$ and $m_2 = \tan\beta$.

Suppose the lines l_1 and l_2 are perpendicular, then $\beta = \alpha + 90°$

$$\tan\beta = \tan(\alpha + 90°) = -\cot\alpha = \frac{-1}{\tan\alpha}$$

i.e., $m_2 = \frac{-1}{m_1}$ or $m_1 m_2 = -1$

Conversely if $m_1 m_2 = -1 \Rightarrow \tan\alpha\tan\beta = -1$

Then $\tan\alpha = \frac{-1}{\tan\beta} = -\cot\beta = \tan(90° + \beta)$

i.e., α and β differ by 90°

$\therefore$ The two lines l_1 and l_2 are perpendicular.

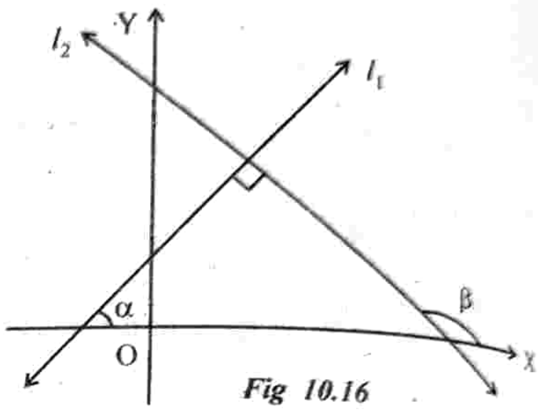


Fig 10.16

NOTE

Two non-vertical lines are perpendicular if and only if their slopes are negative reciprocals of each other. ie, If the slope of a line is m, then the slope of its perpendicular is $\frac{-1}{m}$

Example 7

Find whether the line through (– 3, 2) and (3, 3) is perpendicular or parallel or neither perpendicular nor parallel to the line through (– 2, – 1) and (4, 0). **(March 2006)**

Solution

Let A(– 3, 2), B(3, 3), C(– 2, – 1) and D(4, 0) be the points on the two lines.

Slope of AB $= \frac{3-2}{3-(-3)} = \frac{1}{6}$ and slope of CD $= \frac{0-(-1)}{4-(-2)} = \frac{1}{6}$

Slopes are equal. $\therefore$ The two lines are parallel.

Example 8

i. Find the slope of the line joining (–2, 6) and (4, 8).

ii. Find the value of x, if the above line is perpendicular to the line joining (8, 12) and (x, 24). **(NCERT, March 2010)**

Solution

i. Let A (–2, 6) and B(4, 8) be the points.

Slope of the line AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-6}{4-^-2} = \frac{2}{6} = \frac{1}{3}$

ii. Let C(8, 12) and D(x, 24) be the points.

Slope of the line CD = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{24-12}{x-8} = \frac{12}{x-8}$

Since the line AB is perpendicular to the line CD, we get (slope of AB) (slope of CD) = – 1

i.e., $\frac{1}{3}\left(\frac{12}{x-8}\right) = -1 \Rightarrow \frac{4}{x-8} = -1$

Hence $4 = -x + 8$ or $x = 4$