2nd year Mathematics (Commerce), March 2021

Detailed Solution

PART A.

1) i) $B' = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$

ii) $A + B' = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 5 & 6 \end{bmatrix}$

2) i)

$|A| = 12 - 0 = 12$

$|B| = 1 - 0 = 1$

$LHS = |A| \cdot |B| = 12 \times 1 = 12$

$AB = \begin{bmatrix} 4+0 & 8+0 \\ 1+0 & 2+3 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 1 & 5 \end{bmatrix}$

$RHS = |AB| = (4)(5) - (1)(8) = 20 - 8 = 12$

$\therefore |A| \cdot |B| = |AB|.$

3)

$LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2) = 0 + 2 = 2$

$RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 - 1) = 0^2 - 1 = -1$

$\therefore LHL \neq RHL$

$f(x)$ is not continuous at $x = 0$.

4) Eqn. is $\begin{vmatrix} x-3 & y-1 & z-2 \\ -1-3 & 4-1 & 1-2 \\ 1-3 & 0-1 & 5-2 \end{vmatrix} = 0$

$\begin{vmatrix} x-3 & y-1 & z-2 \\ -4 & 3 & -1 \\ -2 & -1 & 3 \end{vmatrix} = 0$

$(x-3)(9-1) - (y-1)(-12-2) + (z-2)(4+6) = 0$

$(x-3)(8) - (y-1)(-14) + (z-2)(10) = 0$

$(x-3)(4) - (y-1)(-7) + (z-2)(5) = 0$

$4x - 12 + 7y - 7 + 5z - 10 = 0$

$4x + 7y + 5z - 29 = 0.$

5) Let r, A be the radius and area of the circle.

$$\frac{dr}{dt} = 2\,cm/s$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$= 2\pi \times 15 \times 2 = 60\pi \ cm^2/s.$$

6) $f(x) = x(x-1)$ in $[0,1]$

It is a polynomial fun. So it is continuous in the closed interval $[0,1]$

$$f'(x) = x(1) + (x-1)(1)$$
$$= x + x - 1$$
$$= 2x - 1$$

$\therefore f$ is differentiable in $(0,1)$

$$f(0) = 0(0-1) = 0$$
$$f(1) = 1(1-1) = 0$$

$$\therefore f(0) = f(1)$$

Then there exists a constant 'c' such that $f'(c) = 0$

$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2} \in (0,1)$$

Hence verified Rolle's theorem.

7) $a * b = \text{maximum}(a,b)$.

*	1	2	3	4
1	1	2	3	4
2	2	2	3	4
3	3	3	3	4
4	4	4	4	4

8 i) $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \in [0, \pi]$

ii) Put $x = \tan\theta \Rightarrow \tan^{-1}x = \theta$

The expn $= \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$= \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left[\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right]$

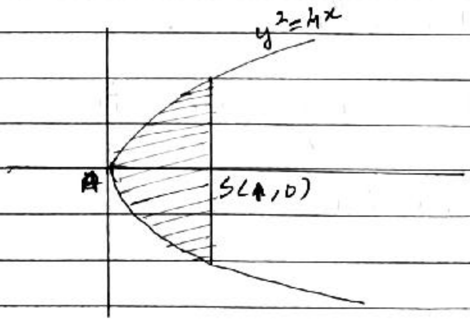
$= \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)$

$= \tan^{-1}\left(\tan\theta/2\right)$

$= \frac{\theta}{2}$

$= \frac{1}{2}\tan^{-1}x.$

9)



Req. area $= 2\int_0^1 y\,dx$

$= 2\int_0^1 2\sqrt{x}\,dx$

$= 4 \times \frac{2}{3}\left(x^{3/2}\right)_0^1$

$= \frac{8}{3}\left[1-0\right] = \frac{8}{3}$ Sq. units.

10) $f(x) = 2x^3 - 6x^2 - 18x + 5$

$f'(x) = 6x^2 - 12x - 18 = 6(x-3)(x+1)$

$f'(x) = 0 \Rightarrow 6x^2 - 12x - 18 = 0$

$\rightarrow x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\therefore x = 3$ (or) $x = -1$

$\therefore$ the intervals may be

$(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$

when $x \in (-\infty, -1)$: Say $x = -2$

$f'(-2) = 6(-2-3)(-2+1) > 0 \uparrow$

when $x \in (-1, 3)$: Say $x = 0$

$f'(0) = 6(0-3)(0+1) = 6(-3)(1) < 0 \downarrow$

when $x \in (3, \infty)$: Say $x = 4$

$f'(4) = 6(4-3)(4+1) = 6(1)(5) > 0 \uparrow$

a) $f(x)$ is increasing in the intervals $(-\infty, -1)$ and $(3, \infty)$

b) $f(x)$ is decreasing in the interval $(-1, 3)$.

PART B

11)

i) $\text{adj } A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

ii) LHS $= A \cdot (\text{adj } A)$

$= \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6-4 & -3+3 \\ 8-8 & -4+6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ —— (1)

$|A| = 6-4 = 2$

$RHS = |A| \cdot I = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ —— (2)

From (1) and (2)

$$A\,(\text{adj} A) = |A| \cdot I$$

12)

i) a) $\tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

ii) $LHS = \tan^{-1}\left(\frac{3}{2}\right) - \tan^{-1}\left(\frac{1}{5}\right)$

$= \tan^{-1}\left[\dfrac{\frac{3}{2} - \frac{1}{5}}{1 + \frac{3}{2} \times \frac{1}{5}}\right]$

$= \tan^{-1}\left[\dfrac{15-2}{10+3}\right] = \tan^{-1}\left(\frac{13}{13}\right)$

$= \tan^{-1}(1) = \dfrac{\pi}{4} = RHS.$

13)

i) a) 2×3

ii) $x = 3$

$x + y = 5$

$y = 5 - 3 = 2$

$z + 4 = 7$

$z = 7 - 4 = 3.$

14 i) b) 2

ii) $y\,dy = \dfrac{1}{x^2+1}\,dx$

$\int y\,dy = \int \dfrac{1}{1+x^2}\,dx$

$\dfrac{y^2}{2} = \tan^{-1}x + C$

15 i)

$$\bar{a} = \hat{i} + 3\hat{j} - 3\hat{k}$$
$$\bar{b} = 2\hat{i} + \hat{j} + \hat{k}$$

VE eqn: $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$$= (\hat{i} + 3\hat{j} - 3\hat{k}) + \lambda\left[(2-1)\hat{i} + (1-3)\hat{j} + (1-^{-}3)\hat{k}\right]$$

$$= (\hat{i} + 3\hat{j} - 3\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 4\hat{k})$$

CE : $\dfrac{x-1}{2-1} = \dfrac{y-3}{1-3} = \dfrac{z-^{-}3}{1-^{-}3}$

$$\Rightarrow \frac{x-1}{1} = \frac{y-3}{-2} = \frac{z+3}{4}$$

16 i) b) $P(A) \cdot P(B)$

ii) $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

$$= 0.5 + 0.3 - (0.5)(0.3)$$
$$= 0.8 - 0.15$$
$$= 0.65$$

17) $I = \displaystyle\int_0^1 x\sqrt{1-x}\, dx$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore I = \int_0^1 (1-x)\sqrt{1-(1-x)}\, dx$$

$$= \int_0^1 (1-x)\sqrt{x}\, dx$$

$$= \int_0^1 \left(\sqrt{x} - x^{3/2}\right) dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^1$$

$$= \left[\frac{3}{2} x^{3/2} - \frac{2}{5} x^{5/2}\right]_0^1$$

$$= \frac{3}{2}(1) - \frac{2}{5}(1) - 0$$

$$= \frac{3}{2} - \frac{2}{5} = \frac{15-4}{10} = \frac{11}{10}$$

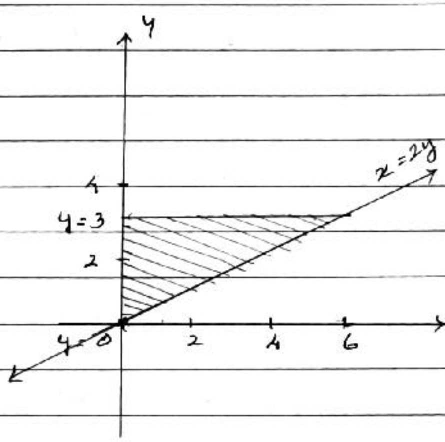
18 i) c) $\int_c^d x\,dy$.

ii) $y = \frac{x}{2}$

$x = 2y$

when $y=0$, $x = 2(0) = 0$

when $y = 3$, $x = 2(3) = 6$



$$\text{Req. area} = \int_{y=0}^{y=3} x\,dy$$

$$= \int_0^3 2y\,dy$$

$$= \left(2 \cdot \frac{y^2}{2}\right)_0^3$$

$$= (y^2)_0^3 = 3^2 - 0 = 9 \text{ Sq. units.}$$

19)

i) $\vec{OA} = 2\hat{i} + \hat{j} + 3\hat{k}$

ii) $\vec{OA} + \vec{OB} = 2\hat{i} + \hat{j} + 3\hat{k} + 3\hat{i} - \hat{j} - 2\hat{k}$

$= 5\hat{i} + 0\hat{j} + \hat{k}$

$= 5\hat{i} + \hat{k}$

$\vec{OA} \cdot \vec{OB} = (2)(3) + (1)(-1) + (3)(-2)$

$= \cancel{6} - 1 - \cancel{6} = -1$

20) $R_2 \to R_1 - R_2$, $R_3 \to R_1 - R_3$

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} (a-b) & (a-b)(a+b) \\ (a-c) & (a-c)(a+c) \end{vmatrix}$$

$$= (a-b)(a-c) \begin{vmatrix} 1 & a+b \\ 1 & a+c \end{vmatrix}$$

$$= (a-b)(a-c)\left[\cancel{a}+c-(\cancel{a}+b)\right]$$

$$= (a-b)(a-c)(c-b)$$

$$= (a-b) \times -(c-a) \times -(b-c)$$

$$= (a-b)(b-c)(c-a)$$

$= RHS.$

21. Let $A = I \cdot A$

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} A$$

$R_2 \to (-1) R_2$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} A$$

$$I = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

$$I A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} A A^{-1}$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

22) Let $y = \sqrt{x}$ —— (1)

$x = 25$
$\Delta x = 0.2$
$\sqrt{x} = \sqrt{25} = 5$

Then $\sqrt{x + \Delta x} = y + \Delta y$

$\sqrt{25 + 0.2} = \sqrt{x} + \Delta y$

$\therefore \sqrt{25.2} = 5 + \Delta y$ — (2)

diff (1) wrt x

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$dy = \frac{1}{2\sqrt{x}} \cdot dx$$

$$\Rightarrow \Delta y = \frac{1}{2\sqrt{x}} \cdot \Delta x = \frac{1}{2 \times 5} \times 0.2$$

$$= \frac{0.2}{10} = 0.02.$$

in (2) $\sqrt{25.2} = 5 + 0.02$
$= 5.02$ (approx.)

Part C

23 i) $A = \begin{bmatrix} 5 & 2 \\ 4 & -1 \end{bmatrix}$

$A' = \begin{bmatrix} 5 & 4 \\ 2 & -1 \end{bmatrix}$

$A + A' = \begin{bmatrix} 10 & 6 \\ 6 & -2 \end{bmatrix}$

$A - A' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

Let $P = \frac{1}{2}(A + A')$

$= \frac{1}{2}\begin{bmatrix} 10 & 6 \\ 6 & -2 \end{bmatrix}$

$P' = \left(\frac{1}{2}\begin{bmatrix} 10 & 6 \\ 6 & -2 \end{bmatrix}\right)' = \frac{1}{2}\begin{bmatrix} 10 & 6 \\ 6 & -2 \end{bmatrix} = P$, Symmetric

Let $Q = \frac{1}{2}(A - A')$

$Q' = \left(\frac{1}{2}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\right)'$

$= \frac{1}{2}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -Q$, is skew-symmetric

Now $P + Q = \frac{1}{2}\left(\begin{bmatrix} 10 & 6 \\ 6 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix} 10 & 4 \\ 8 & -2 \end{bmatrix}$

$= \begin{bmatrix} 5 & 2 \\ 4 & -1 \end{bmatrix} = A$, a square matrix.

Hence proved.

ii) $2x+y = \begin{bmatrix} -1 & 3 \\ 11 & 8 \end{bmatrix}$ ——— (1)

$x - y = \begin{bmatrix} 1 & 0 \\ 1 & 7 \end{bmatrix}$ ——— (2)

$3x = \begin{bmatrix} 0 & 3 \\ 12 & 15 \end{bmatrix}$

$\therefore x = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$

From (2) $y = x - \begin{bmatrix} 1 & 0 \\ 1 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 7 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

24). i. $y = \sqrt{3x-2}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{3x-2}} \times 3 = \frac{3}{2\sqrt{3x-2}}$

ii) $x^2 + y^2 + 3y = 0$

diff wrt x

$2x + 2y \cdot \frac{dy}{dx} + 3\frac{dy}{dx} = 0$

$(2y+3)\frac{dy}{dx} = -2x$

$\therefore \frac{dy}{dx} = \frac{-2x}{2y+3}$.

iii) $y = \tan^{-1}(x^2)$

$\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \times 2x = \frac{2x}{1+x^4}$

25)

i) $y = 3x^2 - 2$

$\frac{dy}{dx} = 6x$

when $x = 2$

$y = 3(2^2) - 2$

$= 10$

Slope of the tangent at $x = 2$, $= \frac{dy}{dx}$ at $x = 2$

$= 6(2) = 12$

ii) Slope of the normal. $= \frac{-1}{\text{Slope of the tangent}}$

$= \frac{-1}{12}$

Eqn of the normal is

$y - 10 = -\frac{1}{12}(x - 2)$

$12y - 120 = -x + 2$

$12y - 120 + x - 2 = 0$

$x + 12y - 122 = 0$.

26 A) $I = \int \frac{1}{(1+x)^3} dx$

Put $1 + x = t$

$dx = dt$

$\therefore I = \int \frac{1}{t^3} dt$

$= \int t^{-3} dt$

$= \frac{t^{-2}}{-2} + c$

$= -\frac{1}{2} \times \frac{1}{t^2} + c$

$= -\frac{1}{2(1+x)^2} + c$

B) $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= (\sin^{-1}x)_0^1 = \sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

c) $I = \int \sin^2 x \cos x \, dx$

put $\sin x = t$

$\cos x \, dx = dt$

$\therefore I = \int t^2 \cdot dt$

$= \frac{t^3}{3} + c$

$= \frac{1}{3} \sin^3 x + c$

27 i) $f(x) = 2x+1 \; ; g(x) = x^2$

$fog = f[g(x)]$

$= f(x^2)$

$= 2(x^2)+1 = 2x^2+1$

$gof = g[f(x)]$

$= g(2x+1)$

$= (2x+1)^2$

ii) $f(x) = 3-2x$ —— (1)

$f(x_1) = f(x_2) \Rightarrow 3-2x_1 = 3-2x_2$

$\Rightarrow -2x_1 = -2x_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-one.

Now $y = 3-2x$

$2x = 3-y$

$\therefore x = \frac{3-y}{2}$

in (1) $\Rightarrow$

$$f\left(\frac{3-y}{2}\right) = 3 - 2\cdot\left(\frac{3-y}{2}\right)$$

$$= 3 - (3-y)$$

$$= 3 - 3 + y$$

$$= y$$

$\therefore f$ is onto

$\therefore f$ is bijective

$\therefore f$ is invertible

$$\therefore f^{-1} = \frac{3-y}{2}$$

$$\therefore f^{-1}(x) = \frac{3-x}{2}.$$

28) The eqns can be written as

$$AX = B$$

where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix}.$$

$$= 1(-1+1) - 1(-2-1) + -1(-2-1)$$

$$= 1(0) - 1(-3) - 1(-3)$$

$$= 3 + 3 = 6 \neq 0$$

$\therefore A$ is non-singular.

$A_{11} = 0$	$A_{21} = 2$	$A_{31} = 2$
$A_{12} = 3$	$A_{22} = 0$	$A_{32} = -3$
$A_{13} = -3$	$A_{23} = 2$	$A_{33} = -1$

$\therefore \text{adj } A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & -3 \\ -3 & 2 & -1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & -3 \\ -3 & 2 & -1 \end{bmatrix}$

$\therefore X = A^{-1}B$

$= \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & 0 & -3 \\ -3 & 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} +12 \\ -6 \\ 18 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

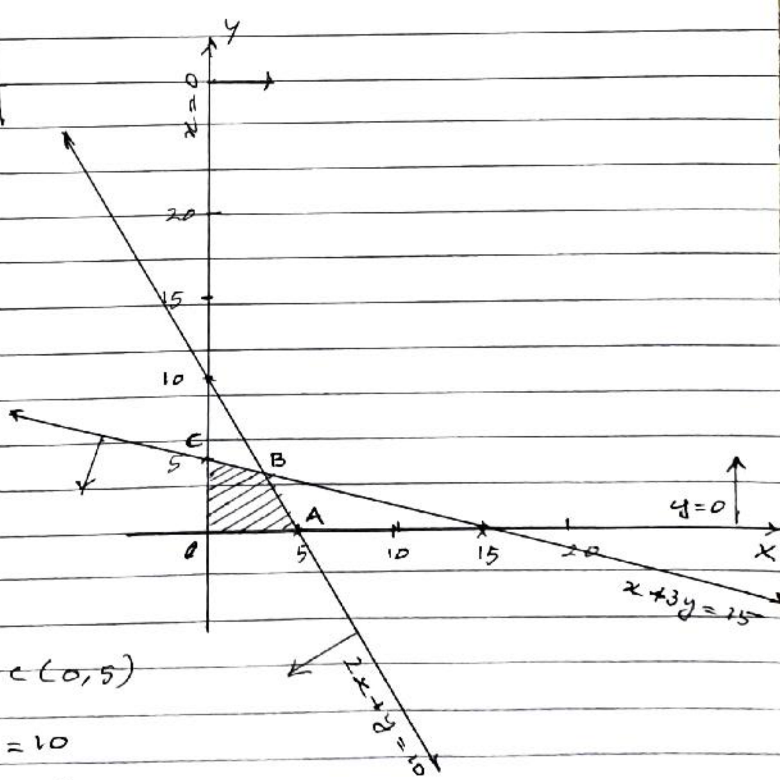
$\Rightarrow x = 2,\ y = -1,\ z = 3$

29) $x + 3y = 15$

x	0	15
y	5	0

$2x + y = 10$

x	0	5
y	10	0



Corner points:

O (0, 0), A (5, 0), C (0, 5)

and B: $\quad 2x + y = 10$

$\Rightarrow 6x + 3y = 30$

$\quad\ \ x + 3y = 15$

$\ (-)\quad (-)\quad (-)$

$5x = 15$

$x = 3$

$\therefore y = 10 - 2x = 10 - 6 = 4$

$\therefore$ B is $(3, 4)$

Corner point	$Z = 4x + 6y$
O $(0,0)$	$Z = 0$
A $(5,0)$	$Z = 4(5) + 6(0) = 20$
B $(3,4)$	$Z = 4(3) + 6(4) = 36 \leftarrow$
C $(0,5)$	$Z = 4(0) + 6(5) = 30$

Hence $Z_{max} = 36$ at $B(3,4)$.