HIGHER SECONDARY

IT MATHS LAB MANUAL

for 2021

Suggested Lab for Class XI

1. Lab 0:Basic Concepts

2. Lab 1: Value of Functions

3. Lab 2: Shifting of Graphs

4. Lab 4: Trigonometric Functions

5. Lab 8: Straight lines

6. Lab 9: Conic Sections

7. Lab 13: Limits

Suggested Lab for Class XII

1. Lab 18 : Functions

2. Lab 19 : Invertible functions

3. Lab 20 : Inverse Trigonometric Functions

4. Lab 31 : Application of Integrals

5. Lab 34 : Vectors

6. Lab 37 : The Plane

Basic Concepts

Aim:

- To familiarise the GeoGebra interface and toolbar

- To familiarise the concept of domain, range and graphs of standard functions

Concepts:

- Domain, range and graph of functions

Discussion :
Many of us are already familiar with the software GeoGebra which leads us to the joy of dynamism of Geometry. In higher secondary mathematics, we deal with concepts like Analytic geometry, Trigonometry, Calculus etc in which GeoGebra can contribute a lot in conceptual understanding. In this lab, we learn some basic tools and commands of GeoGebra which will help us in our learning process.
We also learn about using input command to plot the graph of a function, especially a polynomial function.

Activity 0.1 GeoGebra Interface

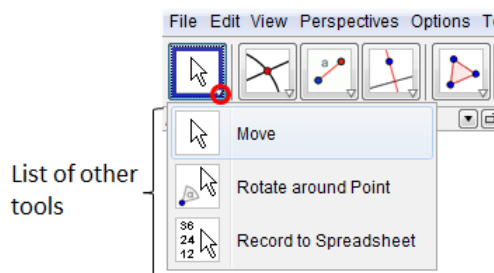**Procedure:**

- Familiarise the interfaces of GeoGebra
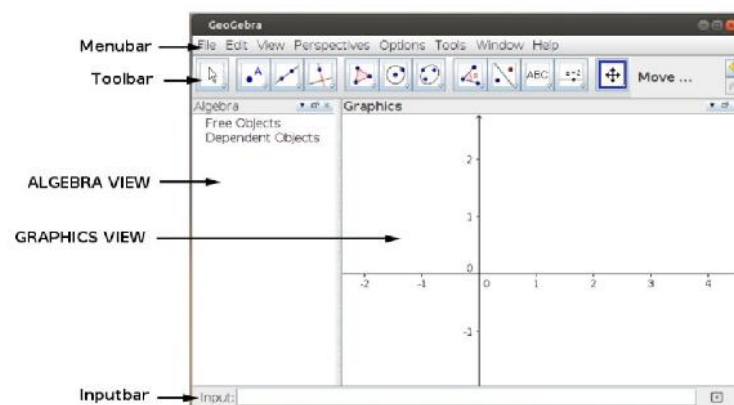
We can draw geometrical figures or graphs in the Graphics View by selecting tools from the toolbar or by giving commands in the input bar.The algebraic form of the figures or graphs drawn in the Graphics view is available in the Algebra View. Apart from the Graphics View and Algebra View, GeoGebra also offers Graphics2, Spreadsheet, CAS (Computer Algebra System) and 3D Graphics. All these views can be shown or hidden using "View" menu.

- Familiarise the Tool bar and some important tools of GeoGebra
  In GeoGebra construction, tools are arranged in 12 sets as shown in the figure below.
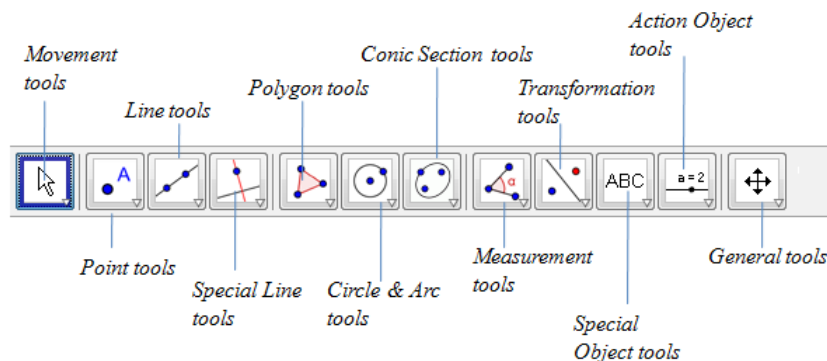
All the tools in each set are obtained by clicking on the small arrow at the bottom right corner of each icon as shown in Figure. Keeping the cursor on the tool, a brief description of the function of the tool is displayed.
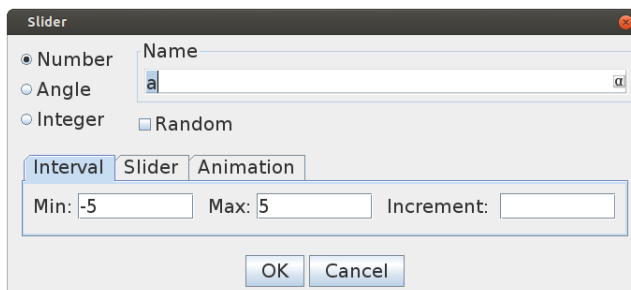

When the 3D Graphics view is enabled, the tools will change accordingly.

Activity 0.2 Graph of a Function

Procedure:

- Create a slider **a** with increment 1 as follows
  Using `Slider` tool click anywhere on the **Graphics** view. We get a window in which we can edit the name, minimum value, maximum value, and increment of the slider.

- Plot the point $A(a, a^2)$. (Input `A=(a,a^2)` or `(a,a^2)`)

- Change the value of the slider and observe the movement of A.

  We can change the value of a slider in different ways.

    - Click and drag the slider point.
    - Click on the slider point and use arrow keys to change the value.
    - Right click on the slider and select **Animation On**
    - Create an input box for the slider and change the value

- Change the increment of the slider to 0.01.(Right click on the slider $\rightarrow$ Object Properties $\rightarrow$ Slider $\rightarrow$ Enter 0.01 in the **Increment** Box.)

- Observe the movement of the point.

- Trace the point A. (Right click on the point$\rightarrow$ Trace On)

Input Box

- To create an Input Box for the slider **a**, using the **Input Box** tool, click anywhere on the Graphics view. We get the 'Input Box' window. Enter any caption (say Slider). From the **Input Box** drop down menu select **a**.

- To create an Input Box for the point A, from the **Input Box** drop down menu select **A**.

Locus Tool

- We can draw the path traced by the point A using **Locus** tool.For this, using the tool click on the slider **a** and on the point A.

- Observe the curve traced. What dose it represent ?

- Create an input box for the point A.

- Change the definition of A as (a, a^3). (In the Input Box enter `(a,a^3)`.)

- Observe the curve traced. What dose it represent ?

- What should be the definition of A so that the curve represents the graph of the function $f(x) = x^4$?

Activity 0.3 Standard Functions

Procedure:

- Draw the graphs of standard functions using inputs

Function	Input
x	x
x^2	x^2
$|x|$	abs(x)
$\sqrt{x}$	sqrt(x)
x^3	x^3
$[x]$	floor(x)
$\frac{1}{x}$	1/x
Signum function	sign(x)

  - We can see the name of each function in the algebra view

  - By clicking the bullets, we can hide/show the graph of the function

- Observe the graph of each function and find its domain and range

- Input `ceil(x)`.

  Observe the graph of the function, compare the graph of this function with the graph of the floor function. Define this function

Activity 0.4 Domain and Range

Procedure:

- Create an integer slider n
  (Using `Slider` tool click anywhere on the Graphics→select Integer→click OK. If we want we can change the minimum , maximum and increment of the slider.)

- Draw the graph of $f(x) = x^n$
  [Input: `f(x) = x^n`]

- Observe the graph of the function x^n and find the domain and range for different values of n

- What happens to the graph of the function x^n between -1 and 1 as n becomes larger and larger ? why ?

n	function	domain	range
1	x		
2	x^2		
3	x^3		

Additional Activities

Activity 0.A Polynomial Function

Discussion:

We discuss how the domain and range of a polynomial function related to its degree.

Procedure:

- Draw the graphs of some polynomial functions.(eg. for getting the graph of
  $f(x) = x^3 + 2x^2 - 3$, Input: `f(x)=x^3+2x^2-3`)

- Draw the graphs of the following functions and find their domain and range

Sl No	function	domain	range
1	$2x^3 - 3x + 4$		
2	$-x^2 + 2x - 3$		
3	$3x^4 + 5$		

- What is your general inference about the domain and range of polynomial functions?

Activity 0.B Functions With Rational Powers

Discussion:

We discuss the nature of the function $f(x) = x^{\frac{1}{n}}$ for integer values of **n**

Procedure:

- Create an integer slider **n** (min=1,max=10)

- Draw the graph of $f(x) = x^{\frac{1}{n}}$
  (Input:`f(x)=x^(1/n)`)

- Move the slider and observe the graph. identify the change in domain,range and the graph when **n** takes even and odd values.

- Also draw the graph of x^n and compare it with the graph of $x^{\frac{1}{n}}$.

Lab 1

Value of functions

Aim:

- To construct an applet to establish geometrically the correspondence of a number and its image under a function.

- To use this applet to find the images of numbers under various functions

- To use an applet to visualise the comparison of a function with an input-output machine.

Concepts:

- Image of a number a under a function f is denoted by $f(a)$

- Graph of a function is a collection of points $(a, f(a))$

Discussion :
For any number a,the ordered pair $(a, f(a))$ are the coordinates of a point on the graph of the function $f(x)$, so its y coordinate gives the value of $f(a)$. We use this idea for constructing our applet. Once such an applet is constructed, we can simply change the function and use it for different functions.

Sometimes we compare a function with a machine which gives an output, according to the definition of the function, for a given input. In Activity 1.3 we use an Applet which helps us to visualise this comparison. By this activity we get a clear idea about the domain of the function.

Activity 1.1 Functions

Procedure:

- Draw the graph of $f(x) = x^2$.

- Create a number slider **a** with increment 0.01

- Plot the points $A(a, 0)$, $B(a, f(a))$, $C(0, f(a))$. (give inputs like `A=(a,0)`).

- Draw the line segments AB and BC using `line segment` tool.

> 👆 To show the coordinates of a point, right click on the point. Go to Object Properties → Basic → Show Label and select the Name and Value option

- Show the coordinates of A, B, and C.

- Now drag the point A along the x axis (either click and drag the point or using slider - click and drag the slider point to change the value of **a**) and observe the movement of C on the y axis.

📖 Using this, find the values of $(2.3)^2$,$(-1.8)^2$,$(0.9)^2$,$(2.9)^2$...
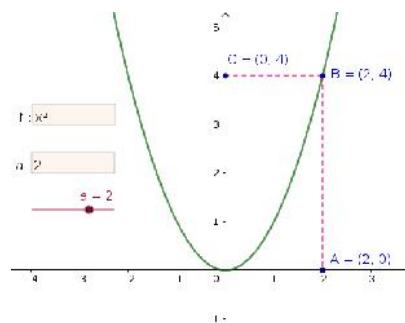
Save the file as Activity 1.1

Activity 1.2 Values of Functions

Procedure:

- Open the file Activity 1.1 and save as Activity 1.2

- Create an input box for f and change the function using it.
  (Select *input box* tool, $\rightarrow$ Click on `Graphics view` $\rightarrow$ give a suitable caption (say function) $\rightarrow$ `linked object` $\rightarrow$ $f(x) = x^2 \rightarrow$ OK)

- Similarly create an input box for the slider.

Change the functions accordingly and find the approximate values corrected to 3 decimal places of the following

	$3^{\frac{1}{3}}$	$\sqrt{1.8}$	$2^{\frac{2}{3}}$	$\sqrt{\sqrt{5}}$	$(3.46)^{\frac{-3}{2}}$
Function	$x^{\frac{1}{3}}$				
Input(x)	3				
Value$(f(x))$					

We can set the number of decimal places as follows;
Options $\rightarrow$ Rounding $\rightarrow$ Select number of decimal places.

Change the function to $f(x) = \dfrac{1}{x}$, and observe how the point C moves as the point A approaches the origin from either side.
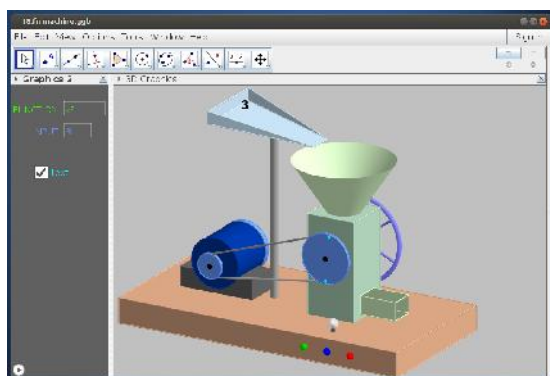
Change the function to $f(x) = [x]$ and observe the movement of C according to A

Activity 1.3 Function Machine

Procedure :

Use Applet ML 1.3

About the Applet

Three switches are provided on the machine

- GREEN :- Click to start the machine.

- RED :- Click to stop the machine.

- BLUE :- Click to reset.

Using input boxes we can change the function and the input number.
The warning light provided on the machine turns red if the input number is out of the domain of the function.

Change the function to $f(x) = \sqrt{x}$ and find the values of the following.

i) $\sqrt{2}$ ii) $\sqrt{1.8}$ iii) $\sqrt{\frac{2}{3}}$

What happens if we give a negative number as the input ?

Change the function to $f(x) = \frac{1}{x}$ and find the values of the following.

i) $\frac{2}{3}$ ii) $\frac{-3}{7}$ iii) $\sqrt{\frac{2}{3}}$

What happens if the input is 0 ?

Additional Activities

Activity 1.A   Temperature Scales

Discussion

There are various scales to measure temperature. Perhaps the most popular ones are the Fahrenheit and the Celsius scales.

$F(C)$ is the Fahrenheit temperature corresponding to the Celsius temperature C and they related to each other as

$$F(C) = \frac{9}{5}C + 32$$

- Plot the graph of the above function (Consider C as the variable x)

- From the graph identify the Celsius temperature at which the Fahrenheit temperature become zero

- From the graph identify the Fahrenheit temperature at which the Celsius temperature become zero

> 👉 While plotting the graph of $F(C)$ we have to use x instead of C. So in order to get the graph input **9x/5+32**

Lab 2

Shifting of Graphs

Aim:

To analyse the changes in the graph of a function according to some slight changes in the definition

Concepts:

- Graph of a function

Discussion :

If we know the graph of the function $f(x)$ we can obtain the graphs of the functions $f(x) + a$, $f(x + a)$, $-f(x)$ and $f(-x)$ by translation or reflection.

This idea helps us to imagine the graphs of some functions if the graph of the base function is known.

This activity throws light on the concept of a family of curves.

Activity 2.1 Shifting of graphs : $\quad f(x) + a$

Procedure:

- Draw the graph of $f(x) = x^2$

- Create a number slider **a** with increment 0.1

- Draw the graph of $g(x) = f(x) + a$
  (Input:**f+a**)

- 📖 Observe how the graph of $f(x)$ changes according to **a**

- Create input boxes for editing function and slider **a**

  📖 Do the above observations for different functions such as $|x|$, $[x]$, x^3 *etc*

- Save this as Activity 2.1

Apply trace to the graph to get a pattern.(rightclick → trace on).To erase the pattern ,press ctrl+F

Change the value of slider by these methods :

- Click on the slider point and move

- Using *Move* tool, select the slider and use arrow keys

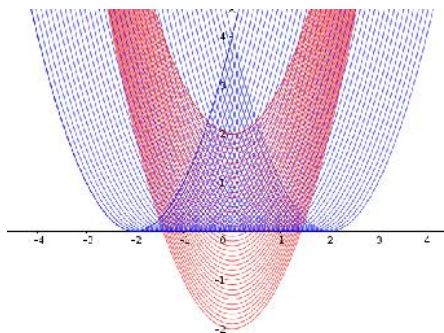- Right click on the slider → and turn on animation

Activity 2.2 Shifting of graphs : $f(x+a)$

Procedure:

- Open a new GeoGebra window.

- Draw the graph of $f(x) = x^2$.

- Create a number slider **a** with increment 0.1

- Draw the graph of $g(x) = f(x+a)$. (Input: `f(x+a)`)

  Observe how the graph of $g(x)$ changes according to **a**.

- Create input boxes for editing function and slider **a**.

  Generalise the above observations with different functions such as $|x|$, $[x]$, x^3 *etc*

- You may use the animation option to change the slider.

- Save this as Activity 2.2

Activity 2.3 Reflection of a Graph : $-f(x)$

Procedure:

- Open a new GeoGebra window.

- Draw the graph of $f(x) = x^2$

- Draw the graph of $g(x) = -f(x)$ (Input: `-f`)

  Compare the graphs of $f(x)$ and $g(x)$.

- Create an input box for f and change the function to

  i) $x^2 + 2$ ii) $x^2 - 1$ iii)$|x| - 1$ iv) $|x - 1|$

  v) $[x]$ vi) $x^2 + 2x + 1$ vii) $\dfrac{1}{x}$

  Compare the graphs of f and g in each case. Write your findings.

- Save this file as Activity 2.3

Activity 2.4 Reflection of a Graph : $f(-x)$

Procedure:

- Open new GeoGebra window

- Draw the graph of $f(x) = x^3$

- Draw the graph of $g(x) = f(-x)$ (Input: f(-x))

  📖   Compare the graph of $f(x)$ and $g(x)$

- Create an input box for f and change the function to
  i) $\dfrac{1}{x}$ ii) $[x]$ iii) $|x|$

  iv) x^2 v) $(x-2)^2$

  📖   Compare the graphs of f and g in each case. Write your findings.

- Save this file as Activity 2.4

> 👉 A function $f(x)$ is an even function if $f(-x) = f(x)$ and an odd function if $f(-x) = -f(x)$.
> What is the speciality of the graphs of odd and even functions?
> Identify odd and even functions discussed in this lab.
> Is there any function which is neither odd nor even?

Additional Activities

Activity 2.A Translations of Graphs:1

Procedure:

- Draw the graph of $f(x) = x^2$.

- Create a number slider **a**, with increment 0.1
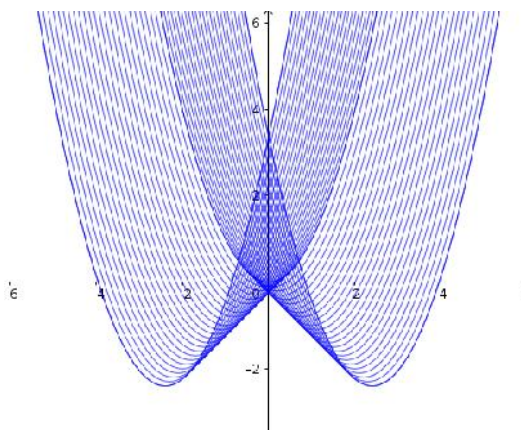
- Draw the graph of $g(x) = f(x-a) + a$.

  📖   Observe how the graph of $g(x)$ changes according to **a**.

- Create an input box for g and change the function to
  i) $f(x-a) - a$ ii) $f(x-a) + 2a$
  iii) $f(x-a) + 3a$ iv) $f(x-a) - 3a$

  📖   Observe the shift in the graph of g according to the change in **a**.

- 📖   Try to draw the pattern given in the figure.

Activity 2.B Translations of Graphs: 2

- Draw the graph of $f(x) = x^2$.

- Create number sliders **a** and **b** , with increment 0.1

- Draw the graph of $g(x) = f(x + a) + b$.

- By adjusting the values of a and b transform the graph of x^2 to that of the following functions.

    i) $(x + 2)^2 - 3$ 　　　　　　 ii) $x^2 + 6x + 9$ 　　　　　　 iii) $x^2 - 4x + 6$

Activity 2.C Family of curves - using sequence command

Using sequence command, we can represent the family of curves obtained by shifting a graph

Procedure:

- Draw the graph of $f(x) = x^2$

- In the input bar, give the command, `Sequence[f+i,i,-3,3,0.2]`, which gives the graphs of the functions

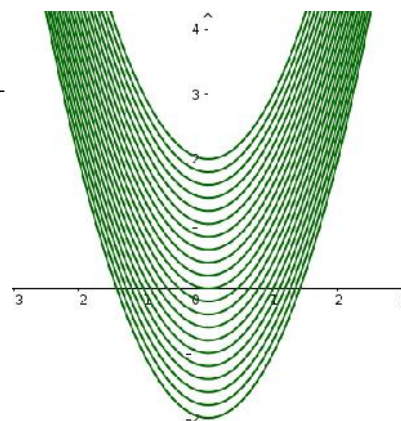 $x^2 - 3, x^2 - 2.8, x^2 - 2.6, \ldots, x^2, \ldots, x^2 + 3$

👉 In the input command Sequence[f+i,i,-3,3,0.2], f is function, i is variable, -3 is start value, 3 is end value and 0.2 is increment

📖　Imagine the family of curves obtained by the following input commands and then draw them.

1. `Sequence[f(x+i),i,-3,3,0.2]`

2. `Sequence[f(x-i)+i,i,-3,3,0.2]`

3. `Sequence[f(x-i)-i,i,-3,3,0.2]`

4. `Sequence[f(x-i)+2i,i,-3,3,0.2]`

📖　Create a slider **a** and input the Command `Sequence[f(x-i)+a*i,i,-3,3,0.2]`

📖　Create an input box for f and observe the pattern for different functions

Lab 4

Trigonometric Functions

Aim:

- To create an applet to find the values of trigonometric functions and plot their graphs

- To establish some behaviours of trigonometric functions in different quadrants

Concepts:

- Concept of circular functions

- Graph of the function f is a collection of points of the form $(a, f(a))$ for all values of a in its domain

Discussion :

If a point is rotated from $(1, 0)$ along the unit circle centred at the origin, by an angle x radians, the x and y coordinates of the point represent $\cos x$ and $\sin x$ respectively. We define all other trigonometric functions in terms of $\cos x$ and $\sin x$. We use this idea to construct our applet.

## Activity 4.1 Values of Trigonometric Functions

Procedure:

Open a new GeoGebra window,do some initial settings as follows
`Options→ Advanced → Angle unit → Radian`

- Plot the point $O(0, 0)$ (input `O=(0,0)`)

- Draw a unit circle centred at the origin O

- Plot the point $A(1, 0)$ (input `A=(1,0)`)

- Create a number slider **a** with min value -10 , max value 10 and increment 0.01. While creating the slider, set its animation as increasing

- Plot another point A' such that $\angle\, AOA' = $ **a** radian

- Rename the point A' as P (right click → Rename)

- Show the coordinates of P

- Join OP using a line segment

- Create an input box for the slider **a**.

☞

- To set the animation of a slider as increasing, right click on the slider and in the object properties, select increasing option from the repeat dropdown menu

- To create an angle AOA' with measurement a, use *Angle with a given size* tool, click on A, O and then give **a** as the angle in the box
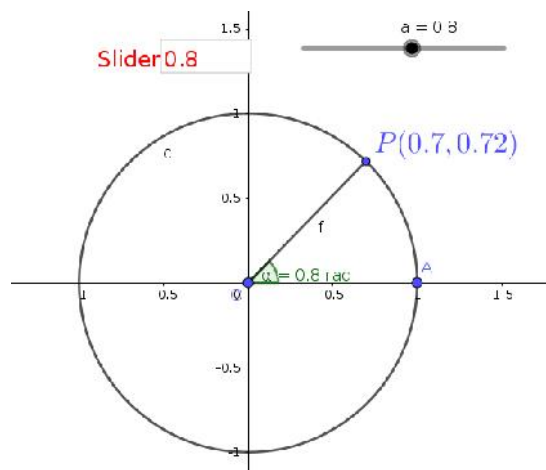
- Animate the slider,observe the coordinates of the point P, hence find the domain and range of $\sin x$ and $\cos x$

- Find the values of $\sin x$ and $\cos x$ for the given values of x

x	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	0.3	0.6	2	-1.5	-3.1	7.5
$\sin x$										
$\cos x$										

👉

For giving π in the input box, input:pi

 Identify the values of x for which $\sin x$ and $\cos x$ become 0,1,-1

- Save this file as Activity 4.1

Activity 4.2 Graphs of Trigonometric Functions - 1

Procedure:

- Save file Activity 4.1 as Activity 4.2 using save as option

- Open Graphics 2 [view $\longrightarrow$ Graphics 2]

- Plot the point $B(a, y(P))$. [$y(P)$ gives the y coordinate of P]

- Give trace to this point and animate the slider

- Observe the path of this point. What does this path represent?

- Save the file.

👉

- We can see the path of the point using locus tool also.To get the path,take the locus tool, click on the point and on the slider

- For analysing the graphs of trigonometric functions, it is more convenient to mark $\cdots \frac{-\pi}{2}, 0, \frac{\pi}{2} \cdots$ on the x axis instead of ...-1 ,0 ,1...
  (For this right click on the Graphics 2. Go to *Object Properties* Change the x Axis distance to $\frac{\pi}{2}$)

- We can draw the graphs of $\sin x$ and $\cos x$ using input commands sin(x), cos(x) *etc.*

Activity 4.3 Graphs of Trigonometric Functions - 2

Procedure:

- Open Activity 4.2 and save as Activity 4.3 using save as option

- Create an input box for the point B

- Change the definition of B as $(a, x(P))$

- 📘 Observe the path of this point

- 📘 What does this path represent?

- 📘 Redefine B as $(a, \frac{1}{y(P)})$ and $(a, \frac{y(P)}{x(P)})$, observe the path of P and identify the functions.

- 📘 What should be the definition of B for getting the graphs of $\sec x$ and $\cot x$?

- 📘 Observe the values of trigonometric functions, write their domain, range and complete the following table.

Function	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \pi\right)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
$\sin x$	Positive			
	Increasing from 0 to 1			
$\cos x$				
$\tan x$				
			Increasing from 0 to ∞	
$\sec x$				
$\cot x$				
$\operatorname{cosec} x$				

Additional Activities

Activity 4.A $k\sin(x)$

Discussion :

We construct an applet similar to that in Activity 4.1, using which we describe the functions $k\sin(x), k\cos(x)$ *etc.* for different values of **k**.

Procedure :

Do the initial settings as in activity 4.1

- Create two sliders, **k** with Min = 0 and **a** with min value -10 , max value 10 and increment 0.01.While creating the slider **a**, set its animation as increasing

- Draw a circle of radius **k** centered at the origin O(0,0)

- Plot the point $A(k,0)$

- Plot another point A' such that $\angle\, AOA' = $ **a** radian

- Rename the point A' as P

- Show the coordinates of P

- Join OP using a line segment

-    What does the coordinates of the point P represent ?

-    Find the domain and range of $k\sin(x)$ and $k\cos(x)$ for different values of **k**

- Open Graphics 2 and plot the graphs of $k\sin(x)$ and $k\cos(x)$ as we done in Activity 4.2.

- Save this file as Activity 4.A

Activity 4.B $k\sin(2x)$

Discussion :

We construct an applet using which we describe the functions $k\sin(2x), k\cos(2x)$ *etc.*

Procedure :

- Open Activity 4.A and save it as Activity 4.B

- edit the rotation of P as **2a** (Double click and edit)

-    What does the coordinates of the point P represent ?

- Open Graphics 2 and plot the graphs of $k\sin(2x)$ and $k\cos(2x)$

- Create an applet to describe $k\sin(ax)$ and $k\cos(ax)$, for different values of **k** and **a**

Lab 8

Straight lines

Aim :

- To establish the role of coefficients and constant in the general equation of a straight line

- To explore geometrically the Normal form of a straight line

- To explore geometrically a family of straight lines

Concepts:

- General equation of a straight line

- Family of straight lines

- Normal form of a straight line

Discussion :

A straight line is represented as $ax + by + c = 0$ where a,b and c are constants. If c is changed, while keeping a and b fixed, the straight line will change in a particular manner. Similar situations are encountered with other constants. In this activity we explore the variations in the coefficients and constant and its effect in the geometry of straight line.

All the above mentioned activities will result in a set of straight lines having a common property. This set is called a family of straight lines.

We also discuss the family of straight lines passing through a point of intersection of given lines.
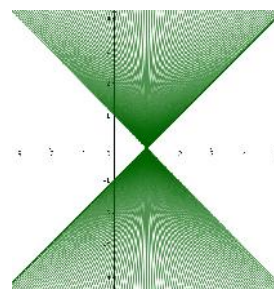
Activity 8.1 General Form of Straight Lines

Procedure:

- Create three sliders **a**,**b** and **c**

- Draw the line $ax + by + c = 0$

- Change the values of **a**,**b** and **c**

- What happens to the line if
  (i) $\mathbf{a} = 0$ (ii) $\mathbf{b} = 0$ (iii) $\mathbf{c} = 0$
  (iv) $\mathbf{a} = \mathbf{b}$ (v) $\mathbf{a} = \mathbf{-b}$

- Make the following changes and observe the corresponding changes in the line (Trace option of the line may be used).

1. Change **a** alone

2. Change **b** alone

3. Change **c** alone

Activity 8.2 Intersection of Two Lines

Procedure:

- Draw two lines say, $3x - 2y + 4 = 0$ and $2x + 5y - 6 = 0$

- Create a slider **k**

- Input the equation $3x - 2y + 4 + \mathbf{k}(2x + 5y - 6) = 0$

- It represents a line (why?)

- Change the value of **k**. Observe how the third line changes

- For what value of **k**, the third line coincides with the first line?

- For what value of **k**, the third line coincides with the second line?

- Edit the equations of the first two lines so that they are parallel

- What happens to the third line for different values of **k**

Activity 8.3 Normal Form

Procedure:

- To find the normal form of a line geometrically, draw the line and the perpendicular from the origin to the line(use `Perpendicular Line` tool).

- Mark the point of intersection of the perpendicular with the line.

- Hide the perpendicular line and draw a line segment from the origin to the line.
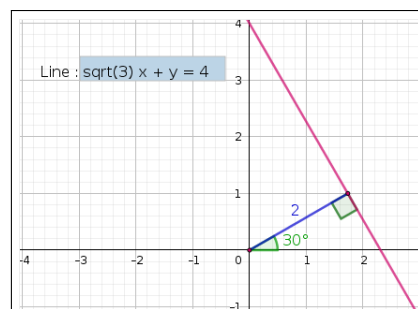
- Show the length of the perpendicular and the angle made by it with the positive direction of the x axis.

- Create an input box for the line

- save as Activity 8.3

> 👆
>
> Using **Angle** tool we can show the angle in the following ways
>
> - Click on the positive direction of x axis and then on the line segment.(Make sure that the line segment has been drawn starting from the origin, otherwise we won't get the required angle.)
>
> - Take a point - say C - on the positive direction of the x axis. Using the tool click on C, the origin and the point of intersection in that order. Hide C

- 📖   Using this applet, write the normal form of the following lines

  1. $x - \sqrt{3}y - 8 = 0$
  2. $x - y - 2 = 0$
  3. $\sqrt{3}x - y + 8 = 0$
  4. $2x - 3y + 4 = 0$
  5. $x + y = 5$
  6. $5x + 2y + 3 = 0$

📖   Write the equation of the lines in normal form for different values of ω and **p** (p is the distance of the line from the origin and ω is the angle made by the normal with the positive direction of x axis). Verify your answer using the above applet

Sl.No	Value of ω	Value of **p**	Equation of the line
1	0°	3	
2	30°	4	$\dfrac{\sqrt{3}x}{2} + \dfrac{y}{2} = 4$
3	30°	5	
4	60°	2	
5	90°	4	
6	120°	4	
7	150°	4	

Activity 8.4 Shifting of Origin

Procedure:

- Using the applet, ML 8.4
  About the applet:
  ○ You can click and drag at the origin to shift the axes.
  ○ You can change the curve and the origin using corresponding input boxes
  ○ You can see the transformed equation using check box
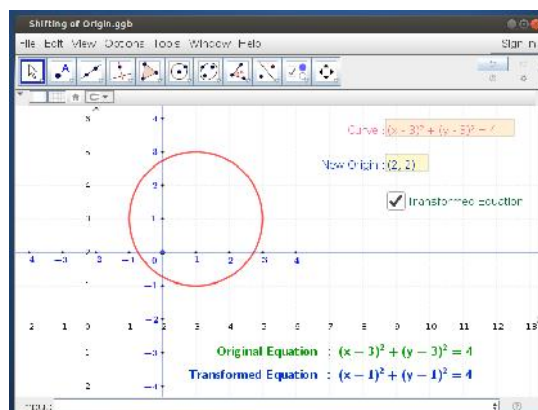
- Shift the origin, parallel to the x axis or y axis and observe the changes in the new equation of the circle

- What should be the new origin to get the transformed equation as $x^2 + y^2 = 4$. Guess the answer and check it.

- Find the transformed equation, if the origin is shifted to the point $(1,3)$. Check the answer.

Lab 9

Conic Sections

Aim :

- To show conics as the section of a cone as well as the locus of a point

Concepts:

- Cone and its section by a plane
- Locus of a point

Discussion :

Conic sections are curves obtained by the intersection a double cone by a plane. The angle at which the plane cuts the cone determines the curve. The semi vertical angle of the cone, the position at which the plane cuts the cone etc.. will determine the shape of the curve.
We treat the curves as the locus of a point moving on a plane subjected to certain constraints.

Activity 9.1 Cutting of a Cone by a Plane

Using the applet ML 9.1

About the applet:
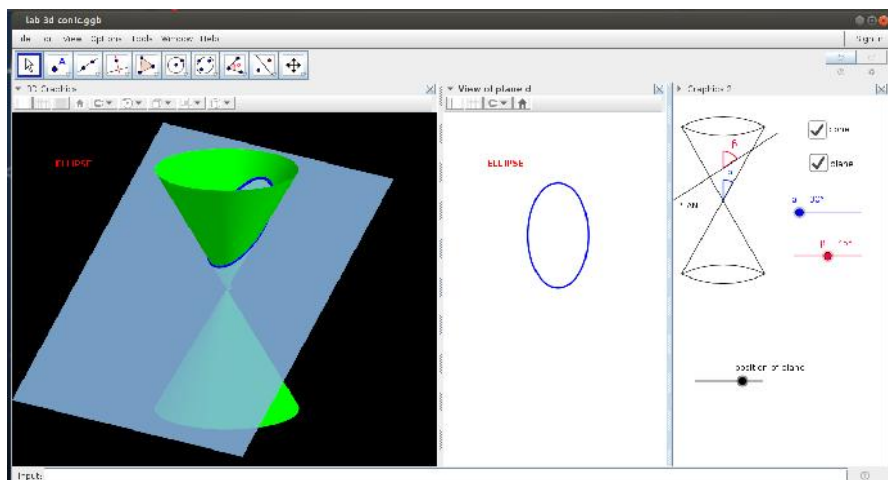
In this applet we can see 3 open windows. **Graphics 2** , **3D Graphics** and the third one is **View of plane d**

Graphics 2

- There are three sliders and two check boxes here.

- Using slider α you can change the semi vertical angle of the cone.

- Using slider β you can tilt the plane

- Using slider "Position of the plane" you can change the position of the plane.

- Using the check boxes we can show or hide the cone and the plane.

3D graphics

You can see the 3D view here, using ***Rotate 3D graphics view*** tool, you can rotate the entire view to see it from a convenient angle.

View of plane d

You can see the curve, obtained by intersecting the cone with the plane.

Procedure:

Change the value of β for a fixed α. Observe the curves for different values of β.

α	Curve	β
25°	Circle	90°
	Parabola	25°
	Ellipse	25° < β < 90°
	Hyperbola	
30°	Circle	
	Parabola	
	Ellipse	
	Hyperbola	
45°	Circle	
	Parabola	
	Ellipse	
	Hyperbola	
50°	Circle	
	Parabola	
	Ellipse	
	Hyperbola	

For what values of β do we get the curves - circle, ellipse, parabola and hyperbola?

Change the position of the plane and observe the corresponding change in the shape of the curve

Change α and observe corresponding change in the curve.

Activity 9.2 Locus of a point moving equidistant from given points

Procedure:

- Plot two points A and B and join them (Using `line segment` tool)

- Create a number slider **a** with min=0 and max=10

- Draw circles of radius **a** centred at A and B

- Plot the points of intersection of the circles and give their trace.

- Give animation to the slider **a**

- 📖 Observe the path of the moving point. Describe the path.

- Save this file as Activity 9.2
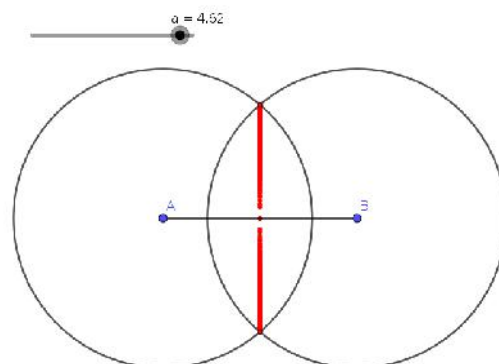
👉 Adjust the increment of the slider as 0.01 or less to get a continuous path. If we use Shift key together with the arrow keys to move the slider, it will decrease the speed of the slider by one tenth. If we use Ctrl key, speed will be increased 10 times

Activity 9.3 Locus of a point the sum of whose distances from two given points is a constant

Procedure:

- As in Activity 9.2, create a slider **a**

- Plot two points A and B and join them.

- Draw a circle of radius **a** centred at A and another circle of radius 10 − **a** centred at B

- Plot the points of intersection of the circles and give their trace.

- Give animation to the slider **a**

- Observe the path of the moving points. Identify the curve traced.
  (We can show the path of the moving point using `Locus` tool. For this, take the `Locus` tool, click on one of the points of intersection and on the slider. Similarly click on the other point of intersection and on the slider)

👉 We can create this applet by editing the radius of the circle centred at B of the previous Activity 9.2

- 📖 Change the distance between A and B and observe the change in the shape of the curve

- 📖 What happens to the curve when B approaches nearer and nearer to A?

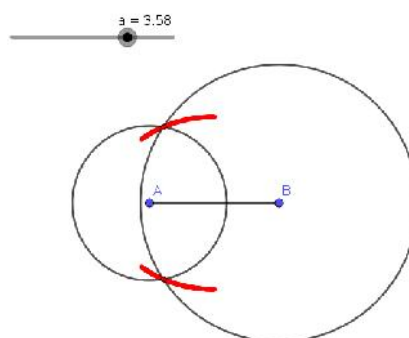- 📖   What is the maximum possible distance between A and B to get a path?

Activity 9.4 Locus of a point the difference of whose distances from two given points is a constant

Procedure:

- As in Activity 9.2, Create a slider **a**

- Plot two points A and B

- Draw a circle of radius **a** centred at A and another circle of radius **a** + 4 centred at B

- Plot the points of intersection of the circles and give their trace.

- Give animation to the slider **a**

- Draw another set of circles of radius **a** centred at B and of radius **a** + 4 centred at A

- Plot the points of intersection of these circles and give their trace.

- Give animation to the slider **a**

- 📖   Observe the path of the moving points.Identify the curve traced. (Here also we can use the Locus tool as in Activity 9.3)
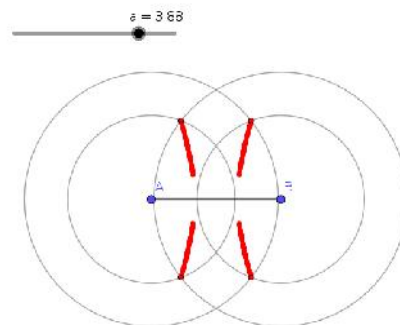
Activity 9.5 Locus of point equidistant from a point and a fixed line

Procedure:

- Draw a line and plot a point C outside the line.

- Draw perpendicular to the line through C.

- Plot the point of intersection D of the line and its perpendicular.

- Create a slider **a** with min = 0, max = 15 and increment 0.01

- Draw a circle of radius **a** centred at D and plot its point of intersection E with the perpendicular line.

- Draw a line through E and parallel to the first line.

- Draw a circle of radius **a** centred at C and plot its points of intersection with the last line drawn. Trace on these points.

- Give animation to the slider **a**

- 📖   Observe the path of the moving points. Identify the curve traced.

- Draw the path using `Locus` tool

- 📖   Change the distance between the point and the first line and observe the change in the shape of the curve

---
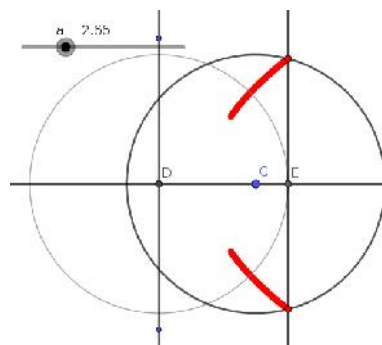
Additional Activities

Activity 9.A Focus - Diretrix Definition

Discussion :

We discuss Parabola, Ellipse and Hyperbola as the locus of a point moving on a plane, keeping a specific ratio of distance from a fixed line and a fixed point. **Procedure :**

- Create two sliders **a**, with min $= 0$, max $= 10$ and increment 0.01 and **b** with min $= 0$, max $= 5$ and increment 0.01

- Draw the line $x = 0$ and plot a point A on the positive side of the x axis.

- Draw the lines $x = a$ and $x = -a$

- Draw a circle of radius ab centred at A.

- Plot the point of intersection of the circle with the lines $x = a$ and $x = -a$ (If necessary, you can change the values of the sliders, so that the circle meets the lines.)

- Trace the points of intersection

- Hide the axes, Set **b**=1 and animate slider **a**
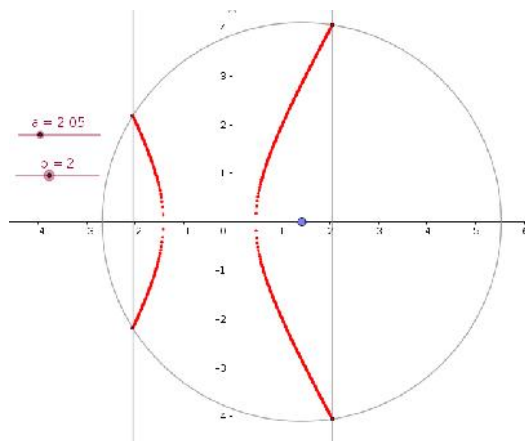
- 　　　Observe the path of the points. Can you identify the curve ?

- Using `Locus` tool, draw the path of the points.

- 　　　Change the value of **b** and observe the path. Can you identify the curves for different values of **b** ?

- 　　　Try to define a parabola,an ellipse and a hyperbola in terms of distances from a fixed line and a fixed point.

Activity 9.B Apollonius Circles

Discussion :

We discuss the locus of a point moving on a plane, keeping a specific ratio of distance from two fixed points.

Procedure :

- Create two sliders **a** and **r** with Min $= 0$ and increment 0.01. Create an input box for **a**.

- Plot two points A and B and join them

- Draw a circle of radius **r** centered at A and another circle of radius **ar** centered at B

- Plot the point of intersections of the circles and trace them. Find the locus of the points using `Locus` tool.

- 　　　Observe the path for different values of **a**

- Can you connect this with internal and external division of a line ?

Lab 13

Limits

Aim :

- To geometrically explore the concept of the limit of a function at a point.

Concepts:

- Value of a function at a point
- Graph of a function

Discussion:

We geometrically explore the concept of limit at a point. We discuss the existence and different cases of non existence of limit, the nature of the graph at a point where limit exists/does not exist, the concept of left limit and right limit etc..
We geometrically interpret some standard limits also.

Activity 13.1 Geometrical Interpretation of Limits

Procedure:

Use the applet 13.1-Limits
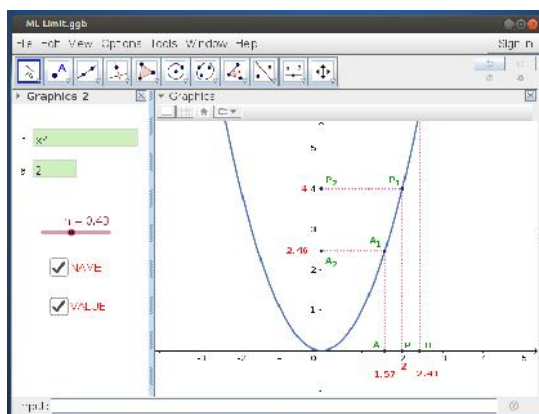
About the applet :

- You can see the graph of a function $f(x)$, 3 points A, B, P on the x axis, corresponding points A_1, B_1, P_1 on the graph and A_2, B_2, P_2 on the y axis

- 'NAME' Check Box: By clicking on it you can show/hide the names of the points

- VALUE Check Box: By clicking on it you can show/hide the x coordinates of the points A, B and P and the y coordinates of the points A_2, B_2 and P_2

- Slider **h**: Using this we can bring the points A and B towards P

- Input box **a**: To change the position of P

- Input box **f**: To change the function

Initial settings

- $f(x) = x^2$

- **a** = 2

- **h** = 1

- Show the names of the points

Gradually change the value of **h** from 1 to 0. Observe the movements of the points.What happens to A_2 and B_2 as A and B approaches P ?

Show the values of the points. Set **h**=1 and gradually bring it to 0. Observe the values.What happens to the x coordinates of the points A and B? What happens to the y coordinates of A_2 and B_2?

> We can record the value of points to spreadsheet as follows. Open spreadsheet view $\longrightarrow$ Spreadsheet.
> Right click on A_1 $\longrightarrow$ record to spreadsheet $\longrightarrow$ tick Row limit(10) $\longrightarrow$ Close. Similarly record the point B_1 to spreadsheet.

We can observe that as the x coordinates of A and B approach to 2, the y coordinates of A_2 and B_2 approach 4.
If we call the x coordinates of A and B as x, then the y coordinates of A_2 and B_2 are $f(x)$
So we observe that as $x \to 2$, $f(x) \to 4$
ie, the limit of $f(x)$ at $x = 2$ is 4

What happens to the points A, B, A_2 and B_2 when **h** = 0

Activity 13.2 Limit of Rational Functions

Procedure:

- In the above applet, change the function to $f(x) = \dfrac{x^2 - 4}{x - 2}$

- Move the slider **h** from 1 to 0

- What is the limit of this function at $x = 2$

- What happens to the points A_2 and B_2 when **h** = 0 (Refer Activity 3.2)

Activity 13.3 Limit of Piecewise Functions

Procedure:

Using the above applet, discuss the limit of the following functions

1. $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x + 1 & \text{if } x > 2 \end{cases}$ at $x = 2$

(Input `If[x<=2,x^2,2x+1]`)

What happens to $f(x)$ as x approaches to 2 from left and right?

2. Change $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$ and discuss the limit at $x = 2$
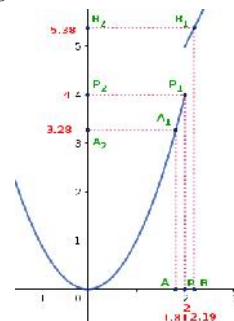
Discuss the existence of limit for the following functions

1. $f(x) = \dfrac{1}{x},$ at $x = 0$

2. $f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$

3. $f(x) = \begin{cases} x - 2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x + 2 & \text{if } x > 0 \end{cases}$
(Input `If[x<0,x-2,x>0,x+2,0]`
Or
`If[x<0,x-2,x==0,0,x+2]`)

4. $f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 1 \\ -x^2 - 1 & \text{if } x > 1 \end{cases}$

Activity 13.4 Limit of Trigonometric Functions

Procedure:

• Plot the graphs of $\sin x$ and x in the same graphics view. Zoom it at the origin. What do you see? What inference do you get from this?

> Plot the graphs of x^2, $\sin(x^2)$, $\sin^2 x$, $\tan(x^2)$ and $\tan^2 x$ on the same graphics view, Zoom it at the origin and what do you see? What inference you get from this?

• Using the applet used in the previous activity, discuss the limit of $\dfrac{\sin x}{x}$ at $x = 0$

Activity 13.5 Limit of Exponential and Logarithmic Functions

Procedure:

• Input **a**=0.We get a slider in the Algebra view. Show it in the graphics view by clicking on it

• Draw the graph of the function $f(x) = e^x - a$

• Input `y = x` to get the line

• Using `Reflect about line` tool, click on the graph and on the line, we get the reflection of the graph of e^x on the line $y = x$, which represents the graph of $log_e(x)$

- Using the slider **a**, move the graph of f downwards until the line becomes tangent to the curve

- What happens to the reflection ?

- What are the definitions of the functions represented by the curves?

- Zoom it at the origin until the three curves seem to be one

- What do you infer from this?

- Write down some limits using this inference

Additional Activities

Activity 13.A Some more problems

Procedure:

With the help of the applet, discuss the limit of the following functions

1. $f(x) = \sin\left(\frac{1}{x}\right)$ at $x = 0$

2. $f(x) = x \sin\left(\frac{1}{x}\right)$ at $x = 0$
   Draw the lines $y = x$ and $y = -x$. Why does the graph of $x \sin\left(\frac{1}{x}\right)$ lie between these lines?

3. $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ at $x = 0$
   Draw the curves $y = x^2$ and $y = -x^2$. Discuss the existence of the limit of $x^2 \sin\left(\frac{1}{x}\right)$ at 0 with the help of these graphs

4. $f(x) = \sqrt{x} \sin\left(\frac{1}{x}\right)$ at $x = 0$
   Draw the parabola $y^2 = x$. Discuss the existence of limit of the above function with the help of this curve

Lab 18

Functions

Aim:

- To visualise one to one and onto function geometrically
- To restrict the domain and codomain of functions so as to make it a bijection
- To visualise composition of two functions geometrically

Concepts:

- One to one and onto functions
- Composition of functions

Discussion :
We discuss the properties of graphs of One to One and onto functions.Discuss about how to determine whether a given function is one to one and onto geometrically,how to restrict the domain and codomain of functions so as to make them bijections.We plot the graph of $g \circ f$ from the graphs of the functions f and g

Activity 18.1 One to One and Onto Functions

Procedure

- Draw the graph of $f(x) = x^2$
- Create a slider a and plot the point $(0, a)$
- Draw a line through the point and perpendicular to the y axis.

    📖 If for any value of a, the line meets the graph of a function at more than one point, can we say whether the function one to one or not. Why ?

    📖 If for all values of a, the line meets the graph of a function at at least one point, can we say whether the function onto or not. Why ?

    📖 Using above applet, say whether the function $f : \mathbb{R} \to \mathbb{R}$ defined above is one to one or onto.

4
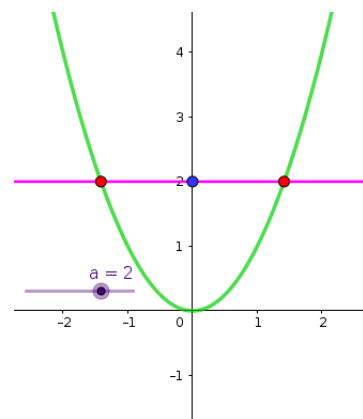
- Keeping above ideas in mind, one can say whether a function is one to one or onto, by observing its graph, even without drawing the horizontal line.

- Save the file as Activity 18.1

Activity 18.2 One to One and Onto function 2

Open Activity 18.1 and save as Activity 18.2

Create Input box for f

Draw the graphs of the following functions defined from $\mathbb{R}$ to $\mathbb{R}$ and say whether they are one to one or onto. Find the range of the function in each case.

	Function	One-One	Onto	Range
i	$x^2 + 2$			
ii	$x^3 - 3x^2 + 3$			
iii	$3\sin(x)$			

The following functions are defined from a subset of the set of real numbers. Say whether they are one to one or onto. Find their range.

i) $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{x-2}{x-3}$

ii) $f : [-1, 1] \to \left[-1, \dfrac{1}{3}\right]$ defined by $f(x) = \dfrac{x}{x+2}$

iii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{1+|x|}$

What is the peculiarity of the range of an onto function ?

Activity 18.3 Bijective Functions

We can make a function one to one by restricting its domain and onto by restricting its co-domain. If we define a function to its range, it becomes onto. From the graph of a function we can easily select a suitable subset of the domain to make it one to one

Procedure

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$. Is it one-one or onto ?

- Draw the graph of above function

  Redefine the function by restricting its co-domain so that it becomes onto

- From the graph of the function we can see that if we restrict the domain to $[0, \infty)$ the function becomes one to one.

- So the function $f : [0, \infty) \to [1, \infty)$ defined by $f(x) = x^2 + 1$ is a bijection.

- Draw the graph of above function. (Use input command `if(x>=0,x^2+1)` or `if(x>=0,f)` or `Function(f,0,infinity)`
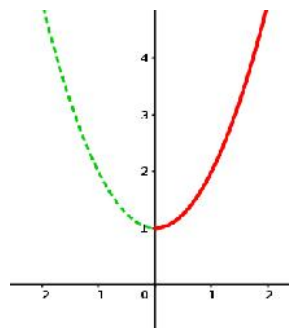
- If we define $f : (-\infty, 0] \to [1, \infty)$ then also it becomes a bijection. Draw the graph of this function.

     Restrict domain and co-domain of the following functions so that they becomes bijections

	Function	Domain	Co-domain
i	$\|x - 2\|$	$[2, \infty)$	
ii	$x^2 - 3x + 3$		
iii	$\|\sin x\|$		

     For each of above functions find one more domain which make them one to one. Draw the corresponding graphs.

Activity 18.4 Composition of functions

We create an applet to describe the composition of two functions geometrically

Procedure

- Draw the graph of $f(x) = x^2$ and $g(x) = \sin x$
- Draw the line $y = x$
- Create a slider a
- Plot the point $A(a, f(a))$
- Draw the line passing through A and parallel to x axis and plot its point of intersection B with the line $y = x$
- Draw the line passing through B and parallel to the y axis and plot its point of intersection C with the graph of the function g
- Draw the following lines and take their point of intersection D

    – Passing through A and parallel to the y axis

    – Passing through C and parallel to the x axis
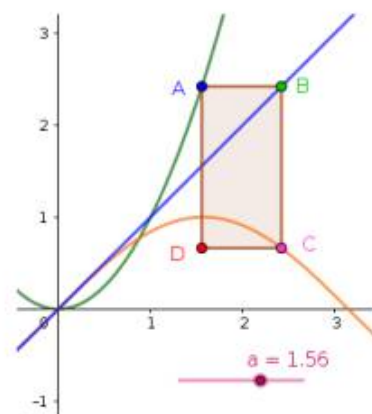
- Complete the rectangle $ABCD$ and hide the lines

     Write the co ordinates of the points B,C and D in terms of a

     Trace the point D and animate the slider.What does the graph represents?Use locus tool to get the clear path.

- Create input boxes for the functions f and g.

     Following functions are of the form $(f \circ g)(x)$. Identify f and g and use above applet to trace the path of the given functions

i)$\sin |x|$ ii) e^{logx}

Additional Activities

Activity 18.A Composition - Physical situation

We discuss a physical situation in which the concept of composition of functions included.

Use Applet ML18.A

About the applet
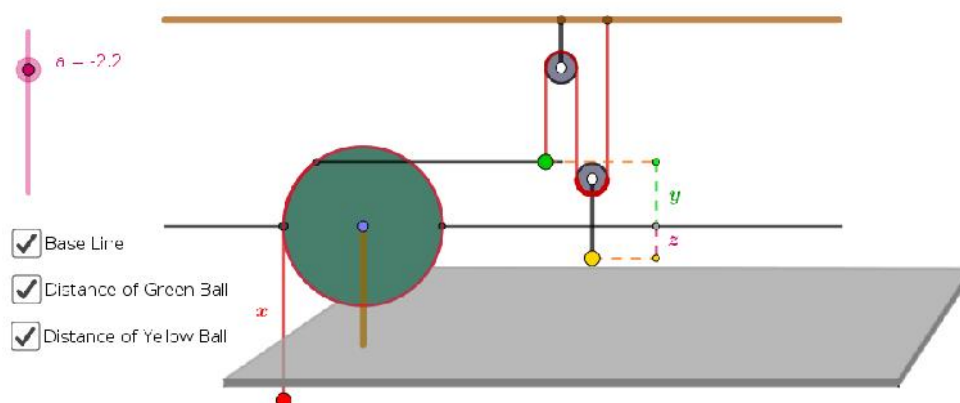
A rope is wound on a disc of radius 1 meter . A red ball is attached at the end of the rope. A green ball is attached to a point on the circumference of the disc by means of a rod which always remains horizontal. A yellow ball is attached to the green ball using two pulleys. We can wind/unwind the rope using the slider **a**. When **a**=0 all the balls lie on the base line, the horizontal line passing through the centre of the disc.

If we unwind the disc so that the red ball moves x meters down wards, let the green ball moves y meters and yellow ball moves z meters from the base line. We take the movement of a balls as positive, if it is above the base line and negative if it is below the base line.

Write y as a function of x.

Write z as a function of y.

Write z as a function of x.

Suppose we pull down the red ball at the rate of 2m/s. Write x, y and z as functions of the time t.

Suppose that the red ball freely falls under gravitational force. Write x, y and z as functions of the time t.

Activity 18.B One-one and onto property of functions

We have seen how to check the one to one and onto properties of functions.Now let us check these properties of composition of such functions.Consider the following functions in an applet

f	one-one	onto	g	one-one	onto	$g \circ f$	one-one	0nto
$2x + 3$			$x - 4$					
$x^2 + 1$			$x - 4$					
$x^{\frac{3}{2}}$			x^2					
$\dfrac{1}{x}$			$x - [x]$					
$\dfrac{1}{x}, x > 0$			x^2					
x^3			x^2					
$|x|$			$x - [x]$					
$3x - 1$			x^2					
$\sin x$			x^2					
$\cos x$			x^3					

and verify the properties

- f is one to one and g is one to one implies $g \circ f$ is one to one

- f is onto and g is onto implies $g \circ f$ is onto

- $g \circ f$ is one to one implies f is one to one and g need not be one to one

- $g \circ f$ is onto implies g is onto and f need not be onto

Lab 19

Invertible Functions

Aim:

- To explore invertible functions

Concepts:

- Bijective functions
- Inverse of a function
- Composition of functions

Discussion :
For a given real number a,we discuss the method of finding the value of $f^{-1}(a)$,from the graph of $f(x)$.Methods of drawing graph of inverse of a function is also discussed.

Activity 19.1 Inverse of a Function 1

If f is an invertible function and (x, y) is a point on the graph of f then $f(x) = y$ and $f^{-1}(y) = x$. We use this idea to find the value of $f^{-1}(a)$, for given values of a, from the graph of $f(x)$.

Procedure :

- Draw the graph of $f(x) = x^2, x \geq 0$

- Create a slider a

- Plot the point $A(0, a)$

- Draw the line passing through A and parallel to the x axis

- Plot the point of intersection B of the above line with the curve

- Draw the line through B, perpendicular to the x axis

- Plot the point of intersection C of the above perpendicular line with the x axis

- Hide the lines and draw the line segments AB and BC

- Show the coordinates of C
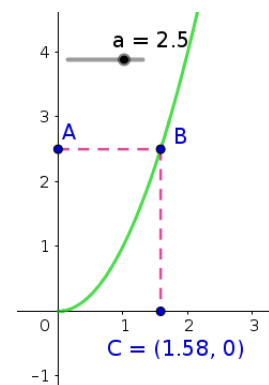
📖   Using this applet, how can we find an approximate value of $\sqrt{a}$ for a given value of a ?

📖   Find approximate values of the following

i) $\sqrt{3}$ ii) $\sqrt{4.5}$

- Note : If we need more accuracy we can increase the number of decimal places. (Options $\rightarrow$ Rounding)

- Save the applet as Activity 19.1

Activity 19.2 Inverse of a Function 2

Procedure :

Use the applet ML19.2

About the applet:

This applet is similar to that we created in the above activity.In addition there are two input boxes to edit the function f and the slider a

Select suitable functions and find approximate values of the following

		Function	Value
i	$\sqrt[3]{4}$		
ii	$\log_2 10$		
iii	$\dfrac{1}{\sqrt{5}}$		

Activity 19.3 Inverse of a Function 3

Procedure

- Find the inverse of the function $f(x) = x^3$

- Draw the graphs of $f(x)$ and $f^{-1}(x)$

- Draw the line $y = x$

Compare the graphs of f and f^{-1} with respect to the above line. What do you observe ?

If we know the graph of function f how can we draw a rough sketch of the graph of f^{-1} ?

- Using the tool *Reflect about line* we can draw the reflection of a graph of a function about a line (Click on the graph of the function and then on the line)

- Draw the graph of function $f(x) = x^2$ and find its reflection on the line $y = x$ Does it represent the graph of a function? Why?.

- To get the graph of inverse of a function ,restrict the domain of the function -if needed-in order to make it one to one.Draw the graph of the function in the restricted domain and find its reflection on the line $y = x$

Find the inverse of the following functions algebraically and draw its graph. Draw the graph of the given function and find its reflection on the line $y = x$. Verify whether it coincides with the graph of the inverse of the function. Find the domain and range of the given function.

	Function	Inverse	Domain	Range
i	$\sqrt{x+2}$			
ii	$x^2 - 4x + 7$			
iii	$\dfrac{x}{x+2}$			

- In algebra view ,the equation of a reflected curve is shown only in its parametric form.

- The command Invert(f) gives the graph of inverse of a function and its equation in explicit form.It is not necessary to restrict the domain to get the inverse.But this command also has some drawbacks

- It works only if the function contains just one x.If f is a quadratic function ,then the command :Invert(CompleteSquare(f)) gives the inverse

- If f is a rational function with first degree polynomial in numerator ,then the command Invert(PartialFractions(f)) gives the inverse

- In CAS view 'invert' command directly works even if the function contains more than one x

Activity 19.4 Composition and Inverse

Procedure :

- * Use the applet Activity 18.4

- Give $f(x) = 3x - 2$ and $g(x) = \dfrac{x+2}{3}$

  Observe the graphs of $f \circ g$ and $g \circ f$. What do you see ? Why it happens so ?

- Find the inverse of the following functions and verify your answer using the above applet.

  [* Or we can verify the answer by directly inputting $g \circ f$ and $f \circ g$ in the input bar,without using the Applet]

	$f(x)$	$f^{-1}(x)$
i	$\dfrac{x-2}{3}$	
ii	$(3 - x^3)^{\frac{1}{3}}$	

Additional Activities

Activity 19.A Self Invertible Functions

We know that the inverse of $f(x) = \dfrac{1}{x}$ is the function itself. We may call it as a self invertible function. Here we discuss a method to generate self invertible functions from existing one. The

logic behind the construction is that the graphs of self invertible functions are symmetric with respect to the line $y = x$ (why ?)

Procedure

- draw the graph of the function $f(x) = \dfrac{1}{x}$ and the line $y = x$

- The graph is symmetric with respect to the line. If we shift it along the line, it keeps its symmetry

- Create a slider a

- Create the function $g(x) = f(x - a) + a$. Find the simplified form of g using the command `Simplify(g)`.

- Create an input box for f and save the applet as Activity 19.A

We get different self invertible functions for different values of a. Find some functions and verify it using the applet Activity 18.4

Starting from the functions $\dfrac{2}{x}, \dfrac{-3}{2x}, \dfrac{2x-5}{2x-2}$ generate more self invertible functions.

Activity 19.B Composition Machine

Last year we have studied functions as input -output machines,which gives outputs corresponding to the given inputs.Here we compare composition of functions with combination of machines in which output of one of the machine is the input of the other.

Use Applet ML19.A

About the applet;
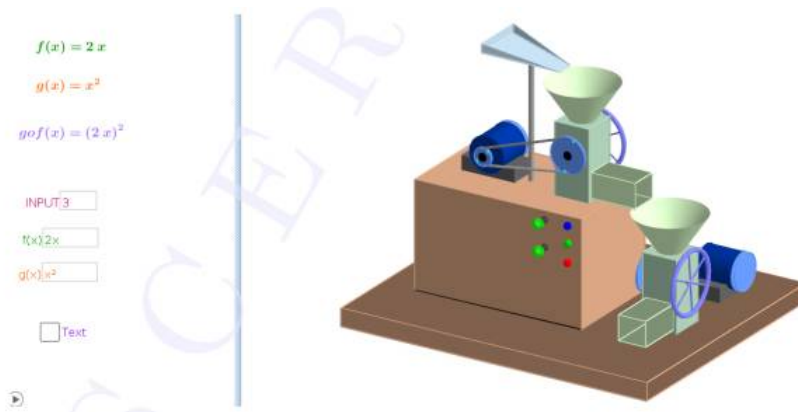
In 3D graphics we can see two machines combined together.Output of the first machine is the input of the second.There are three buttons on the machine, Blue,Green ,and Red,in order to refresh,start or stop the machine.Two indicator lights are provided which turns red if the corresponding input is not suitable for the machines.From the Graphics view we can edit the functions as well as the input number

Procedure

- Set $f(x) = 2x$ and $g(x) = x^2$

Find the values of $(g \circ f)(2)$ and $(g \circ f)(3)$ and $g \circ f(-3)$

- Set $f(x) = 2x$ and $g(x) = \sqrt{x}$

  Find the values of $(g \circ f)(2)$ and $(g \circ f)(3)$

  What happens to the indicators if we give -3 as the input? why?

  What can you say about the domain of the function g in order to define $g \circ f$?

- Set $f(x) = \sqrt{x}$ and $g(x) = x^2$

  Find the values of $(g \circ f)(2)$ and $(g \circ f)(1)$ and $g \circ f(0)$

  What happens to the indicators if we give -2 as the input? why?

  Find the domain in which $(g \circ f)(x) = x$

  Find the domain in which $(f \circ g)(x) = x$.

Lab 20

Inverse Trigonometric Functions 1

Aim:

- To make trigonometric functions invertible by restricting the domain and codomain with the help of their graphs

- To draw the graphs of inverse trigonometric functions

- To find the value of inverse trigonometric functions from graphs of corresponding trigonometric function and from unit circle

Concepts:

- Trigonometric functions

- Inverse trigonometric functions

Discussion :
We discuss how to restrict domain and codomain of a trigonometric function so as to make it invertible

Activity 20.1 Making Invertible

As we have seen in Activity 18.3 we can make a function One-One and Onto by restricting its domain and range. In this activity we discuss trigonometric functions and make them bijections.

Procedure :

- Mark the distance on x axis in terms of $\frac{\pi}{2}$. (Right click on Graphics View $\rightarrow$ Graphics $\rightarrow$ x axis $\rightarrow$ Distance $\rightarrow$ Select $\dfrac{\pi}{2}$ from the drop down menu)

- Draw the graph of $f(x) = \sin x$.

- By observing the graph we can see that if we re define f as $f : [\frac{-\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ it becomes a bijection.

- Draw the graph of the above function (use the command `If(-pi/2<=x<=pi/2,f)`.)
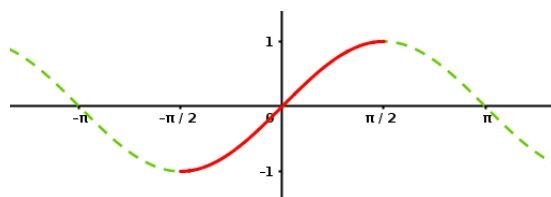
• 📖   By selecting suitable domain and range, make all trigonometric functions invertible. Draw the corresponding graphs and complete the following table (There are infinitely many possible domains which makes a trigonometric function bijective.You may select any one among them)

Function	Domain	Range
$\sin x$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\cos x$		
$\tan x$		
$\operatorname{cosec} x$		
$\sec x$		
$\cot x$		

• Save the GeoGebra file as Activity 20.1

Activity 20.2 Graph of Inverse Trigonometric Functions

We know that the graph of f^{-1} is obtained by reflecting the graph of f on the line $y = x$. We use this idea to draw the graph of inverse trigonometric functions.

• Open the GeoGebra file Activity 20.1

• Mark the distance on x axis as ...-1,0,1,2,3,... and that on y axis in terms of $\frac{\pi}{2}$

• Draw the line $y = x$

• Open the graphs of trigonometric functions (bijective) one by one and reflect them on the line $y = x$ (using `Reflect about Line` tool.)

• save the file as Activity 20.2
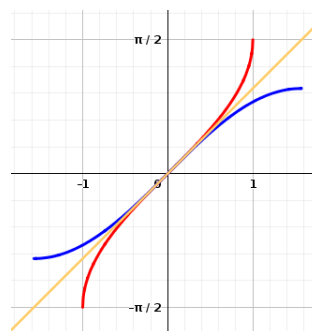
• 📖   Write the domain and range of the of the inverse trigonometric functions.

Function	Damain	Range
$\sin^{-1} x$		
$\cos^{-1} x$		
$\tan^{-1} x$		
$\operatorname{cosec}^{-1} x$		
$\sec^{-1} x$		
$\cot^{-1} x$		

We can draw the graphs of inverse trigonometric functions using input commands

| Function	Command
$\sin^{-1} x$	`sin^-1(x)` or `asin(x)`
$\cos^{-1} x$	`cos^-1(x)` or `acos(x)`
$\tan^{-1} x$	`tan^-1(x)` or `atan(x)`
$\operatorname{cosec}^{-1} x$	`cosec^-1(x)` or `asin(1/x)`
$\sec^{-1} x$	`sec^-1(x)` or `acos(1/x)`
$\cot^{-1} x$	`cot^-1(x)` or `pi/2-atan(x)`

Activity 20.3 Value of Inverse Trigonometric Functions from Graph

In Activity 19.1 we discussed the method of finding $f^{-1}(a)$, from the graph of the function f, for given values of a. We use this method to find the values of inverse trigonometric functions from the graphs of the corresponding trigonometric functions.

Use the applet ML20.3

About the applet:

In this applet graph of a function f is given .Using an input box we can edit the function.Using another input box we can change the value of the slider a.The point A on the y-axis represents $(0, a)$. x co ordinate of the point C represents $f^{-1}(a)$
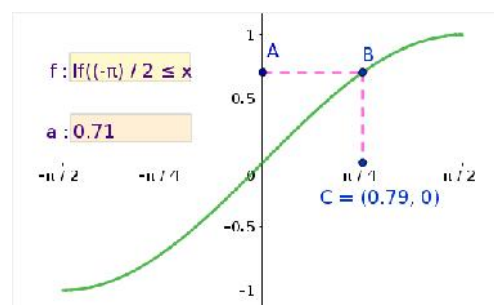
Find the values of $\sin^{-1}(0.5)$ and $\sin^{-1}(-\frac{1}{\sqrt{2}})$

Find the values of the following. If possible write in terms of π otherwise write in decimal form (may be an approximate value)

$\sin^{-1}(\frac{\sqrt{3}}{2})$	$\cos^{-1}(0.5)$	$\tan^{-1}(5)$	$\text{cosec}^{-1}(-2)$	$\sec^{-1}(\frac{2}{\sqrt{3}})$	$\cot^{-1}(-4)$

Activity 20.4 Value of Inverse Trigonometric Functions from Unit Circle

We have defined $\sin x$ and $\cos x$ with the help of a unit circle centered at the origin. Here we use the reverse of that process to find the values of $\sin^{-1} x$ and $\cos^{-1} x$ for given values of x.

Use applet MLActivity 20.4

About the applet

- For a given value of a we can find the values of $\sin^{-1} a$, $\cos^{-1} a$ or $\tan^{-1} a$. We can select the required one using the check boxes.

- We can change the value of a using the slider

- If we select the check box 'sine' we get a semi circle of radius 1 on the positive side of the x axis and a point B, with y co-ordinate a, moves along y axis according to a.

Procedure :

- Suppose we are asked to find $\sin 0.85$ (refer the figure given below), we rotate the point P from $A(1, 0)$ by an arc of length 0.85 (or by an angle of 0.85 radian) in anticlockwise direction. The y coordinate of P gives $\sin 0.85$ that is 0.75

- If we are asked to find $\sin^{-1} 0.75$, then the question is reversed, that is, what should be the angle (or arc length) of rotation of P in order to get its y coordinate as 0.75 ?. So we plot the point $B(0, 0.75)$, the corresponding point on the circle and find the arc length (or angle). So from the figure $\sin^{-1} 0.75 =$).85 (It may be an approximate value). If the rotation of P is in the clockwise direction we take the value as negative.
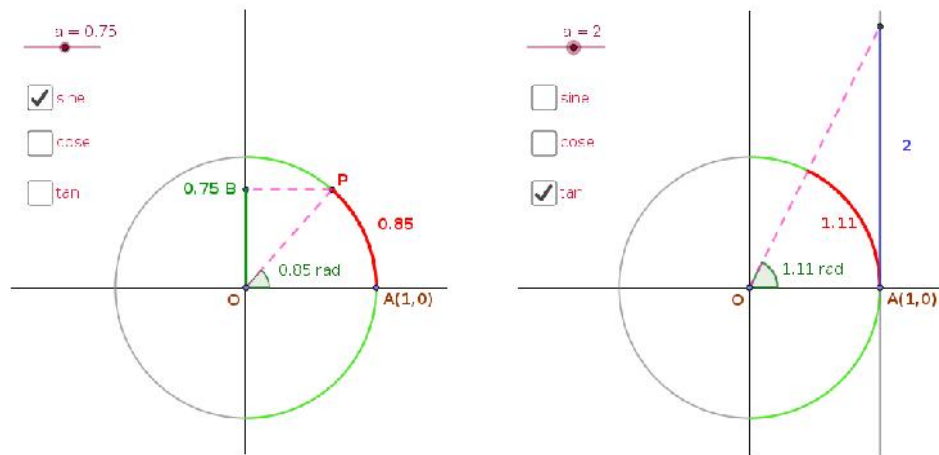
- In a similar manner we can find $\cos^{-1} a$ for give values of a

- 📖 The figure given above shows that $\tan^{-1} 2 = 1.11$. Discuss the reason

- 📖 Discuss a method to find $\tan^{-1} a$, using the applet, for given values of a

- 📖 Using the applet find the following values

$\sin^{-1}(0.4)$	$\sin^{-1}(-0.6)$	$\cos^{-1}(0.8)$	$\cos^{-1}(-0.5)$	$\tan^{-1}(5)$	$\tan^{-1}(-3)$

Lab 31

Application of integrals

Aim

- To find the area enclosed by curves using definite integrals

Concepts

- Definite integrals

Discussion

We discuss different methods of finding area enclosed by curves with the help of GeoGebra

Activity 31.1 Area bounded by x axis

If the curve $y = f(x)$ in $[a, b]$ is above the x axis then the area of the region bounded by the curve, x axis $x = a$ and $x = b$ is given by

$A = \int_a^b f(x)dx$. We can find this using the Input Command `Integral(f,a,b)`

Procedure

- Create two number sliders **a** and **b** and Input boxes for them

- Draw the graph of the function $f(x) = x^2$.

- Find the area enclosed by the curve $y = f(x)$, x axis, $x = 0$ and $x = 2$. [For this set $a = 0$, $b = 2$ and give the Input Command `Integral(f,a,b)`]

    📖 Find the area enclosed by the curve $y = x^2$, x axis, between

    1) $x = -1$ and $x = 3$
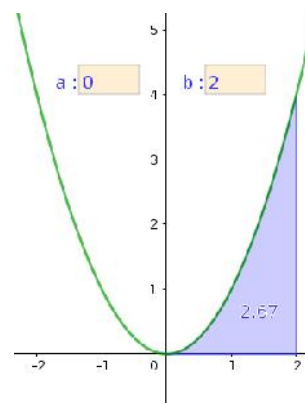
    2) $x = -5$ and $x = 0$

- Create a Input Box for the function $f(x)$.

    📖 Find the area enclosed by the curve $y = x^2 - 1$, x axis between $x = -1$ and $x = 1$.

    📖 Find the required area and complete the following table.

Sl.No.	Enclosed Region	Area
1	$y^2 = x, x$ axis, $x = 1, x = 4$	
2	$\frac{x^2}{16} + \frac{y^2}{9} = 1$	
3	$x^2 + y^2 = 4$	
4	$y = \sin x, x$ axis, $x = \pi, x = 2\pi$	

Hint: For question number 2 and 3 draw the whole region using input bar

• save as Activity33.1

Activity 31.2 Area of a region which is above and below x axis

If a curve lies on both sides of the x axis it is not possible to find its area by direct integration. We discuss different methods of solving such problems.

Procedure

• Find the area enclosed by the curve $y = x^2 - 3, x$ axis $x = -3$ and $x = 3$

• Open the applet Activity33.1. Set $f(x) = x^2 - 3, a = -3$ and $b = 3$.

What is the value of $\int_{-3}^{3} x^2 - 3 \, dx$?. Give the reason.

What is the area of the portion below the x axis?

[To find this find the points of intersection A and B of the curve with the x axis and give the input command `Integral(f,x(A),x(B))`]

What is the area of the portion above the x axis?

Find the total area of the region.
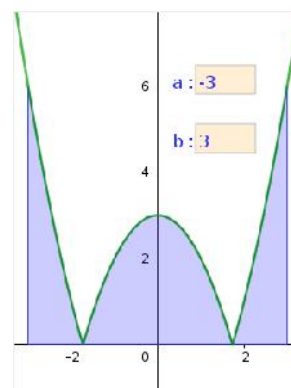
Alternate Method

• Edit the function as $f(x) = |x^2 - 3|$. Set $a = -3$ and $b = 3$

Find the area using the input command `Integral`(f,a,b)

Find the required area and complete the following table.

Sl.No.	Enclosed Region	Area
1	$y = \sin x, x$ axis, $x = 0, x = 2\pi$	
2	$y = x^3, x$ axis, $x = -2, x = 4$	
3	$y = |x| - 2, x$ axis, $x = -1, x = 3$	
4	$y = 3x + 2, x$ axis, $x = -1, x = 1$	

If the area bounded by the curve $y = \sqrt{3x}$ and the line $x = 12$ is divided in the ratio 1:7 by the line $x = a$, then find the value of a. (If necessary change the maximum value of the slider **b**)

Activity 31.3 Combination of curves

We discuss the methods of finding the area bounded by a combination of curves.

📖 Find the area of the region bounded by the curve $x^2 + y^2 = 32$ and the line $y = x$ in the first quadrant.

Procedure

- Draw the circle $x^2 + y^2 = 32$ and the line $y = x$.

- Identify the enclosed region.

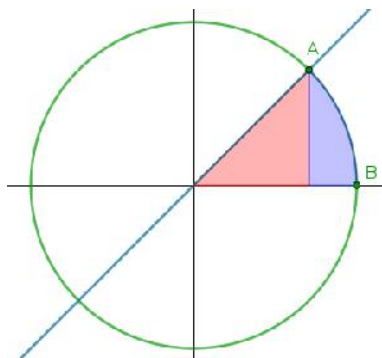- Mark the point of intersection A of the line with the circle. Also mark the point of intersection B of the circle with the x axis.

- Find the area under the line $y = x$ using the input command `a=Integral(x,0,x(A))`.

- Find the area under the circle using the input command `b=Integral(sqrt(32-x^2),x(A),x(B))`.

📖 Find the required area.

Alternate method

- Define the function as $f(x) = \begin{cases} x & \text{if } 0 < x < x(A) \\ \sqrt{32 - x^2} & \text{if } x(A) < x < x(B) \end{cases}$

For this give the input command `f(x)=If(0<x<x(A),x,x(A)<x<x(B),sqrt(32-x^2)`

📖 Find the area using the input command `Integral(f,0,x(B))`

📖 Find the area of the following regions.

1) Region in the first quadrant bounded by x axis, the line $y = \sqrt{3}x$ and the circle $x^2 + y^2 = 4$

2) Region lying above x axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

3) Region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

4) The Region $\{(x, y) : 0 \le y \le x^2, 0 \le y \le x + 1, 0 \le x \le 2\}$

Activity 31.A Availability of medicine in the blood

Medicine can be administered to patients in different ways. For a given method let c(t) be the concentration of the medicine in the blood t hours after the dose is given.Over the time interval $0 \le t \le a$ the area between the grapf of $c(t)$ in the interval $[0, a]$ indicates the availability of the medicine for the patients body over the time.

Since GeoGebra does not have the time axis, we will change the independent variable to x.

Two methods Method 1 and Method II for administrating the medicine is governed by the functions $f(x)$ and $g(x)$ represented as follows

$$f(x) = 5(e^{-0.2x} - e^{-x})$$

$$f(x) = 4(e^{-0.2x} - e^{-3x})$$

In this activity we try to explore the availability of medicine to the patients body

Procedure

- Create a slider a with minimum value 0 and maximum value 24 , corresponding to 24 hour period

- Create the function $f(x)$

- Create the function $g(x)$

- Evaluate the integral of f from 0 to a using integral function .This can be achieved by inputting $Integral(f, 0, a)$

- Evaluate the integral of g from 0 to a as above .In the algebra view we can see that two variables are created for the area of the two figures

- Move the slider a and observe the change in the areas

If 4 is the value of a , which area is more?Which method provides better availability of the medicine?

If 24 is the value of a, which area is more? Which method provides better availability of the medicine?

Move the slider in such a way that the areas of both the functions are same.What does this condition signify in medical perspective?

Lab 34

Vectors

Aim

- To explore the concept of vectors and vector operations

- To explore the concepts of direction angles and direction cosines

- To resolve a vector in given directions

Concepts

- vectors and vector operations

- Direction angles and direction cosines of a vector

- Components of a vector

Discussion

We familiarise some tools and commands for the construction of vectors and vector operations in GeoGebra, using which we discuss some problems on vectors which helps to get the geometry of vectors.Discuss direction angles and direction cosines of vectors geometrically.We also discuss the method of resolving a vector in given directions .

Tools and Commands

Let's familiarise some tools and commands used for the construction of vectors and vector operations in GeoGebra.

- `Vector` tool or the Input Command `Vector(Point,Point)` gives the vector which starts from the first point and ends at the second point.

- The Input Command `Vector(Point)` gives the vector starts from the origin and ends at the point.

- `Vector from Point` tool, by clicking on a point and on a vector gives the vector starts from the point and parallel to the given vector.

- The Input Command `Point(Point,Vector)` Gives the end point of the given vector which starts from the given point.

- `Translate by Vector` tool, by clicking on an object translates it by the magnitude of the vector in its direction.

- `|u|` gives the modulus of $\vec{u}$

- If a is a scalar and u is a vector, then the Input command `au` or `a*u` gives the vector $a\vec{u}$

- If u and v are vectors, the Input command `u+v` gives $\vec{u} + \vec{v}$ and `u-v` gives $\vec{u} - \vec{v}$.

## Activity 34.1 Construction of Vectors

Do the following problems

1. Construct the vector $3\hat{i} + \hat{j} + 2\hat{k}$.
2. Construct the vector $3\hat{i} + \hat{j} + 2\hat{k}$ starting from the point $(2, -2, 3)$.
3. Construct the vector joining the points $(1, -2, 4)$ and $(3, 2, -2)$.
4. Construct three mutually perpendicular vectors with same initial point $(2, 2, 2)$.
5. Construct the parallelogram whose adjacent sides are represented by the vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} - 3\hat{k}$
6. Create a slider **a** and a vector $\vec{u}$. Construct the vector $a\vec{u}$. Observe the vector by changing the value of a.
7. Construct the vector $\vec{u} = 4\hat{i} - 3\hat{j} + 3\hat{k}$. Construct the unit vector in the direction of $\vec{u}$. Also Construct the vector having magnitude 5 in the direction of $\vec{u}$.
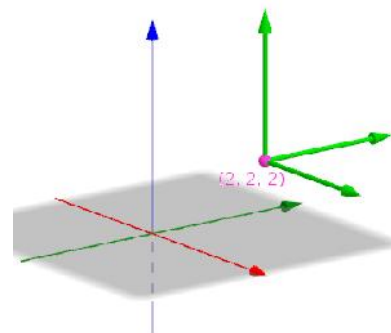
Activity 34.2 Direction ratios and Direction Cosines

Procedure

- Plot the points $O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ and $P(4, 3, 2)$.

- Construct the vector $\overrightarrow{OP}$.

- Find the angles α, β and γ made by $\overrightarrow{OP}$ with the positive direction of x axis, y axis and z axis. [input : α=Angle(A,P), β=Angle(B,P),γ=Angle(C,P)]. Hide the points A, B and C.

- Draw the perpendicular from P to the x axis and plot its point of intersection D with x axis.

  Write the x coordinate of D.

  Find $\cos\alpha$ without using the input command $\cos(\alpha)$. Similarly find $\cos\beta$ and $\cos\gamma$.

  If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ write $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$ in terms of x, y, z.

  Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
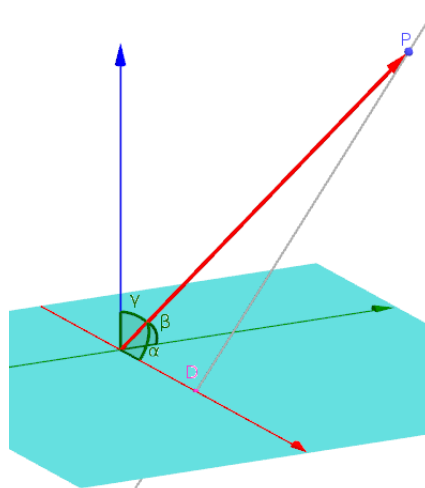
Open Graphics View and create an input box for the point P(It is not possible to create input box in 3D Graphics)

Edit the co ordinates of the point P such that it lies on the x- axis .What happens to the direction cosines?What happens if P lies on the y-axis or z-axis?

📖   Edit the co ordinates of P such that it lies on the xy plane . What happens to n?Write l and m in terms of α only.

Activity 34.3 Components of a vector

Procedure

Construct the components vectors of the vector $\vec{r} = 2\hat{i} + 5\hat{j} - 3\hat{k}$ along the positive direction of the coordinate axes.

- Plot the point $A(2, 5, -3)$ and construct the vector $\vec{r}$ using the input command `r=Vector(A)`.

- Plot the points $B(x(A), 0, 0), C(0, y(A), 0)$ and $D(0, 0, z(A))$. Draw the segments AB, AC and AD.

- Construct the component vectors.

    📖   Create an input box for the point A. Write the vector $\vec{r}$ and its component vectors for the following positions of A, $(3, 2, 0), (0, 5, -2), (0, 4, 0), (0, 0, -3)$

Express the vector $\vec{r} = 4\hat{i} + 4\hat{j}$ as the sum of two vectors which are parallel to $\vec{u} = 3\hat{i} + \hat{j}$ and $\vec{v} = \hat{i} + 6\hat{j}$

- Construct $\vec{r}$ as in the last discussion

- Construct the vectors $\vec{u}$ and $\vec{v}$.

- Draw the lines passing through the origin and parallel to the vectors $\vec{u}$ and $\vec{v}$. For this give the input commands `Line((0,0,0),u)` and `Line((0,0,0),v)`

- Draw the line passing through the point A (end point of $\vec{r}$) and parallel to $\vec{v}$. Plot its point of intersection B with the line parallel to $\vec{u}$

- Draw the vectors $\overrightarrow{OB}$ and $\overrightarrow{BA}$ where O is the origin.
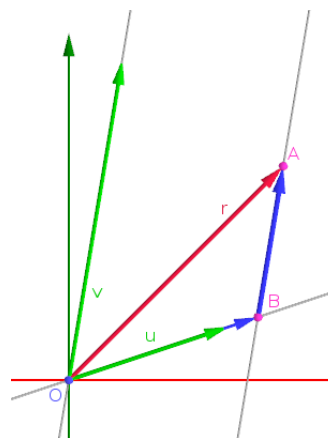
Note: Since all the vectors in the above discussion are in the xy-plane this construction can be done in the graphics view also.

Additional Activities

Activity 34.A Components of a vector in space

Express the vector $\vec{r} = 2\hat{i} + 5\hat{j} + 3\hat{k}$ as the sum of the vectors parallel to $\vec{a} = 4\hat{i} + 2\hat{j}$, $\vec{b} = -\hat{i} + 4\hat{j}$ and $\vec{c} = \hat{i} + \hat{j} + 5\hat{k}$

- Construct the points $P(2,5,3), A(4,2,0), B(-1,4,0)$ and $C(1,1,5)$ hence construct the vectors $\vec{r}, \vec{a}, \vec{b}$ and $\vec{c}$.

- Draw the line passing through P and parallel to $\vec{c}$ and plot its point of intersection D with the xy plane. (xy plane is available in the 3D view by default. Otherwise give the input command z=0.)

- Draw the line passing through O and parallel to $\vec{a}$

- Draw the line passing through D and parallel to $\vec{b}$ and plot its point of intersection E with the line constructed in the last step.

- Draw the vectors $\overrightarrow{OE}, \overrightarrow{ED}$ and $\overrightarrow{DP}$.
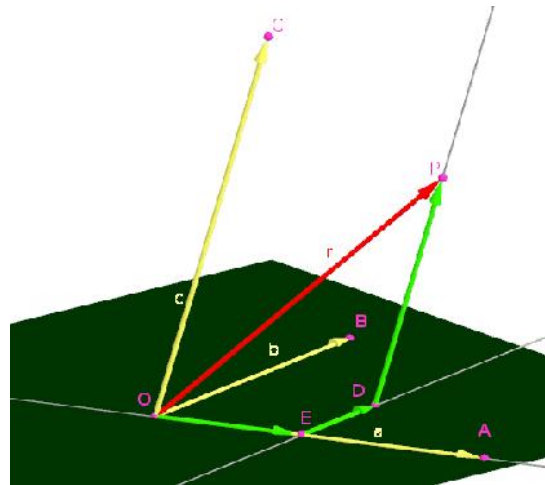
Can you express the vector $\vec{r} = 2\hat{i} + 5\hat{j} + 3\hat{k}$ as the sum of the vectors parallel to $\vec{a} = 4\hat{i} + 2\hat{j}$, $\vec{b} = -\hat{i} + 4\hat{j}$ and $\vec{c} = \hat{i} + \hat{j}$? Why ?

Activity 34.B The Vector Clock

In this activity a clock is viewed from the perspective of vectors. The origin is assumed as the center of the clock and vectors are drawn from the origin to each hour position.

Procedure:

- Create a point A(0,0)

- Create a point B(0,5)

- Using the Sequence command, create a list of points which are a result of rotation of B by an angle $30^0, 60^0, 90^0, \ldots 330^0$. This can be achieved by the command $Sequence(Rotate(B, -k*(360/12)^0), k, 0, 11)$. A list $l1$ is created.

- Draw vectors from A to the list created in the previous step by using Sequence command. This can be achieved by $Sequence(Vector(A, l1(k), k, 1, 12)$. A list of vectors $l2$ is created.

- Translate the vector $l2(2)$ to the tip of $l2(1$ using the command $Translate(l2(2), l2(1))$. A point C is created.

- Draw a vector from the tip of $l2(1)$ to the point C by using the command $Vector(l2(1), C)$

- Translate the vector $l2(3)$ to the tip of C using the command $Translate(l2(3), C)$. A point D is created.

- Draw a vector from the tip of C to the point D by using the command $Vector(C, D)$

- Translate the vector $l2(4)$ to the tip of D using the command $Translate(l2(4), D)$. A point E is created.

- Draw a vector from the tip of D to the point D by using the command $Vector(D, E)$

- Continue this for all the vectors which represent the hands of the clock.

What is the sum of the vectors of the hands of the clock?

The sum of the vectors can be achieved using a single command Sum. In the input bar type $Sum(l2)$. This will create a vector whose sum is the vector sum of all the vectors in the list $l2$.

What is the sum of all the vectors if the vector corresponding to 2:00 is dropped off? Use the command Sum by including all the items in the list $l2$ except the one corresponding to 2:00. This can be achieved by the command

$Sum(l2(1) + l2(2) + l2(4) + l2(5) + l2(6) + l2(7) + l2(8) + l2(9) + l2(10) + l2(11) + l2(12))$

What if the 1:00 hour handle is dropped?

Can you generalise this activity?

If two consecutive vectors are removed, say the vector corresponding to 1:00 and 2:00 are removed, what will happen to the resultant?

If the vectors corresponding to 1:00 and 3:00 are removed, what is the resultant?

If all the twelve vectors start from (0,-5) corresponding to 6:00, what will the resultant be? To do this in the algebra view right click on the point A and in the object properties change the coordinates from (0,0) to (0,-5)

Lab 37

The Plane

Aim

- To derive the equation of a plane

- To explore the geometry of planes satisfying different conditions

Concepts

- Concept of plane

- Concept of vector and vector operations

Discussion
We derive the equation of a plane passing through a given point and perpendicular to a given vector. We also discuss some problems on planes and construct them

We derive the equation of plane passing through a given point and perpendicular to a given direction.

Procedure

- Plot the point $A(3, 2, 2)$ and construct the vector $\vec{a} = 2\hat{i} + \hat{j} + 5\hat{k}$.

- Construct the plane passing through A and perpendicular to $\vec{a}$.

- Take any point B on the plane and draw the vector $\vec{u} = \overrightarrow{AB}$.

  📖   Find $\vec{a} \cdot \vec{u}$ at different position of B. Why it happens so?

Detach the point B from the plane (either use `Attach / Detach Point` tool or edit any one of the coordinates of B). Observe the change in $\vec{a} \cdot \vec{u}$. What happens to it when B is not on the plane? Why?

Write a necessary and sufficient condition for a point to be on the plane. Write the algebraic form of this condition. Hence find the equation of the plane.

Derive the equation of the plane passing through the point (x_1, y_1, z_1) and perpendicular to the vector $a\hat{i} + b\hat{j} + c\hat{k}$.
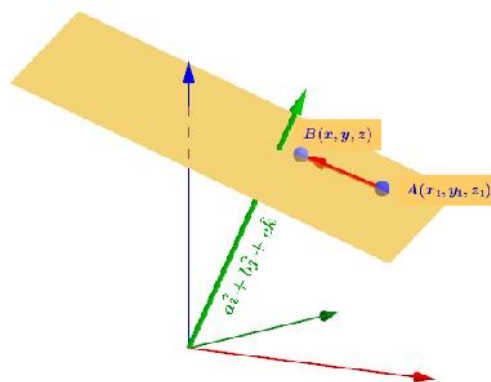
Activity 37.2 Tools and Commands

We discuss some tools and Input Commands for constructing planes

Commands

Sl.No	Input Command	Output Plane
1	Plane(Point,Plane)	Passes through the given point and parallel to the given plane
2	Plane(Point,Line)	Passes through the given point and contains the line
3	Plane(Line,Line)	Contains the lines (if they are intersecting)
4	Plane(Point,Point,Point)	Contains the given points
5	Plane(Point,Vector,Vector)	Contains the given point and parallel to the vectors
6	Plane(Polygon)	Contains the polygon
7	Plane(Conic)	Contains the given conic
8	Perpendicular plane(point,line)	Contains the point and perpendicular to the line
9	Perpendicular plane(point,vector)	Contains the point and perpendicular to the vector

Direct input of the equation gives the corresponding plane

Tools

Sl.No	Name of Tool	Output Plane
1	Plane through 3 Points	Same effect of Command 4
2	Plane	Same effect of Commands 2, 3, 4 and 6
3	Perpendicular Plane	Same effect of commands 8 and 9
4	Parallel Plane	Same effect of Command 1

Construct the planes as per the conditions given in the following problems. You can use any tool to construct vectors and lines but use only the `Perpendicular Plane` tool to construct the plane. Note the method of construction in your observation book. You can verify the result using any direct tool.

1. Plane passes through the point $(2, 2, 3)$ and parallel to the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j}$.

2. Plane passes through the point $(1, -2, 3), (1, 1, 4)$ and $(-3, 2, 2)$

3. Plane contains the line $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k} + \lambda(3\hat{j} + \hat{k})$ and parallel to $\vec{r} = 3\hat{i} + 2\hat{j} + \lambda(2\hat{i} + \hat{j} + \hat{k})$

4. Plane Contains the lines $\vec{r} = \hat{i} - \hat{j} - \hat{k} + \lambda(\hat{i} + 3\hat{j})$ and $\vec{r} = 3\hat{i} - 5\hat{j} - \hat{k} + \lambda(\hat{i} + 3\hat{j})$

5. Plane perpendicular to the vector $2\hat{i} - 2\hat{j} + 2\hat{k}$ and at a distance of 5 units from the origin.

6. Plane passes through the point $(3, 2, 4)$ and parallel to the plane $x + y + 2z = 2$

7. Plane passes through the point $(1, 2, 3)$ and perpendicular to the planes $2x + y + 2z = 3$ and $x - 3y + z = 0$

8. Plane at a distance of 4 units from the plane $3x + 2y - z + 3 = 0$