Example 9

Find $\lim_{x \to 2} \frac{x^4 - 16}{x - 2}$.

Solution

$$\lim_{x \to 2} \frac{x^4 - 16}{x - 2} = \lim_{x \to 2} \frac{x^4 - 2^4}{x - 2} = 4(2)^3 = 32$$

Example 10

Evaluate $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$. **(NCERT, March 2013)**

STUDY TIP

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$$

Solution

$$\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1} = \lim_{x \to 1} \frac{\left(\frac{x^{15} - 1}{x - 1}\right)}{\left(\frac{x^{10} - 1}{x - 1}\right)} = \frac{\lim_{x \to 1}\left(\frac{x^{15} - (1)^{15}}{x - 1}\right)}{\lim_{x \to 1}\left(\frac{x^{10} - (1)^{10}}{x - 1}\right)} = \frac{15(1)^{14}}{10(1)^9} = \frac{15}{10} = \frac{3}{2}$$

Example 11

Evaluate $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$.

Solution

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\left(\frac{\dfrac{x^3-2^3}{x-2}}{\dfrac{x^2-2^2}{x-2}}\right) = \frac{\lim\limits_{x\to 2}\dfrac{x^3-2^3}{x-2}}{\lim\limits_{x\to 2}\dfrac{x^2-2^2}{x-2}} = \frac{3(2)^2}{2(2)} = 3$$

Example 12

Evaluate $\lim\limits_{x\to 1}\dfrac{x^7-1}{x^4-1}$.

Solution

$$\lim_{x\to 1}\frac{x^7-1}{x^4-1} = \lim_{x\to 1}\frac{x^7-(1)^7}{x^4-(1)^4} = \frac{7}{4}(1)^{7-4} = \frac{7}{4}(1)^3 = \frac{7}{4}$$

Example 13

Find $\lim\limits_{x\to -1}\dfrac{x^5+1}{x^3+1}$

Solution

$$\lim_{x\to -1}\frac{x^5+1}{x^3+1} = \lim_{x\to -1}\left(\frac{\dfrac{x^5+1}{x+1}}{\dfrac{x^3+1}{x+1}}\right) = \lim_{x\to -1}\left(\frac{\dfrac{x^5-(-1)^5}{x-(-1)}}{\dfrac{x^3-(-1)^3}{x-(-1)}}\right)$$

$$= \frac{\lim\limits_{x\to -1}\dfrac{x^5-(-1)^5}{x-{}^-1}}{\lim\limits_{x\to -1}\dfrac{x^3-(-1)^3}{x-{}^-1}} = \frac{5(-1)^4}{3(-1)^2} = \frac{5}{3}$$

Example 14

Evaluate $\lim\limits_{x\to 0}\dfrac{(x+5)^2-25}{x}$.

Solution

$$\lim_{x\to 0}\frac{(x+5)^2-25}{x} = \lim_{x\to 0}\frac{(x+5)^2-5^2}{(x+5)-5} = \lim_{(x+5)\to 5}\frac{(x+5)^2-5^2}{(x+5)-5} = 2(5) = 10$$

Another Method

$$\lim_{x\to 0}\frac{(x+5)^2-25}{x} = \lim_{x\to 0}\frac{x^2+10x+25-25}{x}$$

$$= \lim_{x\to 0}\frac{x^2+10x}{x} = \lim_{x\to 0} x+10 = 10$$

Example 15

Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-1}{x}$. **(March 2009, September 2011)**

Solution

Put $y = 1 + x$, as $x \to 0$, $y \to 1$

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{y\to 1}\frac{\sqrt{y}-1}{y-1} = \lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1} = \frac{1}{2}(1)^{\frac{1}{2}-1} = \frac{1}{2}(1)^{\frac{-1}{2}} = \frac{1}{2}(1) = \frac{1}{2}$$

Another method

$$\lim_{x\to 0}\frac{\sqrt{1-x}-1}{x} = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-(1)^{\frac{1}{2}}}{(1+x)-1} = \lim_{(1+x)\to 1}\frac{(1+x)^{\frac{1}{2}}-(1)^{\frac{1}{2}}}{(1+x)-1} = \frac{1}{2}(1)^{\frac{1}{2}-1} = \frac{1}{2}(1) = \frac{1}{2}$$

13.4 LIMITS OF TRIGONOMETRIC FUNCTIONS

The following theorems are helpful in calculating limits of some trigonometric functions.

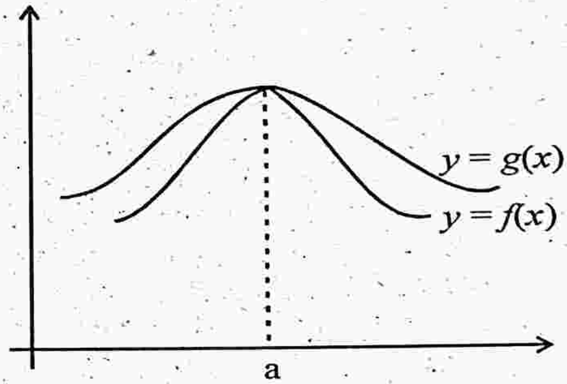

Fig 13.16

Theorem 3

Let f and g be two real valued functions with the same domain such that $f(x) \le g(x)$ for all x in the domain of definition. For some 'a' if both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

Theorem 4 (Sandwich Theorem)

Let f, g, h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the domain of definition. For some real number a, if $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$, then $\lim\limits_{x\to a} g(x) = l$

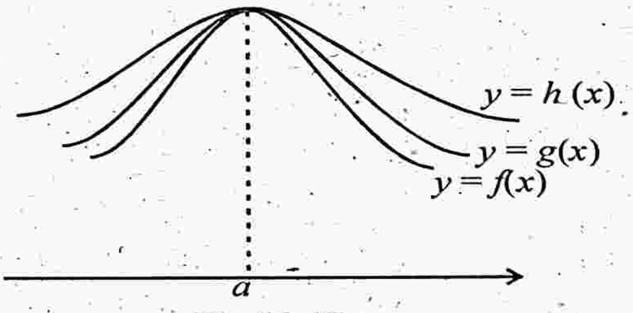

Fig 13.17

Theorem 5

i. $\lim\limits_{x\to 0} \sin x = 0$ **ii.** $\lim\limits_{x\to 0} \cos x = 1$

Example 16

Evaluate $\lim\limits_{x\to 0} \dfrac{\sin \pi x}{\cos 2x}$ **(March 2015)**

Solution

$$\lim_{x\to 0}\frac{\sin \pi x}{\cos 2x}=\frac{\lim\limits_{x\to 0}\sin \pi x}{\lim\limits_{x\to 0}\cos 2x}=\frac{0}{1}=0$$

Example 17

Evaluate i. $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$ ii. $\lim\limits_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Solution

i. $$\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^3 x}$$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\sin^3 x\ \cos x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\cos x\ \sin^2 x}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\cos x\ (1-\cos^2 x)}$$

$$=\lim_{x\to 0}\frac{1-\cos x}{\cos x\ (1-\cos x)(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{1}{\cos x\ (1+\cos x)}=\frac{1}{1(1+1)}=\frac{1}{2}$$

ii. $$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{2\sin^2 x}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{2(1-\cos^2 x)}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{2(1-\cos x)(1+\cos x)}{(1-\cos x)}$$

$$=\lim_{x\to 0}2(1+\cos x)$$

$$=2(1+1)=2(2)=4$$

Theorem 6

If x is measured in radian, $\lim_{x\to0}\frac{\sin x}{x}=1$ **(March 2013)**

Case (i)

Let O be the centre of the unit circle making an acute angle x at the centre.

Let $\angle AOB = x, x > 0$

Draw $BC \perp OX$ and $DA \perp OX$ (to meet D on OB produced)

Area of $\Delta OAB = \frac{1}{2} OA.BC$

Area of $\Delta OAD = \frac{1}{2} OA.AD$

Area of sector OAB $= \frac{1}{2} OA^2 .x$

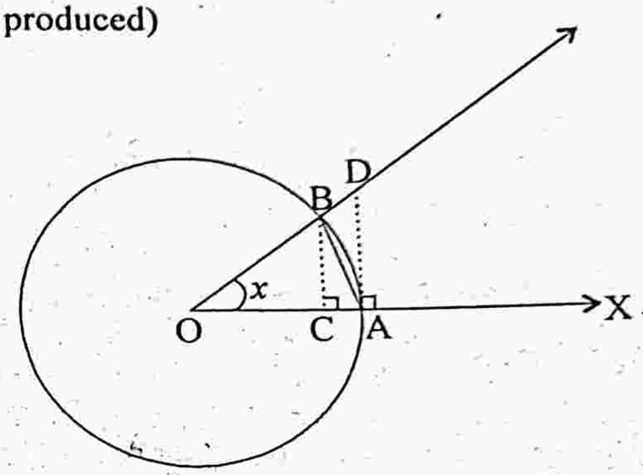

Fig. 13.18

From the figure, it is clear that area of $\Delta OAB <$ area of sector OAB $<$ area of ΔOAD

i.e., $\frac{1}{2} OA .BC < \frac{1}{2} OA^2 x < \frac{1}{2} OA.AD$

i.e., $\frac{BC}{OA} < x < \frac{DA}{OA}$, dividing by $\frac{1}{2} OA^2$

we get $\frac{BC}{OB} < x < \frac{DA}{OA}$ Since OA = OB,

i.e., $\sin x < x < \tan x$

i.e., $\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ Since $\sin x > 0$

i.e., $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

By taking the reciprocals, the inequality is reversed $1 > \frac{\sin x}{x} > \cos x$

$\therefore \lim_{x\to0} 1 \geq \lim_{x\to0}\frac{\sin x}{x} \geq \lim_{x\to0} \cos x$ $\therefore f(x) > g(x) \Rightarrow \lim_{x\to a} f(x) \geq \lim_{x\to a} g(x)$

i.e., $1 \geq \lim_{x\to0}\frac{\sin x}{x} \geq 1$

Thus if $x > 0$, then $\lim_{x\to0}\frac{\sin x}{x} = 1$, when x is positive.

Case (ii)

Let x be negative, then $-x$ is positive.

Put $-x = y$, y is positive

$x \to 0 \Rightarrow y \to 0$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{y\to 0}\frac{\sin(-y)}{-y} = \lim_{y\to 0}\frac{-\sin y}{-y} = \lim_{y\to 0}\frac{\sin y}{y} = 1$$

Thus in any case, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

That is, for any angle x in radian, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Theorem 7

If x is measured in radian $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$ **(March 2011)**

Proof

We have $1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}.\sin\left(\frac{x}{2}\right) = \lim_{\frac{x}{2}\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}.\lim_{\frac{x}{2}\to 0}\sin\left(\frac{x}{2}\right)$$

$$= 1 \times 0 = 0$$

Theorem 8

If x is measured in radian

i. $\lim_{x\to 0}\frac{x}{\sin x} = 1$ **ii.** $\lim_{x\to 0}\frac{\tan x}{x} = 1$

Proof

i. $$\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\left(\frac{1}{\frac{\sin x}{x}}\right) = \frac{\lim_{x\to 0}(1)}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)} = \frac{1}{1} = 1$$

ii. $$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\left(\frac{\sin x}{\cos x}\right)}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}.\frac{1}{\cos x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right).\lim_{x\to 0}\left(\frac{1}{\cos x}\right) = 1\left(\frac{1}{1}\right) = 1$$

Example 18

The value of $\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}$ is

Solution

$$\lim_{x\to 0}\frac{\sin 5x}{5x}=\lim_{5x\to 0}\frac{\sin 5x}{5x}=1$$

Example 19

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sin 2x}$

Solution

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}=\lim_{x\to 0}\left(\frac{\frac{\sin 4x}{x}}{\frac{\sin 2x}{x}}\right)=\frac{\lim\limits_{x\to 0}\frac{\sin 4x}{x}}{\lim\limits_{x\to 0}\frac{\sin 2x}{x}}=\frac{\lim\limits_{4x\to 0}4\times\frac{\sin 4x}{4x}}{\lim\limits_{2x\to 0}\frac{2\times\sin 2x}{2x}}=\frac{4\times 1}{2\times 1}=2$$

Example 20

Evaluate $\lim\limits_{x\to\pi}\dfrac{\sin(\pi-x)}{\pi(\pi-x)}$

Solution

Let $y=\pi-x$, as $x\to\pi$, $y\to 0$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\lim_{y\to 0}\frac{\sin y}{\pi y}=\frac{1}{\pi}\lim_{y\to 0}\frac{\sin y}{y}=\frac{1}{\pi}(1)=\frac{1}{\pi}$$

Another method

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{x\to\pi}\frac{\sin(\pi-x)}{(\pi-x)}$$ when $x\to\pi$, then $\pi-x\to 0$.

$$=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}=\frac{1}{\pi}(1)=\frac{1}{\pi}$$

Example 21

Evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{x^2}$

Solution

$$\lim_{x\to 0}\frac{1-\cos 4x}{x^2}=\lim_{x\to 0}\frac{2\sin^2 2x}{x^2}$$

$$=\lim_{x\to 0}2\times 4\times\frac{\sin^2 2x}{4x^2}=8\left[\lim_{2x\to 0}\frac{\sin 2x}{2x}\right]^2=8\times 1^2=8$$

Example 22

i. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}$ ii. Find $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 7x}$ iii. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sin 3x+7x}{3x+\sin 7x}\right)$

Solution

STUDY TIPS

$\lim\limits_{x\to 0}\dfrac{\sin mx}{x}=m$

$\lim\limits_{x\to 0}\left(\dfrac{\sin mx+nx}{mx+\sin nx}\right)=1$

i. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}=\lim\limits_{x\to 0}3\dfrac{\sin 3x}{3x}=3\lim\limits_{3x\to 0}\dfrac{\sin 3x}{3x}=3\times 1=3$

ii. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\sin 7x}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 7x}{x}}=\dfrac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}}{\lim\limits_{x\to 0}\dfrac{\sin 7x}{x}}=\dfrac{3}{7}$

iii. $\lim\limits_{x\to 0}\left(\dfrac{\sin 3x+7x}{3x+\sin 7x}\right)=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin 3x+7x}{x}}{\dfrac{3x+\sin 7x}{x}}$

$$=\dfrac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}+7}{\lim\limits_{x\to 0}3+\dfrac{\sin 7x}{x}}=\dfrac{\lim\limits_{x\to 0}\dfrac{\sin 3x}{3}+\lim\limits_{x\to 0}7}{\lim\limits_{x\to 0}3+\lim\limits_{x\to 0}\dfrac{\sin 7x}{7}}=\dfrac{3+7}{3+7}=\dfrac{10}{10}=1$$

Example 23

Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x}{\frac{\pi}{2}-x}$

Solution

$\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x}{\left(\frac{\pi}{2}-x\right)}$ Put $y=\dfrac{\pi}{2}-x$ as $x\to\dfrac{\pi}{2},\ y\to 0$

$\therefore \lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x}{\left(\frac{\pi}{2}-x\right)}=\lim\limits_{y\to 0}\dfrac{\cot\left(\frac{\pi}{2}-y\right)}{y}=\lim\limits_{y\to 0}\dfrac{\tan y}{y}=1$

Example 24

Evaluate $\lim\limits_{x\to 0}(\operatorname{cosec}x-\cot x)$ **(NCERT)**

Solution

$\lim\limits_{x\to 0}(\operatorname{cosec}x-\cot x)=\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}\right)=\lim\limits_{x\to 0}\left(\dfrac{1-\cos x}{\sin x}\right)$

$$= \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \lim_{\frac{x}{2}\to 0} \tan\frac{x}{2} = 0$$

Another method

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right) = \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{0}{1} = 0$$

Example 25

Let $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$ and if $\lim_{x\to 1} f(x) = f(1)$ what are the possible values of a and b **(NCERT)**

Solution

$f(1) = 4$

$\lim_{x\to 1} f(x) = f(1) \Rightarrow \lim_{x\to 1-} f(x) = 4$ and $\lim_{x\to 1+} f(x) = 4$

$\lim_{x\to 1-} f(x) = 4 \Rightarrow \lim_{x\to 1-}(a+bx) = 4 \Rightarrow a+b=4$

$\lim_{x\to 1+} f(x) = 4 \Rightarrow \lim_{x\to 1+}(b-ax) = 4 \Rightarrow b-a=4$

Solving $a+b=4$ and $b-a=4$, we get $a=0$ and $b=4$.

Example 26

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2-1, & x\le 1 \\ -x^2-1, & x>1 \end{cases}$ **(NCERT)**

Solution

$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(x^2-1) = 0$

$\lim_{x\to 1+} f(x) = \lim_{x\to 1+}(-x^2-1) = -2$

$\lim_{x\to 1-} f(x) \neq \lim_{x\to 1+} f(x)$

$\therefore$ Limit does not exist at $x=1$.

STUDY TIPS

$\lim_{x\to a} f(x)$ exists ,if and only if both $\lim_{x\to a-} f(x)$, $\lim_{x\to a+} f(x)$ exists and are equal.

Example 27

Choose the most appropriate answer from those given in the bracket. **(September 2010)**

i. If $\lim_{x\to 2}\frac{x^p-2^p}{x-2} = 192$, then $p =$ ....

[2, 4, 6, 10]

ii. $\lim_{x \to 0} \frac{\sin ax}{x \cos bx} =$ ....

[0, a, b, not defined]

iii. $\lim_{z \to -1} \frac{\sqrt{z} - 1}{1 - z} =$ ....

$[0, \frac{-1}{2}, \frac{1}{2}, 1]$

iv. $\lim_{x \to \frac{\pi}{4}} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\left(\frac{\pi}{4} - x\right)} =$ ....

$[0, 1, \frac{\pi}{4}$, not defined]

v. If $\lim_{x \to 0} \frac{f(x)}{x^2} = k$, then $\lim_{x \to 1} f(x) =$ ....

[0, 1, k, not defined]

Solution

i. $\lim_{x \to 2} \frac{x^p - 2^p}{x - 2} = 192$

$$p(2)^{p-1} = 192$$

$$= 6 \times 2^5 = 6\ (2)^{6-1}$$

$$\therefore p = 6$$

ii. $\lim_{x \to 0} \frac{\sin ax}{x \cos bx} = \lim_{x \to 0} \frac{\sin ax}{x} . \lim_{x \to 0} \frac{1}{\cos bx} = a \lim_{x \to 0} \frac{\sin ax}{ax} . \lim_{x \to 0} \frac{1}{\cos bx}$

$$= a\left(\lim_{ax \to 0} \frac{\sin ax}{ax}\right)\left(\lim_{bx \to 0} \frac{1}{\cos bx}\right) = a\,(1)\left(\frac{1}{1}\right) = a$$

iii. $\lim_{z \to 1} \frac{\sqrt{z} - 1}{1 - z} = \lim_{z \to 1} \frac{z^{\frac{1}{2}} - 1^{\frac{1}{2}}}{1 - z} = (-1) \lim_{z \to 1} \frac{z^{\frac{1}{2}} - 1^{\frac{1}{2}}}{z - 1}$

$$= (-1)\frac{1}{2}(1)^{\frac{1}{2}-1} = -1 \times \frac{1}{2} = \frac{-1}{2}$$

iv. $\lim_{x\to\frac{\pi}{4}} \dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\left(\frac{\pi}{4}-x\right)} = \lim_{x-\frac{\pi}{4}\to 0} \dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\left(\frac{\pi}{4}-x\right)} = \lim_{y\to 0} \dfrac{\tan y}{y}$, where $y = \dfrac{\pi}{4} - x = 1$

v. $\lim_{x\to 0} \dfrac{f(x)}{x^2} = k$

$\therefore f(x) = kx^2$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} kx^2 = k \lim_{x\to 1}(x^2) = k\,(1)^2 = k$