Differentiation from first principles

The process of finding the derivative of a function by using the definition of derivative is called **differentiation from first principles.**

The process for finding the derivatives of the differentiable functions from first principles is as follows.

WORKING RULE

1. Name the given function as $f(x)$
2. Find $f(x+h)$
3. Evaluate $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Derivative of Algebraic functions and trigonometric functions

1. Derivative of constant c

Let $f(x) = c$ Then $f(x+h) = c$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0 \qquad \therefore \frac{d}{dx}(c) = 0$$

2. Derivative of x

Let $f(x) = x$. Then $f(x+h) = x + h$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\left(\frac{h}{h}\right) = \lim_{h\to 0} 1 = 1 \qquad \therefore \frac{d}{dx}(x)=1$$

3. Derivative of ax

Let $f(x) = ax$. Then $f(x+h) = a(x+h)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a(x+h)-ax}{h}$$

$$= \lim_{h\to 0}\frac{ax+ah-ax}{h} = \lim_{h\to 0}\frac{ah}{h} = \lim_{h\to 0} a = a \qquad \therefore \frac{d}{dx}(ax)=a$$

4. Derivative of x^2

Let $f(x) = x^2$. Then $f(x+h) = (x+h)^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0} 2x+h = 2x \qquad \therefore \frac{d}{dx}(x^2)=2x$$

5. Derivative of $\frac{1}{x}$ **(September 2010, March 2011)**

Let $f(x) = \frac{1}{x}$ Then $f(x+h) = \frac{1}{x+h}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{-h}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2} \qquad \therefore \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

6. Derivative of x^n

Derivative of $f(x) = x^n$ is nx^{n-1} for any positive integer n.

Proof

Let $f(x) = x^n$. Then $f(x+h) = (x+h)^n$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n-x^n}{h} = \lim_{h\to 0}\frac{(x+h)^n-x^n}{(x+h)-x} = nx^{n-1}$$

$$\therefore \frac{d}{dx}(x^n) = n\,x^{n-1}$$

7. **Derivative of sinx**

Let $f(x) = \sin x$

Then $f(x + h) = \sin (x + h)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\qquad \text{since } \sin x-\sin y=2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$=\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}=\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}=\cos x$$

$$\therefore \frac{d}{dx}(\sin x)=\cos x$$

8. **Derivative of cosx**

Let $f(x) = \cos x$. Then $f(x + h) = \cos (x + h)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

$$=\lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right).\sin\left(\frac{h}{2}\right)}{h},\quad \left(\cos x-\cos y=\left(-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right)\right)$$

$$=\lim_{h\to 0}-\sin\left(x+\frac{h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}=\lim_{h\to 0}-\sin\left(x+\frac{h}{2}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}=-\sin x$$

$$\therefore \frac{d}{dx}(\cos x)=-\sin x$$

9. **Derivative of tanx**

Let $f(x) = \tan x$, x is not an odd multiple of $\frac{\pi}{2}$. Then $f(x + h) = \tan (x + h)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{\cos(x+h).\cos x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right] = \lim_{h\to 0} \frac{\sin h}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0}\frac{\sin h}{h} \cdot \lim_{h\to 0}\left(\frac{1}{\cos(x+h)\cos x}\right) = 1 \cdot \frac{1}{\cos x \cdot \cos x} = \sec^2 x \quad \therefore \frac{d}{dx}(\tan x) = \sec^2 x$$

10. Derivative of secx **(September 2012)**

Let $f(x) = \sec x$, x is not an odd multiple of $\frac{\pi}{2}$

$\therefore f(x+h) = \sec(x+h)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(x+\frac{h}{2}\right).\sin\left(\frac{-h}{2}\right)}{\cos(x+h)\cos x}\right] = \lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)}{\cos(x+h).\cos x}\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)}{\cos(x+h).\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \frac{\sin x}{\cos x.\cos x}.1 = \tan x.\sec x = \sec x.\tan x$$

11. Derivative of cosecx **(September 2013)**

Let $f(x) = \text{cosec}\,x$, when x is not an even multiple of $\frac{\pi}{2}$.

$f(x+h) = \text{cosec}\,(x+h)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\text{cosec}(x+h)-\text{cosec}x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h).\sin x}\right] = \lim_{h\to 0}\left[\frac{-2\sin\frac{h}{2}}{h}\cdot\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$= -\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h).\sin x}\right) = -\lim_{\frac{h}{2}\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right).\lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)}{\sin(x+h)\sin x}$$

$$= -1\times\frac{\cos x}{\sin x.\sin x} = -\cot x\ \text{cosec}x = -\text{cosec}x\ \cot x$$

12. Derivative of cotx

Let $f(x) = \cot x$, when x is not an even multiple of $\frac{\pi}{2}$.

$f(x + h) = \cot(x + h)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{\sin x.\cos(x+h) - \sin(x+h)\cos x}{h\sin(x+h).\sin x}$$

$$= \lim_{h\to 0}\frac{-[\sin(x+h).\cos x - \cos(x+h).\sin x]}{h\sin(x+h)\sin x}$$

$$= -\lim_{h\to 0}\frac{\sin(x+h-x)}{h\sin(x+h)\sin x}$$

$$= -\lim_{h\to 0}\frac{\sin h}{h}.\frac{1}{\sin(x+h).\sin x}$$

$$= -\lim_{h\to 0}\frac{\sin h}{h}.\lim_{h\to 0}\frac{1}{\sin(x+h)\sin x}$$

$$= -1\times\frac{1}{\sin x.\sin x} = -\text{cosec}^2 x$$

Example 36

Find the derivative of $x^2 + x + 1$ from first principles.

Solution

Let $f(x) = x^2 + x + 1$, then

$f(x + h) = (x + h)^2 + (x + h) + 1$

$= x^2 + 2hx + h^2 + x + h + 1.$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2hx+h^2+x+h+1)-(x^2+x+1)}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2+h}{h} = \lim_{h\to 0} \frac{h(2x+h+1)}{h}$$

$$= \lim_{h\to 0} (2x+h+1) = 2x+1$$

Example 37

Find the derivative of $\frac{2x+3}{x-2}$ from first principle. **(NCERT)**

Solution

Let $f(x) = \frac{2x+3}{x-2}$, Then $f(x+h) = \frac{2(x+h)+3}{(x+h)-2} = \frac{2x+3+2h}{x-2+h}$

$$f'(x)=\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{2x+3+2h}{x-2+h}-\frac{2x+3}{x-2}}{h}$$

$$=\lim_{h\to 0} \frac{(x-2)(2x+3+2h)-(2x+3)(x-2+h)}{h(x-2)(x-2+h)}$$

$$=\lim_{h\to 0} \frac{2x^2+3x+2hx-4x-6-4h-2x^2+4x-2hx-3x+6-3h}{h(x-2)(x-2+h)}$$

$$=\lim_{h\to 0} \frac{-7h}{h(x-2)(x-2+h)} = \lim_{h\to 0} \frac{-7}{(x-2)(x-2+h)} = \frac{-7}{(x-2)(x-2)} = \frac{-7}{(x-2)^2}$$

$$\therefore \frac{d}{dx}\left(\frac{2x+3}{x-2}\right) = \frac{-7}{(x-2)^2}$$

Example 38

Find the derivative of f from the first principles, where f is given by $f(x) = x+\frac{1}{x}$ **(NCERT)**

Solution

The function is not defined at $x = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(x+h+\frac{1}{x+h}\right)-\left(x+\frac{1}{x}\right)}{h} = \lim_{h\to 0} \frac{1}{h}\left[h+\frac{1}{x+h}-\frac{1}{x}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[h + \frac{x-x-h}{x(x+h)}\right] = \lim_{h\to 0} \frac{1}{h}\left[h\left(1-\frac{1}{x(x+h)}\right)\right] = \lim_{h\to 0}\left[1-\frac{1}{x(x+h)}\right] = 1-\frac{1}{x^2}$$

Again, note that the function f' is not defined at $x = 0$.

Example 39

Find the derivative of $\sin x + \cos x$ from first principle **(NCERT)**

Solution

Let $f(x) = \sin x + \cos x$

Then $f(x + h) = \sin (x + h) + \cos (x + h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)+\cos(x+h)-\sin x-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x+\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}+-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\left[\cos\left(x+\frac{h}{2}\right)-\sin\left(x+\frac{h}{2}\right)\right]}{h}$$

$$= \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \lim_{h\to 0}\left[\cos\left(x+\frac{h}{2}\right)-\sin\left(x+\frac{h}{2}\right)\right] = 1.\ [\cos x - \sin x] = \cos x - \sin x$$

$$\therefore \frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$$

Example 40

Find the derivative of $x \sin x$ from first principle **(NCERT)**

Solution

Let $f(x) = x \sin x$

Then $f(x + h) = (x + h) \sin (x + h) = x \sin (x + h) + h \sin (x + h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{x\sin(x+h)+h\sin(x+h)-x\sin x}{h} = \lim_{h\to 0} \frac{x[\sin(x+h)-\sin x]+h\sin(x+h)}{h}$$

$$= \lim_{h \to 0} \frac{x\left[2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right] + h\sin(x+h)}{h} = \lim_{h \to 0} x \frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} + \lim_{h \to 0} \frac{h\sin(x+h)}{h}$$

$$= x.\lim_{h \to 0} \cos\left(x+\frac{h}{2}\right).\ \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} + \lim_{h \to 0} \sin(x+h) = x\cos x\,.\,1 + \sin x = x\cos x + \sin x$$

$$\therefore \frac{d}{dx}(x\sin x) = x\cos x + \sin x$$