MATHEMATICAL REASONING

14

14.1 INTRODUCTION

In this chapter, we shall learn about some basics of mathematical reasoning. The ability of reasoning makes human being *superior* to other species. This reasoning varies from person to person. We shall discuss the process of reasoning in the context of mathematics. There are two kinds of reasoning. **Inductive reasoning and Deductive reasoning**

We have already discussed the inductive reasoning in Mathematical Induction. Now we shall discuss some fundamentals of deductive reasoning.

14.2 STATEMENTS

In our daily life, we express our ideas and thoughts with the help of sentences. A **sentence** is a group of words arranged according to the laws of grammar, which conveys a complete idea.

Illustration 1

Consider the following sentences.

i. Delhi is a city ii. $3 + 8 = 11$

iii. Sum of the angles in a triangle is two right angles

iv. Every square is a rhombus

We see that each of the sentences given above are true.

Here each sentence is considered as a ***statement***

Illustration 2

Consider the following sentences

i. $5 + 3 = 11$ ii. Paris is in India iii. Sun is a planet

iv. Ernakulam is the capital of Kerala

We see that each of the sentences given above are false.

Here each sentence is considered as a ***statement***

Illustration 3

Consider the following sentences

i. Give me a pen.

This is a *request* and is an *imperative sentence.*

ii. Open the door.

This is a *command* and is an *imperative sentence.*

iii. What is your name?

This is a *question* and is an *interrogative sentence.*

iv How beautiful it is!

This is an *exclamatory* sentence.

In this illustration,we cannot say whether the statements are true or false.

Hence each sentence is not considered as a statement.

Definition

A sentence is called a mathematically acceptable statement if it is either true or false but not both. When we mention about a statement, it is a mathematically acceptable statement. The statements are usually denoted by the letters *p, q, r, s,* .....

Open sentence

Sentence involving variable(s) is known as an ***open sentence.***

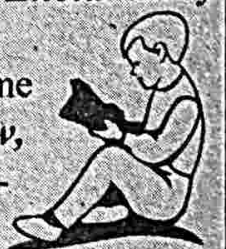
"Tommorrow is Friday" is an open sentence. The truth of this sentence depends on the day on which it is spoken.

$x + 2 = 5$ is also an open sentence. The truth of this sentence depends on the value of x.

Open sentences are not mathematical statements.

STUDY TIPS

- ***Imperative, Interrogative, Exclamatory*** sentences are not statements.
- Sentences involving variable time such as ***Today, Tommorrow, Yesterday*** etc are not statements.
- Sentences with ***Pronouns*** (unless a particular person is referred to) is not a statement.
- Sentences involving variable place such as ***Here, There*** etc are not statements.



Paradox

A statement which is both true and false simultaneously is known as a **Paradox.**

Paradoxes are not statements.

Caution !

'This sentence is false' is not a statement.

Truth value of a statement

The truth or falsehood of a statement is called its ***truth value***. The truth value of a true statement is ***true*** and is denoted by T. The truth value of a false statement is ***false*** and is denoted by F.

Example 1

Which of the following sentences are statements? Give reasons for your answer. **(NCERT)**

i. There are 35 days in a month.
ii. Mathematics is difficult.
iii. The sum of 5 and 7 is greater than 10.
iv. The square of a number is an even number.
v. The sides of a quadrilateral have equal length.
vi. Answer this question.
vii. The product of (–1) and 8 is 8.
viii. The sum of all interior angles of a triangle is 180°.
ix. Today is a windy day.
x. All real numbers are complex numbers.

Solution

i. In this sentence, "There" refers to a variable time. i.e., any of the twelve months. The maximum number of days in any month is 31. So the sentence is always false. Therefore it is a statement
ii. This is not a statement because for some people mathematics can be easy and for some others it can be difficult.
iii. This sentence is always true because the sum is 12 and it is greater than 10. Therefore, it is a statement.
iv. This sentence is sometimes true and sometimes not true. For example the square of 2 is even number and the square of 3 is an odd number. Therefore, it is not a statement.
v. This sentence is sometimes true and sometimes false. For example, squares and rhombus have equal length whereas rectangles and trapezium have unequal length. Therefore, it is not a statement.
vi. It is an order and therefore, is not a statement.
vii. This sentence is false as the product is (–8). Therefore, it is a statement.
viii. This sentence is always true and therefore, it is a statement.
ix. It is not clear from the context which day is referred and therefore, it is not a statement.
x. This is a true statement because all real numbers can be written in the form $a + i \times 0$.

Example 2

Give three examples of sentences which are not statements. **(NCERT, March 2010)**

Solution

i. Tomorrow is Monday. The truth of this sentence depends on the day in which it is spoken. So it is not a statement.

ii. Real numbers are positive. Let x be a real number. The truth of this sentence depends on the value of x. So it is not a statement.

iii. The three sides of a triangle are equal in length. This sentence is true for equilateral triangles and false for right triangles. So it is not a statement.

SOLUTIONS TO NCERT TEXT BOOK EXERCISE 14.1

1. Which of the following sentences are statements? Give reasons for your answer.
   i. There are 35 days in a month.
   ii. Mathematics is difficult.
   iii. The sum of 5 and 7 is greater than 10.
   iv. The square of a number is an even number.
   v. The sides of a quadrilateral have equal length.
   vi. Answer this question.
   vii. The product of (–1) and 8 is 8.
   viii. The sum of all interior angles of a triangle is 180°.
   ix. Today is a windy day.
   x. All real numbers are complex numbers.

Solution ***Refer Example 1***

2. Give three examples of sentences which are not statements. Give reasons for the answers.

Solution

Refer Example 2

14.3. NEW STATEMENTS FROM OLD

In the following section we are discussing the method of forming new statements from the old statements

14.3.1. Negation of a statement

The contradiction of a given statement is called its ***negation***. The negation of a statement is the denial of the statement. Consider the statement.

***p* : Everyone in Tamil Nadu speaks Tamil**

The negation of this statement p means that not all persons in Tamilnadu speaks Tamil. The statement 'No person in Tamil Nadu speaks Tamil' is not the negation of the statement p. The negation of the statement p means that **there is atleast one person in Tamil Nadu who does not speak Tamil.**

Illustration 4

Let us consider the statement *Thiruvananthapuram is a city.*

The negation of this statement is *It is not the case that Thiruvananthapuram is a city.*

This can also be written as *It is false that Thiruvananthapuram is a city*

This can simply be written as *Thiruvananthapuram is not a city.*

Definition

If p is a statement, then the **negation of** p is also a statement and is denoted by $\sim p$ and read as '**not** p'.

WORKING RULE

To write the negation of a statement

If p is a statement, then the negation of p is formed as follows.

Insert the phrases "***It is not the case that***" or '***It is false that***' before p

OR

Insert the word '**not**' at proper place in the statement p.

Example 3

Write the negation of the following statements: **(NCERT)**

i. Chennai is the capital of Tamil Nadu

ii. $\sqrt{2}$ is not a complex number

iii. All triangles are not equilateral triangle. **(March 2013)**

iv. The number 2 is greater than 7. **(March 2011)**

v. Every natural number is an integer. **(March 2012, 2013)**

Solution

i. Chennai is not the capital of Tamil Nadu. ***or***

It is false that Chennai is the capital of Tamil Nadu

ii. $\sqrt{2}$ is a complex number. ***or***

It is not the case that $\sqrt{2}$ is not a complex number

iii. All triangles are equilateral tringles.

iv. The number 2 is not greater than 7.

v. Every natural number is not an integer.

Example 4

Write the negation of the following statements. **(NCERT)**

i. Both the diagonals of a rectangle have the same length.

ii. $\sqrt{7}$ is rational. **(March 2010, March 2015)**

Solution

i. It is false that both diagonals of a rectangle have the same length.

ii. $\sqrt{7}$ is not rational. ***or*** It is not the case that $\sqrt{7}$ is rational.

Example 5

Write the negation of the following statements: **(NCERT)**

i. *p:* For every real number x, $x^2 > x$.

ii. *q:* There exists a rational number x such that $x^2 = 2$.

iii. *r:* All birds have wings.

iv. *s:* All students study mathematics at the elementary level.

Solution

i. The negation of p is "It is false that p" **or**
$\sim p$: There exists a real number x such that $x^2 < x$.

ii. Negation of q is "It is false that q" **or**
$\sim q$: There does not exist a rational number x such that $x^2 = 2$ **or**
$\sim q$: For all real numbers x, $x^2 \neq 2$

iii. The negation of r is "It is false that r" **or**
$\sim r$: There exists a bird which have no wings.

iv. The negation of s is "It is false that s"
or
$\sim s$: There exists a student who does not study mathematics at the elementary level.

STUDY TIPS

If p is true then $\sim p$ is false.
If p is false then $\sim p$ is true

Example 6

Are the following pairs of statements negations of each other? Justify your answer.

i. The number x is not a rational number **(NCERT)**
The number x is not an irrational number

ii. The number x is a rational number
The number x is an irrational number

Solution

i. The negation of the first statement is *'The number x is a rational number'*. It is the same as the second statement. Therefore the pair of statements are negations of each other.

ii. The negation of the first statement is *'The number x is not a rational number'*. It is same as the second statement. Therefore the pair of statements are negations of each other.

14.3.2. Compound statement

In mathematical reasoning, we come across statements which can be broken into two or more statements.

Illustration 5

Consider the statement

p : A square is a quadrilateral and its four sides are equal.

The statement p is formed by the following two statements.

q : A square is a quadrilateral r : The four sides of a square are equal

q, r are connected by '**and**'

Illustration 6

Consider the statement

p : All integers are either even or odd

The statement p is formed by the following two statements

q : All integers are even r : All integers are odd

q, r are connected by '**or**'

Definition

A **compound statement** is a statement which can be broken into two or more statements. Here, each statement is called a ***component statement***.

Many mathematical compound statements are obtained by combining two or more statements using some connecting words like ***and, or*** etc.

A statement which cannot be broken into component statements is called a **simple statement.**

'The sky is blue', '$\sqrt{2}$ is a rational number' are examples of simple statements.

The truth or falsehood of a compound statement depends on the truth and falsehood of its component statements.

Example 7

Write the component statement of the compound statement "All living things have two legs and two eyes". **(NCERT, March 2010)**

Solution

The component statements are

p : All living things have two legs.

q : All living things have two eyes.

Example 8

Write the component statements of the compound statement "A line is straight and extends indefinitely in both directions". Also check whether the compound statement is true or false. (NCERT)

Solution

The component statements are

p : A line is straight

q : A line extends indefinitely in both directions.

Here both p and q are true. Hence the compound statement is true.

Example 9

Write the component statements of the following compound statements. Mention the connecting word.

i. It is raining and it is cold ii. Zero is a positive or negative number

Solution

The component statements are

i. *p: It is raining* *q: It is cold*

Here the connecting word is *and*

ii. *p: Zero is a positive number* *q: Zero is a negative number*

Here the connecting word is *or*.

14.4. SPECIAL WORDS OR PHRASES

In section 14.3 we have studied that the compound statement is formed by combining simple statements by using the words **and, or** etc.

The words which combine simple statements to form compound statements are called special words/phrases or connectives. The basic connectives are the words *'And* and *Or'*

14.4.1. The word 'and'

Two simple statements can be connected by the word *'And'* to form a compound statement.

Illustration 7

Consider the compound statement with *'and'*

p: All rational numbers are real numbers and all real numbers are complex numbers.

This statement can be broken into two component statements as

q: All rational numbers are real numbers

r: All real numbers are complex numbers

We observe that both the statements *q* and *r* are true. Therefore, the compound statement is true.

Let us consider another statement.

p: 36 is divisible by 3, 6 and 5

This statement is broken into 3 component statements as

q: 36 is divisible by 3 *r: 36 is divisible by 6* *s: 36 is divisible by 5*

We observe that the third statement is false, while the other two statements are true. Therefore, the compound statement is false.

Rule for the compound statement with "and"

i. The compound statement with *'and'* is true, if all its component statements are true.

ii. The compound statement with *'and'* is false, if any or all of its component statements are false.

Example 10

Write the component statements of the following compound statements and check whether the compound statement is true or false.

i. Kerala is in India and Trivandrum is in Kerala.

ii. Zero is less than every positive integer and every negative integer **(NCERT)**

iii. All living things have two legs and two eyes.

Solution

The component statements are

i. *p: Kerala is in India* *q: Trivandrum is in Kerala.*

Both these statements are true. Therefore the compound statement is true.

ii. *p: Zero is less than every positive integer*

q: Zero is less than every negative integer

The first statement is true and the second statement is false. Therefore the compound statement is false.

STUDY TIPS

$\sim (p \text{ and } q) = \sim p \text{ or } \sim q$

iii. *p: All living things have two legs.* *q: All living things have two eyes*

Both these statements are false. Therefore the compound statement is false.

14.4.2. The word 'Or'

Illustration 8

Consider the compound statement with *or*.

p: All integers are positive or negative

This statement can be broken into two statements as

q: All integers are positive *r: All integers are negative*

Consider the statement *p: The sun shines or it rains.*

This statement can be broken into two component statements as

q: The sun shines *r: It rains*

Consider the statement

p: Two lines in a plane either intersect at one point or they are parallel

This statement can be broken into two component statements as

q: Two lines in a plane intersect at a point *r: Two lines in a plane are parallel*

NOTE

It should be noted that the word ***'or'*** is used with other meanings in English language.

For example: In the statement two or three children are playing in the playground, here the word *'or'* is used to indicate an approximate number of children. It is not used as a connective. As a connective the word ***'or'*** is used in two distinct ways. Sometimes it is used in the sense of *'p **or** q **or** both'*. That is, atleast one of the two alternatives occurs, then we call it as ***'inclusive or'***. Sometimes ***'or'*** is used in the sense of *p **or** q but not both'*. That is, exactly one of the two alternatives occurs. Then we call it as ***exclusive or***

Example 11

For each of the following statements, determine whether an 'inclusive or' or 'exclusive or' is used. Give reasons for your answer.

i. The school is closed if it is a holiday or Sunday.

ii. To apply for a driving licence, you need a Ration Card or a Voter Identity Card.

iii. The students can take Malayalam or Hindi as their second language.

iv. All Integers are positive or negative. **(NCERT)**

Solution

i. Here *'or'* is inclusive since the school is closed on holiday as well as Sunday. If a holiday falls on Sunday, then also the school is closed.

ii. Here *"or"* is inclusive, since a person can apply for driving licence if he has a Ration Card or Voter Identity Card or both.

iii. Here *'or'* is exclusive, since a student can take Malayalam or Hindi as second language but not both.

iv. Here *'or'* is exclusive since all integers can be positive or negative but not both.

Rule for the compound statement with 'Or'

i. A compound statement with *'or'* is true when any or all of its component statements are true.

ii. A compound statement with *'or'* is false when all of its component statements are false.

Example 12

Check whether "Or" used in the following compound statement is exclusive or inclusive? Write the component statements of the compound statements and use them to check whether the compound statement is true or not. Justify your answer. **(NCERT)**

t: You are wet when it rains or you are in a river.

Solution

"Or" used in the given statement is inclusive because it is possible that it rains and you are in the river.

The component statements of the given statement are

p : You are wet when it rains. *q : You are wet when you are in a river.*

Here both the component statements are true and therefore, the compound statement is true.

Example 13

Consider the statement "$\sqrt{5}$ is a rational number or an irrational number".

i. Find the component statements and check whether they are true or false.

ii. Check whether the compound statement is true or false.

(March 2010, October 2011)

Solution

i. The component statements are

p : $\sqrt{5}$ is a rational number q : $\sqrt{5}$ is an irrational number

Here p is false and q is true

ii. The compound statement is true since q is true.

Example 14

Write the component statements of the following compound statements and check whether the compound statement is true or false.

i. Two lines in a plane intersect at a point or they are parallel

ii. 125 is a multiple of 7 or 8

iii. 36 is divisible by 6 or 9

iv. 48 is divisible by 3 or 6 **(March 2012)** **(NCERT)**

Solution

i. The component statements are

p: Two lines in a plane intersect at a point *q: Two lines in a plane are parallel*

Here we observe that when p is true, q is false and when q is true, p is false

$\therefore$ The compound statement is true.

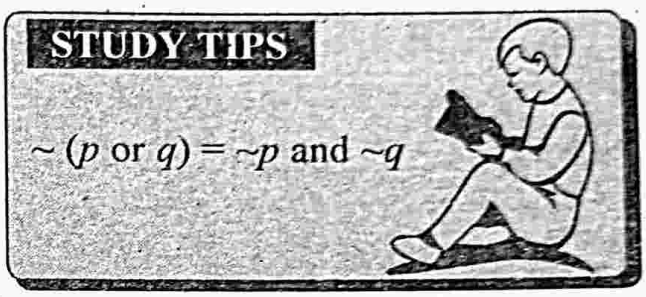
ii. The component statements are

p: 125 is a multiple of 7

q: 125 is a multiple of 8

Here we observe that both p and q are false statements.

$\therefore$ The compound statement is false.



iii. The component statements are

p: 36 is divisible by 6 *q: 36 is divisible by 9*

Here we observe that both statements p and q are true statements.

$\therefore$ The compound statement is true.

iv. The component statements are

p : 48 is divisible by 3 q : 48 is divisible by 6

Here we observe that both p and q are true statements.

Hence the compound statement is true.

Example 15

i. Write the component statements, "All prime numbers are either even or odd.

ii. Check whether these component statements are true or false.

iii. Write the negation of the statement. **(March 2011)**

Solution

i. The component statements are

p: all prime numbers are odd numbers.

q : all prime numbers are even numbers.

ii. p is false, since 2 is a prime number which is even.

q is false, since 3 is a prime number which is odd.

iii. $\sim p$: All prime numbers are not odd numbers.

$\sim q$: All prime numbers are not even numbers.